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Related fixed point on two metric spaces via C-class functions

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Abstract The aim of this paper is to introduce the notions of (I, ψ)-contractions and present a related fixed point theorem for this type of contraction in the setting of metric spaces. This result extends and generalizes [12]. We give examples to explain our findings. Also we discuss an application to nonlinear integral equations.

1 Introduction and Preliminaries

Metric fixed point theory plays a central role in many areas of mathematics and other scientific branches (see [1], [3], [4], [8]). Many researchers have extended the classic metric fixed point theorems to single and multi-valued mappings ([6], [7], [9], [11], [15], [17]). Recently, many authors showed an interest for so-called related fixed point theorems ([5], [11], [14],[15]). Lately, Hamaizia et al. [12] have extended results in [13] for two pairs of mappings of two metric spaces.

The following result is Theorem 2.1 in [12].

Theorem 1.1. *Let (X, d) and (Y, ρ) be complete metric spaces, let A, B be mappings of X into Y , and let S, T be mappings of Y into X to satisfy*

$$\begin{aligned} d(Sy, Ty')d(SAx, TBx') &\leq c\max\{d(Sy, Ty')\rho(Ax, Bx'), d(x', Sy)\rho(y', Ax), \\ &\quad d(x, x')d(Sy, Ty'), d(Sy, SAx)d(Ty', TBx')\}, \\ \rho(Ax, Bx')\rho(BSy, ATy') &\leq c\max\{d(Sy, Ty')\rho(Ax, Bx'), d(x', Sy)\rho(y', Ax), \\ &\quad \rho(y, y')\rho(Ax, Bx'), \rho(Ax, BSy)\rho(Bx', ATy')\}, \end{aligned}$$

for all x, x in X and y, y' in Y , where $0 \leq c < 1$. If one of the mappings A, B, S and T is continuous then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y . Further, $Az = Bz = w$ and $Sw = Tw = z$.

The aim of this paper is to make use of C-class functions to provide a new condition on the mappings A, B, S and T that guarantees the existence of related fixed points in two metric spaces. Our results generalize those in [12] and some older ones. First, we start with the definition of a C-class function introduced in 2014, by A. H. Ansari [2].

Definition 1.2. [2] A continuous function $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if for any $s, t \in [0, \infty)$; the following conditions hold

$c1$ $F(s, t) \leq s$,

$c2$ $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $F(0, 0) = 0$ could be imposed in some cases if required. The letter C will denote the class of all C- functions.

Example 1.3. The following examples show that the class C is nonempty:

1. $F(s, t) = s - t$.
2. $F(s, t) = ms$; for some $m \in (0, 1)$.
3. $F(s, t) = \frac{s}{(1+t)^r}$ for some $r \in (0, 1)$.
4. $F(s, t) = \frac{\log(t+a^s)}{(1+t)}$, for some $a > 1$.
5. $F(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$,

6. $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to (0,1)$, and $\beta(s)$ is continuous,
7. $F(s,t) = s - \frac{t}{k+t}$.

Let Φ_u denotes the class of the functions $\varphi : [0,\infty) \to [0,\infty)$ that satisfy the following conditions:
a) φ is continuous,
b) $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.

Definition 1.4. [10] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:
i) ψ is non-decreasing and continuous,
ii) $\psi(t) = 0$ if and only if $t = 0$.

Let us suppose that Ψ denote the class of the altering distance functions.

Definition 1.5. A triplet (ψ, φ, F) where $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in C$ is said to be monotone if for any $x, y \in [0,\infty)$;

$$x \leq y \Rightarrow F(\psi(x), \varphi(x)) \leq F(\psi(y), \varphi(y)).$$

The next example shows that the class of monotone triplets (ψ, φ, F) is nonempty.

Example 1.6. Let $F(s,t) = s - t$, $\varphi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & if \quad 0 \leq x \leq 1 \\ x^2 & if \quad x > 1 \end{cases},$$

then (ψ, φ, F) is monotone.

Lemma 1.7. *[16] Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of positive integers such that $n_k > m_k > 0$ and
the following sequences tend to ε^+ when $k \to \infty$

$$d(x_{n_k}, x_{m_k}), d(x_{n_k+1}, x_{m_k}), d(x_{n_k}, x_{m_k-1}, d(x_{n_k+1}, x_{m_k-1}).$$

Lemma 1.8. *[16] Let (X,d) be a metric space and let $\{y_n\}$ be a sequence in X such that $d(y_n, y_{n+1}) = 0$ is nonincreasing and*

$$\lim_{n\to+\infty} d(y_n, y_{n+1}) = 0.$$

If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tend to ε when $k \to \infty$

$$d(x_{2n_k}, x_{2m_k}), d(x_{2n_k+1}, x_{2m_k}), d(x_{2n_k}, x_{2m_k-1}, d(x_{2n_k+1}, x_{2m_k-1}), d(x_{2n_k+1}, x_{2m_k+1}), ...$$

Our result extends Theorem 2.1 of Hamaizia et al [12] . Examples are provided to illustrate the validity of our results.

2 Main Results

Now we present our main result.

Theorem 2.1. *Let (X,d) and (Y,ρ) be complete metric spaces, let A, B be mappings of X into Y , and let S, T be mappings of Y into X satisfying the inequalities*

$$d(Sy, Ty')d(SAx, TBx') \leq F(m_d(x,x',y,y'), \varphi(m_d(x,x',y,y'))), \tag{2.1}$$
$$\rho(Ax, Bx')\rho(BSy, ATy') \leq F(m_\rho(x,x',y,y'), \varphi(m_\rho(x,x',y,y'))), \tag{2.2}$$

for all x, x in X and y, y' in Y, where $\varphi \in \Phi_u$ and $F \in C$ such that (I, φ, F) is monotone and

$$\begin{aligned} m_d(x, x', y, y') &= \max\{d(Sy, Ty')\rho(Ax, Bx'), d(x', Sy)\rho(y', Ax), \\ &\quad d(x, x')d(Sy, Ty'), d(Sy, SAx)d(Ty', TBx')\} \\ m_\rho(x, x', y, y') &= \max\{d(Sy, Ty')\rho(Ax, Bx'), d(x', Sy)\rho(y', Ax), \\ &\quad \rho(y, y')\rho(Ax, Bx'), \rho(Ax, BSy)\rho(Bx', ATy')\}. \end{aligned}$$

If one of the mappings A, B, S and T is continuous then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y . Further, $Az = Bz = w$ and $Sw = Tw = z$.

Proof. Let's consider x an arbitrary point in X, we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y as

$Sy_{2n-1} = x_{2n-1}$, $Bx_{2n-1} = y_{2n}$, $Ty_{2n} = x_{2n}$, $Ax_{2n} = y_{2n+1}$

Applying inequality (2.1), we get

$$d(Sy_{2n-1}, Ty_{2n})d(SAx_{2n}, TBx_{2n-1}) \le F(m_d(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})), \varphi(m_d(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}))),$$

where

$$\begin{aligned} m_d(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) &= \max\{d(Sy_{2n-1}, Ty_{2n})\rho(Ax_{2n}, Bx_{2n-1}), d(x_{2n-1}, Sy_{2n-1})\rho(y_{2n}, Ax) \\ &\quad , d(x_{2n}, x_{2n-1})d(Sy, Ty_{2n}), d(Sy_{2n-1}, SAx_{2n})d(Ty_{2n}, TBx_{2n-1})\}), \end{aligned}$$

Then, we obtain

$$\begin{aligned} d(x_{2n-1}, x_{2n})d(x_{2n+1}, x_{2n}) &\le F(\max\{d(x_{2n-1}, x_{2n})\rho(y_{2n+1}, y_{2n}), d(x_{2n}, x_{2n-1})d(x_{2n-1}, x_{2n})\}), \\ &\quad \varphi(\max\{d(x_{2n-1}, x_{2n})\rho(y_{2n+1}, y_{2n}), d(x_{2n}, x_{2n-1})d(x_{2n-1}, x_{2n})\})), \end{aligned}$$

Thus,

$$d(x_{2n+1}, x_{2n}) \le F(\max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n}, x_{2n-1})\}); \varphi(\max\{\rho(y_{2n+1}, y_{2n}), d(x_{2n-1}, x_{2n})\})). \tag{2.3}$$

Similar, applying inequality (2.2) , we get

$$\begin{aligned} \rho(y_{2n}, y_{2n+1})\rho(y_{2n}, y_{2n+1}) &\le F(\max\{d(x_{2n-1}, x_{2n})\rho(y_{2n}, y_{2n+1}), \rho(y_{2n-1}, y_{2n})\rho(y_{2n}, y_{2n+1})\}), \\ &\quad \varphi(c\max\{d(x_{2n-1}, x_{2n})\rho(y_{2n}, y_{2n+1}), \rho(y_{2n-1}, y_{2n})\rho(y_{2n}, y_{2n+1})\})). \end{aligned}$$

Then

$$\rho(y_{2n}, y_{2n+1}) \le F(\max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n})\}), \varphi(\max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n})\})). \tag{2.4}$$

By(2.3),(2.4) and from n, it follow

$$\begin{aligned} d(x_{n+1}, x_n) &\le F(\max\{\rho(y_{n+1}, y_n), d(x_n, x_{n-1})\}), \varphi(\max\{\rho(y_{n+1}, y_n), d(x_{n-1}, x_n)\})). \\ \rho(y_n, y_{n+1}) &\le F(\max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\}), \varphi(\max\{d(x_{n-1}, x_{2n}), \rho(y_{n-1}, y_n)\})), \end{aligned}$$

witch implies

$$\begin{aligned} d(x_{n+1}, x_n) &\le F(\max\{\rho(y_{n+1}, y_n), d(x_n, x_{n-1})\}), \varphi(\max\{\rho(y_{n+1}, y_n), d(x_{n-1}, x_n)\})) \\ &\le \psi(\max\{\rho(y_{n+1}, y_n), d(x_{n-1}, x_n)\} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \rho(y_n, y_{n+1}) &\le F(\max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\}); \varphi(\max\{d(x_{n-1}, x_{2n}), \rho(y_{n-1}, y_n)\})) \\ &\le \psi(\max\{d(x_{n-1}, x_{2n}), \rho(y_{n-1}, y_n)\}). \end{aligned} \tag{2.6}$$

So, from (2.4)and(2.5), respectively, it becomes

$$d(x_{n+1}, x_n) \leq \max\{\rho(y_{n+1}, y_n), d(x_{n-1}, x_n)\} \tag{2.7}$$
$$\rho(y_n, y_{n+1}) \leq \max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\}, \tag{2.8}$$

also from this, we have

$$d(x_{n+1}, x_n) \leq \max\{\max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\}, d(x_{n-1}, x_n)\} = \max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\}.$$

Therefore

$$\max\{\rho(y_{n+1}, y_n), d(x_{n+1}, x_n)\} \leq \max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\} \to h \geq 0, \tag{2.9}$$

from (2.4)and(2.5)

$$\begin{aligned} \max\{\rho(y_{n+1}, y_n), d(x_{n+1}, x_n)\} &\leq F(\max\{\rho(y_{n+1}, y_n), d(x_n, x_{n-1})\}), \\ &\quad \varphi(\max\{\rho(y_{n+1}, y_n), d(x_{n-1}, x_n)\})) \\ &\leq F(\max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\}), \\ &\quad \varphi(\max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\})). \end{aligned} \tag{2.10}$$

We prove now that $h = 0$. If we take $h > 0$ letting $n \to +\infty$, we obtain in (2.10) with $\max\{\rho(y_{n+1}, y_n), d(x_{n+1}, x_n)\} \to h$,we conclude that

$$h \leq F(h, \varphi(h)) \leq h$$

that is hold $F(h, \varphi(h)) = h$, F is of C-class, thus $h = 0$ or $\varphi(h) = 0$, we get a contradiction. Hence

$$\lim_{n\to+\infty} d(x_{n+1}, x_n) = 0 \tag{2.11}$$

and

$$\lim_{n\to+\infty} \rho(y_n, y_{n+1}) = 0. \tag{2.12}$$

Now proving that $\{x_n\}$ and $\{y_n\}$ are the Cauchy sequences with the limits z in X and w in Y.

- Lets $\{x_n\}$ and $\{y_n\}$ are not the Cauchy sequence. For this, there exists ε for which we can find subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k}\}$ of $\{x_n\}$ with $n_{2k} > m_{2k} > k$ such that

$$d(x_{2n_k}, x_{2m_k}) \geq \varepsilon, \tag{2.13}$$

if we take n_{2k} is a smallest, so

$$d(x_{2n_k-1}, x_{2m_k}) < \varepsilon, \tag{2.14}$$

and $\{y_{2n_k}\}$,$\{y_{2m_k}\}$ of $\{y_n\}$ with $n_{2k} > m_{2k} > k$ such that

$$\rho(y_{2n_k}, y_{2m_k}) \geq \varepsilon, \tag{2.15}$$

if we take n_{2k} is a smallest, so

$$\rho(y_{2n_k-1}, y_{2m_k}) < \varepsilon, \tag{2.16}$$

Then, taking into consideration the inequalities we have (2.13) , (2.14) and (2.15) , (2.16) respevtively, we have

$$0 < \varepsilon \leq d(x_{2n_k}, x_{2m_k}) \leq d(x_{2n_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2m_k}) < d(x_{2n_k}, x_{2n_k-1}) + \varepsilon$$
$$0 < \varepsilon \leq \rho(y_{2n_k}, x_{2m_k}) \leq \rho(y_{2n_k}, y_{2n_k-1}) + \rho(y_{2n_k-1}, y_{2m_k}) < \rho(y_{2n_k}, y_{2n_k-1}) + \varepsilon.$$

Letting $k \to +\infty$ and using (2.11) and (2.12) , we find

$$\begin{aligned}\lim_{k\to+\infty} d(x_{2n_k}, x_{2m_k}) &= \varepsilon. \\ \lim_{k\to+\infty} \rho(y_{2n_k}, y_{2m_k}) &= \varepsilon.\end{aligned}$$

Take $Sy_{2n_k} = x_{2n_k+1}$, $Bx_{2m_k} = y_{2m_k}$, $Ty_{2m_k} = x_{2m_k+1}$, $Ax_{2n_k} = y_{2n_k}$ in (2.1), we obtain

$$\begin{aligned}d(x_{2n_k+1}, y_{2m_k})d(x_{2n_k+1}, x_{2m_k+1}) \le\ & F(m_d(x_{2n_k}, x_{2m_k}, y_{2n_k}, y_{2m_k}), \\ & \varphi(m_d(x_{2n_k}, x_{2m_k}, y_{2n_k}, y_{2m_k}))),\end{aligned}$$

where

$$\begin{aligned}m_d(x_{2n_k}, x_{2m_k}, y_{2n_k}, y_{2m_k}) =\ & \max\{d(x_{2n_k+1}, x_{2m_k+1})\rho(y_{2n_k}, y_{2m_k}), d(x_{2m_k}, x_{2n_k+1})\rho(y_{2m_k}, y_{2n_k}), \\ & d(x_{2n_k}, x_{2m_k})d(x_{2n_k+1}, x_{2m_k+1}), d(x_{2n_k+1}, Sy_{2n_k})d(x_{2m_k+1}, x_{2m_k+1})\}.\end{aligned}$$

Letting $k \to \infty$

$$\begin{aligned}\varepsilon\varepsilon &\le F(\max\{\varepsilon\varepsilon, \varepsilon\varepsilon, \varepsilon\varepsilon, 0\}, \varphi(\max\{\varepsilon\varepsilon, \varepsilon\varepsilon, \varepsilon\varepsilon, 0\})) \\ &\le \varepsilon\varepsilon.\end{aligned}$$

Analogously, we can calculate with the same manner in (2.2) , we deduce

$$\begin{aligned}\varepsilon &\le F(\max\{\varepsilon, \varepsilon, \varepsilon\}, \varphi(\max\{\varepsilon, \varepsilon, \varepsilon\})) \le \varepsilon \\ \varepsilon &\le F(\max\{\varepsilon, \varepsilon, \varepsilon\}, \varphi(\max\{\varepsilon, \varepsilon, \varepsilon\})) \le \varepsilon,\end{aligned}$$

that is $\varepsilon = 0$, which is a contradiction. Since (X, d) is a complete metric space it follows that: the sequence $\{x_n\}$ is a Cauchy sequence with limit z in X and $\{y_n\}$ is a Cauchy sequence with limit w in Y.

By using inequality (2.1) , we get

$$\begin{aligned}d(SAz, TBx_{2n-1}) \le\ & F((\max\{\rho(Az, Bx_{2n-1}), \rho(y_{2n}, Az), d(x_{2n-1}, x_{2n})), \\ & \varphi(\max\{\rho(Az, Bx_{2n-1}), \rho(y_{2n}, Az), d(x_{2n-1}, x_{2n})\})) \\ \le\ & \max\{\rho(Az, Bx_{2n-1}), \rho(y_{2n}, Az), d(x_{2n-1}, x_{2n})\}).\end{aligned}$$

Implies that

$$d(SAz, TBx_{2n-1}) \le \max\{\rho(Az, Bx_{2n-1}), \rho(y_{2n}, Az), d(x_{2n-1}, x_{2n})\}.$$

Letting n grow to infinity, we deduce

$$d(Sw, z) \le \max\{\rho(Az, w), \rho(w, Az), 0\}.$$

Then $Sw = z = SAz$.

Similarly, using inequality (2.2) , we get

$$\begin{aligned}\rho(BSy_{2n-1}, ATy_{2n}) \le\ & F(\max\{d(Sy_{2n-1}, Ty_{2n}), d(x_{2n-1}, Sy_{2n-1}), \rho(y_{2n}, y_{2n-1}), \rho(Az, BSy_{2n-1})\}), \\ & \varphi(\max\{d(Sy_{2n-1}, Ty_{2n}), d(x_{2n-1}, Sy_{2n-1}), \rho(y_{2n}, y_{2n-1}), \rho(Az, BSy_{2n-1})\})) \\ \le\ & \max\{d(Sy_{2n-1}, Ty_{2n}), d(x_{2n-1}, Sy_{2n-1}), \rho(y_{2n}, y_{2n-1}), \rho(Az, BSy_{2n-1})\}).\end{aligned}$$

Thus,

$$\rho(BSy_{2n-1}, ATy_{2n}) \le \max\{d(Sy_{2n-1}, Ty_{2n}), d(x_{2n-1}, Sy_{2n-1}), \rho(y_{2n}, y_{2n-1}), \rho(Az, BSy_{2n-1})\}.$$

Taking $n \to \infty$, we have

$$\rho(w, Az) \le \max\{d(z, Tw), d(z, Sw), 0, \rho(Az, w)\}.$$

Then $Tw = z = TBz$.

By symmetry, a similar calculation again hold if one of the mappings B, S, T is continuous instead of A.

To establish uniqueness, suppose that TB and SA have a second distinct common fixed point z_0. Then, using inequality (2.1), we obtain

$$\begin{aligned} d(Sy, Ty')d(SAz, TBz') \leq\ & F(\max\{d(Sy, Ty')\rho(Az, Bz'), d(z', Sy)\rho(y', Az), \\ & d(z, z')d(Sy, Ty'), d(Sy, SAz)d(Ty', TBz')\} \\ & , \varphi(\max\{d(Sy, Ty')\rho(Az, Bz'), d(z', Sy)\rho(y', Az), \\ & d(z, z')d(Sy, Ty'), d(Sy, SAz)d(Ty', TBz')\})). \end{aligned}$$

So,

$$\begin{aligned} d(z, z')d(SAz, TBz') \leq\ & F(\max\{d(z, z')\rho(Az, Bz'), d(z', z)\rho(Bz', Az), \\ & d(z, z')d(z, z'), d(z, z)d(z', z')\} \\ & , \varphi(\max\{d(z, z')\rho(Az, Bz'), d(z', z)\rho(Bz', Az), \\ & d(z, z')d(z, z'), d(z, z)d(z', z')\}. \end{aligned}$$

This implies:

$$d(z, z') \leq F(\rho(Az, Bz'), \varphi(\rho(Az, Bz'))). \tag{2.17}$$

Hence, by the same manner, applying (2.2) , it follow

$$\rho(Az, Bz') \leq F(d(z, z'), \varphi(d(z, z'))). \tag{2.18}$$

From inequalities (2.17) and (2.18) which implies the uniqueness.

So $z = z'$. The uniqueness of w is proved similary. This complete the proof of the theorem. □

If we assume $A = B$ and $S = T$ in Theorem 2.1, we deduce the following corollary:

Corollary 2.2. *Let (X, d) and (Y, ρ) be complete metric spaces, let A be mapping of X into Y and let S be mapping of Y into X satisfying the inequalities*

$$\begin{aligned} d(Sy, Sy')d(SAx, SAx') &\leq F(m_d(x, x', y, y'), \varphi(m_d(x, x', y, y'))), \\ \rho(Ax, Ax')\rho(ASy, ASy') &\leq F(m_\rho(x, x', y, y'), \varphi(m_\rho(x, x', y, y'))), \end{aligned}$$

for all x, x in X and y, y' in Y, where $\varphi \in \Phi_u$ and $F \in C$ such that (I, φ, F) is monotone and

$$\begin{aligned} m_d(x, x', y, y') &= \max\{d(Sy, Sy')\rho(Ax, Ax'), d(x', Sy)\rho(y', Ax), \\ & \quad d(x, x')d(Sy, Sy'), d(Sy, SAx)d(Sy', SAx')\} \\ m_\rho(x, x', y, y') &= \max\{d(Sy, Sy')\rho(Ax, Ax'), d(x', Sy)\rho(y', Ax), \\ & \quad \rho(y, y')\rho(Ax, Ax'), \rho(Ax, ASy)\rho(Ax', ASy')\}. \end{aligned}$$

If one of the mappings A and S is continuous then SA has a unique fixed point z in X and AS has a unique fixed point w in Y,Further, $Az = w$ and $Sw = z$.

By setting $A = S$ and $(X, d) = (Y, \rho)$ in Corollary 2.2, then we have the following corollary:

Corollary 2.3. *Let (X, d) be complete metric spaces, let S be a continuous mapping of X into X satisfying the inequality*

$$d(Sy, Sy')d(S^2x, S^2x') \leq F(m_d(x, x', y, y'), \varphi(m_d(x, x', y, y'))),$$

for all x, x in X and y, y' in Y, where $\varphi \in \Phi_u$ and $F \in C$ such that (I, φ, F) is monotone and

$$\begin{aligned} m_d(x, x', y, y') &= \max\{d(Sy, Sy')d(Sx, Sx'), d(x', Sy)d(y', Sx), \\ &\quad d(x, x')d(Sy, Sy'), d(Sy, S^2x)d(Sy', S^2x')\}. \end{aligned}$$

Then S has a unique fixed point z in X.

Remark 2.4. - Putting in (2.1) and (2.2) : $\psi(t) = t$, $\varphi(t) = (1-c)t$, $0 \le c < 1$ and $F(s,t) = s - t$, we get a well-known Hamaizia's result from [12].
- Theorem 1.2 of [13] is a special case of Corollary 2.2.
- Corollary 1.3 of [13] is a special case of Corollary 2.3

Now, we give some examples which satisfies all the conditions of Theorem 2.1 and to demonstrate the validity of the hypotheses of our result.

Example 2.5. Let $F(s,t) = s - t$, $X = [0,1]$ and $Y = [1,2]$ be complete metric spaces with $d = \rho = |x - y|$ If $A, B : X \to Y$, and $S, T : Y \to X$ be a mappings , where

$$A(x) = \begin{cases} \frac{1}{2} & if \quad 0 \le x \le \frac{3}{4} \\ \frac{1}{2}x & if \quad \frac{3}{4} < x \le 1 \end{cases}, \qquad B(x) = \frac{1}{2},$$

$$S(y) = \begin{cases} \frac{3}{4} & if \quad 1 \le y \le \frac{3}{2} \\ 1 & if \quad \frac{3}{2} < y \le 2 \end{cases}, \qquad T(y) = \begin{cases} \frac{3}{4} & if \quad 1 \le y \le \frac{3}{2} \\ \frac{3}{2} & if \quad \frac{3}{2} < y \le 2 \end{cases}$$

The altering functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ are defined by $\varphi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & if \quad 0 \le x \le 1 \\ x^2 & if \quad x > 1 \end{cases},$$

Thus, $\frac{3}{4}$ is the unique common fixed point of the maps SA, TB and $\frac{1}{2}$ is the unique common fixed point of the maps AT, BS since all the conditions of Theorem 2.1 are satisfied

Example 2.6. Suppose $X = \{0, 1, 2\}$, $Y = \{0, 1\}$,

$d(x,y)$	0	1	2
0	0	$\frac{5}{6}$	$\frac{7}{6}$
1	$\frac{5}{6}$	0	1
2	$\frac{7}{6}$	1	0

$\rho(x,y)$	0	1
0	0	$\frac{1}{5}$
1	$\frac{1}{5}$	0

We define the mappings A , B, S and T as

$\backslash y$	0	1
S	1	0
T	1	0

$\backslash x$	0	1	2
A	1	0	1
B	1	1	0

Let also $F(s,t) = \frac{99}{100}s$, $\varphi(x) = \psi(x) = \sqrt{x}$.
Case 1: If $x = 0, x' = 1, y = 0$ and $y' = 1$
$\psi(d(Sy, Ty')d(SAx, TBx')) = \psi(d(1,0)d(0,0)) =$
$\psi(\rho(Ax, Bx')\rho(BSy, ATy')) = \psi(\rho(1,1)\rho(BS0, AT1))$
Case 2: If $x = 0, x' = 2, y = 0$ and $y' = 1$

$$\begin{aligned}\psi\left(d(Sy,Ty')d(SAx,TBx')\right) &= \psi\left(d(1,0)d(0,1)\right)=\frac{5}{6}\\ &\leq F(\psi\left(\max\{d(1,0)\rho(1,0),d(2,1)\rho(1,1),d(0,2)d(1,0),d(1,0)d(0,1)\}\right);\\ &\quad \varphi\left(\max\{d(1,0)\rho(1,0),d(2,1)\rho(1,1),d(0,2)d(1,0),d(1,0)d(0,1)\}\right)\\ &= F\left(\psi\left(d(0,2)d(1,0)\right),\varphi\left(\max\left(d(0,2)d(1,0)\right)\right)\right)\\ &= F\left(\psi\left(\frac{5}{6}.\frac{7}{6}\right),\varphi\left(\frac{5}{6}.\frac{7}{6}\right)\right)\\ &= F\left(\left(\sqrt{\frac{5}{6}.\frac{7}{6}}\right),\left(\sqrt{\frac{5}{6}.\frac{7}{6}}\right)\right)=\frac{99}{100}.\sqrt{\frac{35}{36}}.\end{aligned}$$

$\psi\left(\rho(Ax,Bx')\rho(BSy,ATy')\right)=\psi\left(\rho(1,0)\rho(1,1)\right)=0$

Case 3: If $x=1,x'=2,y=0$ and $y'=1$

$\psi\left(d(Sy,Ty')d(SAx,TBx')\right)=\psi\left(d(1,0)d(1,1)\right)=0$

$\psi\left(\rho(Ax,Bx')\rho(BSy,ATy')\right)=\psi\left(\rho(0,0)\rho(BS1,AT1)\right)=0.$

Thus, 0 is the unique common fixed point of the maps SA,TB and 1 is the unique common fixed point of the maps AT,BS since all the conditions of Theorem 2.1 are satisfied .

3 Application to nonlinear integral equations

Let $X=C[a,b]$ be the space of all real valued continuous functions on $[a,b]$, a closed bounded interval in $\mathbb{R}$: The metric of uniform convergence: $d(x,y)=\max_{t\in[a,b]}|x-y|$ is complete.

In this section. In this section, we apply our theorem 2.1 to establish the existence of common solutions of a system of nonlinear integral equations defined by:

$$x(t)=g(t)+\int_a^b \xi(t,\tau,x(\tau))d\tau, \tag{3.1}$$

where $x\in C[a,b]$ is the unknown function, $t,\tau\in[a,b]$, $\xi:[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are given continuous functions.

Theorem 3.1. *Assume that the following conditions*

(i) There exists a continuous functions $\theta_1,\theta_2:[a,b]\times[a,b]\to\mathbb{R}_+$ such that for all $x,y\in X$, and $t,\tau\in[a,b]$, we get

$$|(\xi(t,\tau,y(\tau))-\xi(t,\tau,y'(\tau)))|\leq\theta_1(t,\tau)\left(F(m_d(x,x',y,y'),\varphi(m_d(x,x',y,y')))\right)^{\frac{1}{2}}$$

$$\left|(\xi(t,\tau,x^2(\tau))-\xi(t,\tau,x'^2(\tau)))\right|\leq\theta_2(t,\tau)\left(F(m_d(x,x',y,y'),\varphi(m_d(x,x',y,y')))\right)^{\frac{1}{2}},$$

where $\varphi\in\Phi_u$ and $F\in C$ such that (I,φ,F) is monotone such that

$$\begin{aligned}m_d(x,x',y,y') &= \max\{d(Sy,Sy')d(Sx,Sx'),d(x',Sy)d(y',Sx),\\ &\quad d(x,x')d(Sy,Sy'),d(Sy,S^2x)d(Sy',S^2x')\}.\end{aligned}$$

(ii)

$$\left(\max_{\tau\in[a,b]}\int_a^b\theta_1(t,\tau)d\tau\right)\times\left(\max_{\tau\in[a,b]}\int_a^b\theta_2(t,\tau)d\tau\right)\leq 1.$$

Then, the equation (3.1) has a unique solution $z\in C[a,b]$.

Proof. Define the mapping $S:X\to X$ by:

$$Sx(t)=g(t)+\int_a^b\xi_1(t,\tau,x(\tau))d\tau,$$

for all $t \in [a,b]$. So, the existence of a solution of (3.1) is equivalent to the existence of a fixed point of S, we have

$$\begin{aligned}
d(Sy,Sy')d(S^2x,S^2x') &= |Sy(t)-Sy'(t)| \times |S^2x(t)-S^2x'(t)| \\
&\le \max_{t\in[a,b]}\left(\int_a^b |(\xi(t,\tau,y(\tau))-\xi(t,\tau,y'(\tau)))|\,d\tau\right). \\
&\quad \times \max_{t\in[a,b]}\left(\int_a^b \left|(\xi(t,\tau,x^2(\tau))-\xi(t,\tau,x'^2(\tau)))\right|d\tau\right) \\
&\le \left(\max_{t\in[a,b]}\int_a^b \theta_1(t,\tau)d\tau\right)(F(m_d(x,x',y,y'),\varphi(m_d(x,x',y,y'))))^{\frac{1}{2}} \\
&\quad \times \left(\max_{t\in[a,b]}\int_a^b \theta_2(t,\tau)d\tau\right)(F(m_d(x,x',y,y'),\varphi(m_d(x,x',y,y'))))^{\frac{1}{2}} \\
&\le (F(m_d(x,x',y,y'),\varphi(m_d(x,x',y,y'))))^{\frac{1}{2}} \\
&\quad \times (F(m_d(x,x',y,y'),\varphi(m_d(x,x',y,y'))))^{\frac{1}{2}} \\
&\le F(m_d(x,x',y,y'),\varphi(m_d(x,x',y,y'))).
\end{aligned}$$

Thus

$$d(Sy,Sy')d(S^2x,S^2x') \le F(m_d(x,x',y,y'),\varphi(m_d(x,x',y,y'))).$$

Then, all the conditions of theorem 2.1 hold. Consequently, the equation (3.1) has a solution $z \in C[a,b]$. □

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