Palestine Journal of Mathematics

Vol. 10(2)(2021) , 547–561

SOME FIXED POINT RESULTS UNDER CONTRACTIVE TYPE MAPPINGS IN CONE S_b-METRIC SPACES

G. S. Saluja

Communicated by Thabet Abdeljawad

MSC 2010 Classifications: 47H10, 54H25.

Keywords and phrases: Fixed point, contractive type condition, cone S_b-metric space, cone.

The author would like to thanks the anonymous referee for his/her careful reading and valuable suggestions to improve the manuscript.

Abstract. In this paper, we establish fixed point results using some contractive type mappings in the setting of cone S_b-metric spaces. Our results extend, unify and generalize several results given in the current existing literature.

1 Introduction and Preliminaries

The Banach contraction principle [3] is a basic tool in studying the existence of solutions to many problems in mathematics and many different fields. In recent times, the contraction principle has been extended in various directions. The Banach fixed point theorem (or Banach contraction principle) is stated as follows.

Theorem 1.1. *(Banach contraction principle) ([3]) Let (X, d) be a complete metric space and $\mathcal{T}$ be a mapping of X into itself satisfying:*

$$d(\mathcal{T}(x), \mathcal{T}(y)) \leq \lambda\, d(x, y) \;\; \forall\, x, y \in X, \tag{1.1}$$

where λ is a constant in $(0, 1)$. Then $\mathcal{T}$ has a fixed point $v \in X$.

Since then, fixed point theory has had a rapid development. There is a great number of generalization of the Banach contraction principle. The underlying metric space can be generalized in many ways. In addition to the improvement of Banach's contractive condition, more and more attention is devoted itself to the generalization of metric spaces such as 2-metric spaces, D_*-metric spaces, partial metric spaces, cone metric spaces, b-metric spaces, G-metric spaces and cone b-metric spaces etc.

The notion of a b-metric space was introduced by Bakhtin [2] and then extensively used by Czerwik in [5]. Since then b-metric fixed point theory grew up in the classical metric fixed point theory to obtain a generalization of some known metric version of fixed point results. On the other hand, some authors are interested and have tried to give generalizations of metric spaces in different way.

In 2007, Huang and Zhang [8] introduced the concept of cone metric spaces as a generalization of metric spaces by replacing the set of real numbers by a general Banach space E which is partially ordered with respect to a cone $P \subset E$ and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [1, 10, 15, 22] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space.

In 2011, Hussain and Shah [9] introduced the concept of cone b-metric space as a generalization of b-metric space and cone metric spaces. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space.

In 2012, Sedghi et al. [16] introduced the concept of S-metric space which is different from other space and proved fixed point theorems in S-metric space. They also give some examples of S-metric space which shows that S-metric space is different from other spaces.

In 2016, Souayah and Mlaiki [19] introduced the concept of S_b-metric space. The concept of S_b-metric space is further used in many other research papers. In 2017, Dhamodharan and Krishnakumar [6] further extended the notion of S-metric spaces to cone S-metric spaces and proved fixed point results using different contractive type mappings.

Recently, K. A. Singh and M. R. Singh [18] have introduced the concept of cone S_b-metric space and proved some fixed point results via different contractive conditions which extend various results in the existing literature.

The importance of cone S_b-metric space is to further study of topological properties like continuity and compactness and to extend and generalise several standard results, for example, Banach contraction principle, Kannan contraction, Chatterjae contraction etc. in the said space.

The main interest of this paper is to establish some existence of fixed point theorems under various contractive type conditions in the setting of cone S_b-metric spaces. Our results extend, generalize and unify several results from the existing literature.

We need the following definitions and properties in the sequel.

Definition 1.2. ([8]) Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:

(c_1) P is closed, nonempty and $P \neq \{0\}$;
(c_2) $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;
(c_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ in E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where P^0 stands for the interior of P. If $P^0 \neq \emptyset$ then P is called a solid cone (see [21]).

There exist two kinds of cones- normal (with the normal constant K) and non-normal ones ([7]).

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by P. Then P is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ imply } \|x\| \leq K\|y\|, \tag{1.2}$$

or equivalently, if $(\forall n)$ $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{1.3}$$

The least positive number K satisfying (1.2) is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose E is a Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is partial ordering in E with respect to P.

Example 1.3. ([12]) Let $K > 1$ be given. Consider the real vector space

$$E = \Big\{ax + b : a, b \in R; x \in \Big[1 - \frac{1}{K}, 1\Big]\Big\}$$

with supremum norm and the cone

$$P = \big\{ax + b \in E : a \geq 0, b \geq 0\big\}$$

in E. The cone P is regular and so normal.

Definition 1.4. ([8, 23]) Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to E$ satisfies:

(CM_1) $0 \leq d(x,y)$ for all $x,y \in X$ with $x \neq y$ and $d(x,y) = 0 \Leftrightarrow x = y$;
(CM_2) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(CM_3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.
Then d is called a cone metric [8] on X and (X,d) is called a cone metric space [8] or simply CMS.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Lemma 1.5. *[15] Every regular cone is normal.*

Example 1.6. ([8]) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space with normal cone P where $K = 1$.

Example 1.7. ([14]) Let $E = \ell^2$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \text{for all } n\}$, (X,ρ) a metric space, and $d\colon X \times X \to E$ defined by $d(x,y) = \{\rho(x,y)/2^n\}_{n\geq 1}$. Then (X,d) is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

Definition 1.8. ([16, 13]) Let X be a nonempty set and $S\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all x, y, z, $t \in X$:
(SM_1) $S(x,y,z) \geq 0$;
(SM_2) $S(x,y,z) = 0$ if and only if $x = y = z$;
(SM_3) $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$.
Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

Example 1.9. ([20]) Let X be a nonempty set and d be the ordinary metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.

Example 1.10. ([16]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an S-metric on X.

Example 1.11. ([16]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|x-z\| + \|y-z\|$ is an S-metric on X.

Definition 1.12. ([19]) Let X be a nonempty set and $b \geq 1$ be a given real number. A function $S_b\colon X^3 \to [0,\infty)$ is said to be S_b-metric if and only if for all $x,y,z,t \in X$ the following conditions are satisfied:
(S_bM_1) $S_b(x,y,z) = 0$ if and only if $x = y = z$,
(S_bM_2) $S_b(x,y,z) \leq b[S_b(x,x,t) + S_b(y,y,t) + S_b(z,z,t)]$.

The pair (X,S_b) is called an S_b-metric space.

Remark 1.13. Note that the class of S_b-metric spaces is larger than the class of S-metric spaces. Indeed every S-metric space is an S_b-metric space with $b = 1$. However, the converse is not always true.

Example 1.14. ([19]) Let X be a non-empty set and $card(X) \geq 5$. Suppose $X = X_1 \cup X_2$ a partition of X such that $card(X_1) \geq 4$. Let $s \geq 1$. Then

$$S_b(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ 3s & \text{if } (x,y,z) \in X_1^3, \\ 1 & \text{if } (x,y,z) \notin X_1^3, \end{cases}$$

for all $x,y,z \in X$, S_b is a S_b-metric on X with the coefficient $s \geq 1$.

Proof. (i) If $x = y = z$ then $S_b(x, y, z) = 0$. Thus the first assertion of the definition of S_b-metric space is satisfied.

(ii) Now to prove the triangle inequality:

$$S_b(x, y, z) \le s[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)] \quad (*).$$

($\mathbf{a_1}$) Case 1. If $(x, y, z) \notin X_1^3$, we have $S_b(x, y, z) = 1$, $S_b(x, x, t) \ge 1$, $S_b(y, y, t) \ge 1$ and $S_b(z, z, t) \ge 1$ for all $t \in X$. Thus $(*)$ holds $(1 \le 3s)$.

($\mathbf{a_2}$) Case 2. If $(x, y, z) \in X_1^3$, we have the following two sub-cases:

($\mathbf{b_1}$) if $t \in X_1$, $(*)$ is satisfied since $S_b(x, y, z) = S_b(x, x, t) = S_b(y, y, t) = S_b(z, z, t) = 3s$.

($\mathbf{b_2}$) if $t \notin X_1$, we have $S_b(x, x, t) = S_b(y, y, t) = S_b(z, z, t) = 1$ and $S_b(x, y, z) = 3s$. Then $(*)$ holds. □

Example 1.15. ([17]) Let (X, S) be an S-metric space and $S_*(x, y, z) = \Big\{S(x, y, z)\Big\}^p$, where $p > 1$ is a real number. Then S_* is an S_b-metric on X with $b = 2^{2(p-1)}$.

Example 1.16. ([18]) Let $X = \mathbb{R}$ and let the function $S\colon X^3 \to \mathbb{R}$ be defined as $S(x, y, z) = |x - z| + |y - z|$. Then S is an S-metric on X. Therefore, the function $S_b(x, y, z) = \Big\{S(x, y, z)\Big\}^2 = \Big\{|x - z| + |y - z|\Big\}^2$ is an S_b-metric on X with $b = 2^{2(2-1)} = 4$.

Definition 1.17. ([6]) Suppose that E is a real Banach space, P is a cone in E with $int\, P \neq \emptyset$ and $\le$ is partial ordering with respect to P. Let X be a nonempty set and let the function $S\colon X^3 \to E$ satisfy the following conditions:

(1) $S(x, y, z) \ge 0$;

(2) $S(x, y, z) = 0$ if and only if $x = y = z$;

(3) $S(x, y, z) \le S(x, x, a) + S(y, y, a) + S(z, z, a), \forall x, y, z, a \in X$.

Then the function S is called a cone S-metric on X and the pair (X, S) is called a cone S-metric space or simply CSMS.

Example 1.18. ([6]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \ge 0, y \ge 0\}$, $X = \mathbb{R}$ and d be the ordinary metric on X. Then the function $S\colon X^3 \to E$ defined by $S(x, y, z) = \Big(d(x, z) + d(y, z), \alpha(d(x, z) + d(y, z))\Big)$, where $\alpha > 0$ is a constant, is a cone S-metric on X.

Lemma 1.19. *([6]) Let (X, S) be a cone S-metric space. Then we have $S(x, x, y) = S(y, y, x)$.*

Definition 1.20. ([6]) Let (X, S) be a cone S-metric space.

(i) A sequence $\{u_n\}$ in X converges to u if and only if $S(u_n, u_n, u) \to 0$ as $n \to \infty$, that is, there exists $n_0 \in N$ such that for all $n \ge n_0$, $S(u_n, u_n, u) \ll c$ for each $c \in E$, $0 \ll c$. We denote this by $\lim_{n\to\infty} u_n = u$ or $\lim_{n\to\infty} S(u_n, u_n, u) = 0$.

(ii) A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$, that is, there exists $n_0 \in N$ such that for all $n, m \ge n_0$, $S(u_n, u_n, u_m) \ll c$ for each $c \in E$, $0 \ll c$.

(iii) The cone S-metric space (X, S) is called complete if every Cauchy sequence in X is convergent in X.

In the following lemma, we see the relationship between a cone metric and a cone S-metric.

Lemma 1.21. *([6]) Let (X, S) be a cone S-metric space. Then, the following properties are satisfied:*

(1) *$S(u, v, z) = d(u, z) + d(v, z)$ for all $u, v, z \in X$, is a cone S-metric on X.*

(2) *$u_n \to u$ in (X, d) if and only if $u_n \to u$ in (X, S_d).*

(3) *$\{u_n\}$ is Cauchy in (X, d) if and only if $\{u_n\}$ is Cauchy in (X, S_d).*

(4) *(X, d) is complete if and only if (X, S_d) is complete.*

The notion of cone S_b-metric space is introduced by K. Anthony Singh and M. R. Singh [18] in 2018 as follows:

Definition 1.22. ([18]) Suppose that E is a real Banach space, P is a cone in E with $int\, P \neq \emptyset$ and $\leq$ is partial ordering in E with respect to P. Let X be a nonempty set and let the function $S\colon X^3 \to E$ satisfy the following conditions:

(CS_bM_1) $S(x,y,z) \geq 0$;
(CS_bM_2) $S(x,y,z) = 0$ if and only if $x = y = z$;
(CS_bM_3) $S(x,y,z) \leq b\big[S(x,x,a) + S(y,y,a) + S(z,z,a)\big], \forall x,y,z,a \in X$;

where $b \geq 1$ is a constant. Then the function S is called a cone S_b-metric on X and the pair (X,S) is called a cone S_b-metric space or simply CS_bMS.

We note that cone S_b-metric spaces are generalizations of cone S-metric spaces since every cone S-metric space is a cone S_b-metric space with $b = 1$.

Example 1.23. ([18]) Let $E = \mathbb{R}^2$, the Euclidean plane and $P = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, a normal cone in E. Let $X = \mathbb{R}$ and $S\colon X^3 \to E$ be such that $S(x,y,z) = \Big(\alpha S_*(x,y,z), \beta S_*(x,y,z)\Big)$, where $\alpha, \beta > 0$ are constants and S_* is an S_b-metric on X. Then S is a cone S_b-metric on X.

In particular, the function $S_*(x,y,z) = \Big\{|x-z|+|y-z|\Big\}^2$ for all $x,y,z \in X$ is an S_b-metric on X with $b = 4$.

Therefore, the function $S(x,y,z) = \Big[\{|x-z|+|y-z|\}^2, \frac{1}{4}\{|x-z|+|y-z|\}^2\Big]$ for all $x,y,z \in X$ is a cone S_b-metric on X with $b = 4$.

Definition 1.24. ([18]) Let (X,S) be a cone S_b-metric space.

(1) A sequence $\{u_n\}$ in X converges to u if and only if $S(u_n,u_n,u) \to 0$ as $n \to \infty$, that is, there exists $n_0 \in N$ such that for all $n \geq n_0$, $S(u_n,u_n,u) \ll c$ for each $c \in E$, $0 \ll c$. We denote this by $\lim_{n\to\infty} u_n = u$ or $u_n \to u$ as $n \to \infty$.

(2) A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n,u_n,u_m) \to 0$ as $n,m \to \infty$, that is, there exists $n_0 \in N$ such that for all $n,m \geq n_0$, $S(u_n,u_n,u_m) \ll c$ for each $c \in E$, $0 \ll c$.

(3) The cone S_b-metric space (X,S) is called complete if every Cauchy sequence in X is convergent in X.

Lemma 1.25. *([18]) Let (X,S) be a cone S_b-metric space, P be a normal cone with normal constant K. Then a sequence $\{u_n\}$ in X converges to u if and only if $S(u_n,u_n,u) \to 0$ as $n \to \infty$.*

Lemma 1.26. *([18]) Let (X,S) be a cone S_b-metric space, P be a normal cone with normal constant K. Let $\{u_n\}$ be a sequence in X. If the sequence $\{u_n\}$ converges to u_1 and u_2, then $u_1 = u_2$, that is, the limit of a convergent sequence is unique.*

Lemma 1.27. *([18]) Let (X,S) be a cone S_b-metric space, P be a normal cone with normal constant K. Then a sequence $\{u_n\}$ in X is a Cauchy sequence if and only if $S(u_n,u_n,u_m) \to 0$ as $n,m \to \infty$.*

Lemma 1.28. *([18]) Let (X,S) be a cone S_b-metric space, P be a normal cone with normal constant K. Let $\{u_n\}$ be a sequence in X. If the sequence $\{u_n\}$ converges to u, then $\{u_n\}$ is a Cauchy sequence, that is, every convergent sequence is a Cauchy sequence.*

2 Main Results

In this section, we shall prove some existence of fixed point results under contractive type conditions in the setting of cone S_b-metric spaces.

Theorem 2.1. *Let (X,S) be a complete cone S_b-metric space with the coefficient $b \geq 1$ and P be a normal cone with normal constant K. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$\begin{aligned} S(Tu,Tu,Tv) \leq\ & h_1\left[S(u,u,Tu) + S(v,v,Tv)\right] + h_2\left[S(u,u,Tv) + S(v,v,Tu)\right] \\ & + h_3 \max\Big\{S(u,u,Tu), S(v,v,Tv), S(v,v,Tu)\Big\} \\ & + h_4\left[S(u,u,v) + S(u,u,Tv)\right] \end{aligned} \tag{2.1}$$

for all $u, v \in X$, where $h_1, h_2, h_3, h_4 > 0$ are constants such that $2h_1 + b(b+2)h_2 + h_3 + (b+1)^2 h_4 < 1$. Then T has a unique fixed point w in X and we have $\lim_{n\to\infty} T^n(u) = w$, for each $u \in X$.

Proof. Let $u_0 \in X$ and a sequence $\{u_n\}$ be defined by $T^n(u_0) = u_n$. Suppose that $u_n \neq u_{n+1}$ for all n. Using the inequality (2.1) and the condition (CS_bM_3) of definition 1.22, we get

$$\begin{aligned}
S(u_n, u_n, u_{n+1}) &= S(Tu_{n-1}, Tu_{n-1}, Tu_n) \\
&\leq h_1 \left[S(u_{n-1}, u_{n-1}, Tu_{n-1}) + S(u_n, u_n, Tu_n)\right] \\
&\quad + h_2 \left[S(u_{n-1}, u_n + Tu_{n-1}) + S(u_n, u_n, Tu_{n-1})\right] \\
&\quad + h_3 \max\Big\{S(u_{n-1}, u_{n-1}, Tu_{n-1}), S(u_n, u_n, Tu_n), S(u_n, u_n, Tu_{n-1})\Big\} \\
&\quad + h_4 \left[S(u_{n-1}, u_{n-1}, u_n) + S(u_{n-1}, u_{n-1}, Tu_n)\right] \\
&= h_1 \left[S(u_{n-1}, u_{n-1}, u_n) + S(u_n, u_n, u_{n+1})\right] \\
&\quad + h_2 \left[S(u_{n-1}, u_{n-1}, u_{n+1}) + S(u_n, u_n, u_n)\right] \\
&\quad + h_3 \max\Big\{S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1}), S(u_n, u_n, u_n)\Big\} \\
&\quad + h_4 \left[S(u_{n-1}, u_{n-1}, u_n) + S(u_{n-1}, u_{n-1}, u_{n+1})\right] \\
&= (h_1 + h_4)\, S(u_{n-1}, u_{n-1}, u_n) + h_1\, S(u_n, u_n, u_{n+1}) \\
&\quad + h_3 \max\Big\{S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1})\Big\} \\
&\quad + (h_2 + h_4)\, S(u_{n-1}, u_{n-1}, u_{n+1}) \\
&\leq (h_1 + h_4)\, S(u_{n-1}, u_{n-1}, u_n) + h_1\, S(u_n, u_n, u_{n+1}) \\
&\quad + h_3 \max\Big\{S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1})\Big\} \\
&\quad + 2b(h_2 + h_4)\, S(u_{n-1}, u_{n-1}, u_n) \\
&\quad + b^2(h_2 + h_4)\, S(u_n, u_n, u_{n+1}) \\
&= [h_1 + 2bh_2 + (2b+1)h_4]\, S(u_{n-1}, u_{n-1}, u_n) \\
&\quad + [h_1 + b^2(h_2 + h_4)]\, S(u_n, u_n, u_{n+1}) \\
&\quad + h_3 \max\Big\{S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1})\Big\}. \qquad (2.2)
\end{aligned}$$

Here we consider the following cases.

Case I: If $\max\Big\{S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1})\Big\} = S(u_{n-1}, u_{n-1}, u_n)$, then from equation (2.2), we obtain

$$\begin{aligned}
S(u_n, u_n, u_{n+1}) &\leq [h_1 + 2bh_2 + h_3 + (2b+1)h_4]\, S(u_{n-1}, u_{n-1}, u_n) \\
&\quad + [h_1 + b^2(h_2 + h_4)]\, S(u_n, u_n, u_{n+1}).
\end{aligned}$$

or

$$\begin{aligned}
[1 - h_1 - b^2(h_2 + h_4)]S(u_n, u_n, u_{n+1}) &\leq [h_1 + 2bh_2 + h_3 + (2b+1)h_4] \times \\
&\quad S(u_{n-1}, u_{n-1}, u_n) \\
\Rightarrow S(u_n, u_n, u_{n+1}) &\leq \Big(\frac{h_1 + 2bh_2 + h_3 + (2b+1)h_4}{1 - h_1 - b^2(h_2 + h_4)}\Big) \times \\
&\quad S(u_{n-1}, u_{n-1}, u_n) \\
&= t\, S(u_{n-1}, u_{n-1}, u_n), \qquad (2.3)
\end{aligned}$$

where $t = \Big(\frac{h_1 + 2bh_2 + h_3 + (2b+1)h_4}{1 - h_1 - b^2(h_2 + h_4)}\Big) < 1$ as $2h_1 + b(b+2)h_2 + h_3 + (b+1)^2 h_4 < 1$.

Case II: If $\max\Big\{S(u_{n-1},u_{n-1},u_n), S(u_n,u_n,u_{n+1})\Big\} = S(u_n,u_n,u_{n+1})$, then from equation (2.2), we obtain

$$\begin{aligned} S(u_n,u_n,u_{n+1}) &\le [h_1+2bh_2+(2b+1)h_4]\,S(u_{n-1},u_{n-1},u_n) \\ &\quad +[h_1+b^2(h_2+h_4)+h_3]\,S(u_n,u_n,u_{n+1}). \end{aligned}$$

or

$$\begin{aligned} [1-h_1-b^2(h_2+h_4)-h_3]S(u_n,u_n,u_{n+1}) &\le [h_1+2bh_2+(2b+1)h_4]\times \\ &\quad S(u_{n-1},u_{n-1},u_n) \\ \Rightarrow S(u_n,u_n,u_{n+1}) &\le \Big(\frac{h_1+2bh_2+(2b+1)h_4}{1-h_1-b^2(h_2+h_4)-h_3}\Big)\times \\ &\quad S(u_{n-1},u_{n-1},u_n) \\ &= q\,S(u_{n-1},u_{n-1},u_n), \end{aligned} \tag{2.4}$$

where $q=\Big(\frac{h_1+2bh_2+(2b+1)h_4}{1-h_1-b^2(h_2+h_4)-h_3}\Big)<1$ as $2h_1+b(b+2)h_2+h_3+(b+1)^2h_4<1$.

Combining above two cases, we have

$$S(u_n,u_n,u_{n+1}) \le \mu\,S(u_{n-1},u_{n-1},u_n), \tag{2.5}$$

where $0<\mu<1$ and $\mu\in\{t,q\}<1$ as $2h_1+b(b+2)h_2+h_3+(b+1)^2h_4<1$.

Let $V_n=S(u_n,u_n,u_{n+1})$ and $V_{n-1}=S(u_{n-1},u_{n-1},u_n)$, then from equation (2.5), we get

$$V_n \le \mu\,V_{n-1}. \tag{2.6}$$

Using equation (2.6), we obtain

$$V_n \le \mu\,V_{n-1} \le \mu^2\,V_{n-2} \le \mu^3\,V_{n-3} \le \cdots \le \mu^n\,V_0. \tag{2.7}$$

Now for $m,n\ge 1$ and $m>n$, we have

$$\begin{aligned} S(u_n,u_n,u_m) &\le b[2S(u_n,u_n,u_{n+1})+S(u_m,u_m,u_{n+1})] \\ &\le 2bS(u_n,u_n,u_{n+1})+b^2S(u_m,u_m,u_{n+1}) \\ &\le 2bS(u_n,u_n,u_{n+1})+2b^3S(u_{n+1},u_{n+1},u_{n+2}) \\ &\quad +b^4S(u_{n+2},u_{n+2},u_m) \\ &\le 2bS(u_n,u_n,u_{n+1})+2b^3S(u_{n+1},u_{n+1},u_{n+2}) \\ &\quad +2b^5S(u_{n+2},u_{n+2},u_{n+3})+\ldots \\ &\quad +2b^{2(m-n-1)}S(u_{m-1},u_{m-1},u_m) \\ &< 2b\Big\{S(u_n,u_n,u_{n+1})+b^2S(u_{n+1},u_{n+1},u_{n+2}) \\ &\quad +b^4S(u_{n+2},u_{n+2},u_{n+3})+\ldots \\ &\quad +b^{2(m-n-1)}S(u_{m-1},u_{m-1},u_m)\Big\} \\ &\le 2b\Big\{\mu^n+b^2\mu^{n+1}+b^4\mu^{n+2}+\cdots+b^{2(m-n-1)}\mu^{m-1}\Big\}S(u_0,u_0,u_1) \\ &= 2b\mu^n\Big\{1+b^2\mu+(b^2\mu)^2+\cdots+(b^2\mu)^{m-n-1}\Big\}V_0 \\ &\le \Big(\frac{2b\mu^n}{1-b^2\mu}\Big)V_0. \end{aligned}$$

This implies that

$$\|S(u_n,u_n,u_m)\| \le \Big(\frac{2b\mu^nK}{1-b^2\mu}\Big)\|V_0\|.$$

Taking limit as $n, m \to \infty$, we get

$$\|S(u_n, u_n, u_m)\| \to 0,$$

since $0 < \mu < 1$. Thus, we have $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$.

Therefore, the sequence $\{u_n\}$ is a Cauchy sequence in X. By the completeness of the space, there exists $w \in X$ such that $\lim_{n\to\infty} u_n = w$, i.e., $\lim_{n\to\infty} T^n u_0 = w$.

Also, we have

$$\begin{aligned}
S(Tw, Tw, w) &\leq 2bS(Tw, Tw, Tu_n) + bS(w, w, Tu_n) \\
&\leq 2b\Big\{h_1[S(w, w, Tw) + S(u_n, u_n, Tu_n)] \\
&\quad +h_2[S(w, w, Tu_n) + S(u_n, u_n, Tw)] \\
&\quad +h_3 \max\{S(w, w, Tw), S(u_n, u_n, Tu_n), S(u_n, u_n, Tw)\} \\
&\quad +h_4[S(w, w, u_n) + S(w, w, Tu_n)]\Big\} \\
&\quad +bS(w, w, Tu_n) \\
&= 2b\Big\{h_1[S(w, w, Tw) + S(u_n, u_n, u_{n+1})] \\
&\quad +h_2[S(w, w, u_{n+1}) + S(u_n, u_n, Tw)] \\
&\quad +h_3 \max\{S(w, w, Tw), S(u_n, u_n, u_{n+1}), S(u_n, u_n, Tw)\} \\
&\quad +h_4[S(w, w, u_n) + S(w, w, u_{n+1})]\Big\} \\
&\quad +bS(w, w, u_{n+1}) \\
&\leq 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_{n+1}) \\
&\quad +b^2(2h_2 + 2h_4 + 1)S(u_{n+1}, u_{n+1}, w) \\
&\quad +2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w) \\
&\quad +2bh_3 \max\{S(w, w, Tw), S(u_n, u_n, u_{n+1}), S(u_n, u_n, Tw)\}
\end{aligned} \tag{2.8}$$

We consider the following cases.

Case I: If $\max\{S(w, w, Tw), S(u_n, u_n, u_{n+1}), S(u_n, u_n, Tw)\} = S(w, w, Tw)$, then from equation (2.8), we obtain

$$\begin{aligned}
S(Tw, Tw, w) &\leq 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_{n+1}) \\
&\quad +b^2(2h_2 + 2h_4 + 1)S(u_{n+1}, u_{n+1}, w) \\
&\quad +2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w) \\
&\quad +2bh_3S(w, w, Tw) \\
&\leq 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_{n+1}) \\
&\quad +b^2(2h_2 + 2h_4 + 1)S(u_{n+1}, u_{n+1}, w) \\
&\quad +2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w) \\
&\quad +2b^2h_3S(Tw, Tw, w) \\
&= 2b^2(h_1 + h_3)S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_{n+1}) \\
&\quad +b^2(2h_2 + 2h_4 + 1)S(u_{n+1}, u_{n+1}, w) \\
&\quad +2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w).
\end{aligned} \tag{2.9}$$

From equation (2.9), we obtain

$$\begin{aligned} S(Tw,Tw,w) &\leq \frac{1}{1-2b^2(h_1+h_3)}\Big[2bh_1S(u_n,u_n,u_{n+1})+b^2(2h_2+2h_4+1)\times \\ &\quad S(u_{n+1},u_{n+1},w)+b^2(2h_2+2h_4+1)S(u_{n+1},u_{n+1},w) \\ &\quad +2b^2h_2S(Tw,Tw,u_n)+2b^2h_4S(u_n,u_n,w)\Big]. \end{aligned}$$

This implies that

$$\begin{aligned} \|S(Tw,Tw,w)\| &\leq \frac{K}{1-2b^2(h_1+h_3)}\Big[2bh_1\|S(u_n,u_n,u_{n+1})\| \\ &\quad +b^2(2h_2+2h_4+1)\|S(u_{n+1},u_{n+1},w)\| \\ &\quad +b^2(2h_2+2h_4+1)\|S(u_{n+1},u_{n+1},w)\| \\ &\quad +2b^2h_2\|S(Tw,Tw,u_n)\|+2b^2h_4\|S(u_n,u_n,w)\|\Big]. \end{aligned}$$

Taking the limit as $n\to\infty$, we get

$$\|S(Tw,Tw,w)\| \leq \frac{2b^2h_2K}{1-2b^2(h_1+h_3)}\|S(Tw,Tw,w)\|,$$

or

$$\begin{aligned} \Big(1-\frac{2b^2h_2K}{1-2b^2(h_1+h_3)}\Big)\|S(Tw,Tw,w)\| &\leq 0 \\ \Rightarrow \|S(Tw,Tw,w)\| &\leq 0. \end{aligned}$$

Thus, we have $S(Tw,Tw,w)=0$. Hence $Tw=w$. This shows that w is a fixed point of T.

Case II: If $\max\{S(w,w,Tw),S(u_n,u_n,u_{n+1}),S(u_n,u_n,Tw)\}=S(u_n,u_n,u_{n+1})$, then from equation (2.8), we obtain

$$\begin{aligned} S(Tw,Tw,w) &\leq 2b^2h_1S(Tw,Tw,w)+2bh_1S(u_n,u_n,u_{n+1}) \\ &\quad +b^2(2h_2+2h_4+1)S(u_{n+1},u_{n+1},w) \\ &\quad +2b^2h_2S(Tw,Tw,u_n)+2b^2h_4S(u_n,u_n,w) \\ &\quad +2bh_3S(u_n,u_n,u_{n+1}) \\ &= 2b^2h_1S(Tw,Tw,w)+2b(h_1+h_3)S(u_n,u_n,u_{n+1}) \\ &\quad +b^2(2h_2+2h_4+1)S(u_{n+1},u_{n+1},w) \\ &\quad +2b^2h_2S(Tw,Tw,u_n)+2b^2h_4S(u_n,u_n,w). \end{aligned} \tag{2.10}$$

From equation (2.10), we obtain

$$\begin{aligned} S(Tw,Tw,w) &\leq \frac{1}{1-2b^2h_1}\Big[2b(h_1+h_3)S(u_n,u_n,u_{n+1})+b^2(2h_2+2h_4+1)\times \\ &\quad S(u_{n+1},u_{n+1},w)+b^2(2h_2+2h_4+1)S(u_{n+1},u_{n+1},w) \\ &\quad +2b^2h_2S(Tw,Tw,u_n)+2b^2h_4S(u_n,u_n,w)\Big]. \end{aligned}$$

This implies that

$$\begin{aligned} \|S(Tw,Tw,w)\| &\leq \frac{K}{1-2b^2h_1}\Big[2b(h_1+h_3)\|S(u_n,u_n,u_{n+1})\|+b^2(2h_2+2h_4+1)\times \\ &\quad \|S(u_{n+1},u_{n+1},w)\|+b^2(2h_2+2h_4+1)\|S(u_{n+1},u_{n+1},w)\| \\ &\quad +2b^2h_2\|S(Tw,Tw,u_n)\|+2b^2h_4\|S(u_n,u_n,w)\|\Big]. \end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\|S(Tw, Tw, w)\| \leq \frac{2b^2h_2K}{1 - 2b^2h_1}\|S(Tw, Tw, w)\|,$$

or

$$\begin{aligned}\Big(1 - \frac{2b^2h_2K}{1 - 2b^2h_1}\Big)\|S(Tw, Tw, w)\| &\leq 0 \\ \Rightarrow \|S(Tw, Tw, w)\| &\leq 0.\end{aligned}$$

Thus, we have $S(Tw, Tw, w) = 0$. Hence $Tw = w$. This shows that w is a fixed point of T.

Case III: If $\max\{S(w, w, Tw), S(u_n, u_n, u_{n+1}), S(u_n, u_n, Tw)\} = S(u_n, u_n, Tw)$, then from equation (2.8), we obtain

$$\begin{aligned}S(Tw, Tw, w) &\leq 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_{n+1}) \\ &\quad +b^2(2h_2 + 2h_4 + 1)S(u_{n+1}, u_{n+1}, w) \\ &\quad +2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w) \\ &\quad +2bh_3S(u_n, u_n, Tw) \\ &\leq 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_{n+1}) \\ &\quad +b^2(2h_2 + 2h_4 + 1)S(u_{n+1}, u_{n+1}, w) \\ &\quad +2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w) \\ &\quad +2b^2h_3S(Tw, Tw, u_n) \\ &= 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_{n+1}) \\ &\quad +b^2(2h_2 + 2h_4 + 1)S(u_{n+1}, u_{n+1}, w) \\ &\quad +2b^2(h_2 + h_3)S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w).\end{aligned} \tag{2.11}$$

From equation (2.11), we obtain

$$\begin{aligned}S(Tw, Tw, w) &\leq \frac{1}{1 - 2b^2h_1}\Big[2bh_1S(u_n, u_n, u_{n+1}) + b^2(2h_2 + 2h_4 + 1) \times \\ &\quad S(u_{n+1}, u_{n+1}, w) + b^2(2h_2 + 2h_4 + 1)S(u_{n+1}, u_{n+1}, w) \\ &\quad +2b^2(h_2 + h_3)S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w)\Big].\end{aligned}$$

This implies that

$$\begin{aligned}\|S(Tw, Tw, w)\| &\leq \frac{K}{1 - 2b^2h_1}\Big[2bh_1\|S(u_n, u_n, u_{n+1})\| + b^2(2h_2 + 2h_4 + 1) \times \\ &\quad \|S(u_{n+1}, u_{n+1}, w)\| + b^2(2h_2 + 2h_4 + 1)\|S(u_{n+1}, u_{n+1}, w)\| \\ &\quad +2b^2(h_2 + h_3)\|S(Tw, Tw, u_n)\| + 2b^2h_4\|S(u_n, u_n, w)\|\Big].\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\|S(Tw, Tw, w)\| \leq \frac{2b^2(h_2 + h_3)K}{1 - 2b^2h_1}\|S(Tw, Tw, w)\|,$$

or

$$\begin{aligned}\Big(1 - \frac{2b^2(h_2 + h_3)K}{1 - 2b^2h_1}\Big)\|S(Tw, Tw, w)\| &\leq 0 \\ \Rightarrow \|S(Tw, Tw, w)\| &\leq 0.\end{aligned}$$

Thus, we have $S(Tw, Tw, w) = 0$. Hence $Tw = w$. This shows that w is a fixed point of T.

From above three cases we see that w is a fixed point of T.

Now to show that the fixed point of T is unique.

Let w_0 be another fixed point of T in X such that $Tw_0 = w_0$. Then,

$$\begin{aligned}
S(w, w, w_0) &= S(Tw, Tw, Tw_0) \\
&\leq h_1[S(w, w, Tw) + S(w_0, w_0, Tw_0)] \\
&\quad +h_2[S(w, w, Tw_0) + S(w_0, w_0, Tw)] \\
&\quad +h_3 \max\{S(w, w, Tw), S(w_0, w_0, Tw_0), S(w_0, w_0, Tw)\} \\
&\quad +h_4[S(w, w, w_0) + S(w, w, Tw_0)] \\
&= h_1[S(w, w, w) + S(w_0, w_0, w_0)] \\
&\quad +h_2[S(w, w, w_0) + S(w_0, w_0, w)] \\
&\quad +h_3 \max\{S(w, w, w), S(w_0, w_0, w_0), S(w_0, w_0, w)\} \\
&\quad +h_4[S(w, w, w_0) + S(w, w, w_0)] \\
&= (h_2 + 2h_4)S(w, w, w_0) + (h_2 + h_3)S(w_0, w_0, w) \\
&\leq (h_2 + 2h_4)S(w, w, w_0) + b(h_2 + h_3)S(w, w, w_0) \\
&= [(b + 1)h_2 + bh_3 + 2h_4]S(w, w, w_0),
\end{aligned}$$

or

$$\begin{aligned}
[1 - (b + 1)h_2 - bh_3 - h_4]S(w, w, w_0) &\leq 0 \\
\Rightarrow S(w, w, w_0) &\leq 0,
\end{aligned}$$

since $1 - (b + 1)h_2 - bh_3 - h_4 < 1$. Therefore, we have $S(w, w, w_0) = 0$. Hence $w = w_0$. This shows that the fixed point of T is unique. This completes the proof. □

If we take $h_1 = h$ and $h_2 = h_3 = h_4 = 0$ in Theorem 2.1, then we have the following result as corollary.

Corollary 2.2. *([18], Theorem 2.2) Let (X, S) be a complete cone S_b-metric space with the coefficient $b \geq 1$ and P be a normal cone with normal constant K. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$S(Tu, Tu, Tv) \leq h\,[S(u, u, Tu) + S(v, v, Tv)]$$

for all $u, v \in X$, where $0 \leq h < \frac{1}{2}$, that is, $h \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point w in X and we have $\lim_{n\to\infty} T^n(u) = w$, for each $u \in X$.

Remark 2.3. The contractive condition of Corollary 2.2 is similar with the well known Kannan [11] contraction type condition in the setting of complete cone S_b-metric space.

If we take $h_2 = h$ and $h_1 = h_3 = h_4 = 0$ in Theorem 2.1, then we have the following result as corollary.

Corollary 2.4. *([18], Theorem 2.3) Let (X, S) be a complete cone S_b-metric space with the coefficient $b \geq 1$ and P be a normal cone with normal constant K. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$S(Tu, Tu, Tv) \leq h\,[S(u, u, Tv) + S(v, v, Tu)]$$

for all $u, v \in X$, where $0 \leq h < \frac{1}{b(b+2)} < \frac{1}{2}$, that is, $h \in [0, \frac{1}{b(b+2)})$ is a constant. Then T has a unique fixed point w in X and we have $\lim_{n\to\infty} T^n(u) = w$, for each $u \in X$.

Remark 2.5. The contractive condition of Corollary 2.4 is similar with the well known Chatterjae [4] contraction type condition in the setting of complete cone S_b-metric space.

Theorem 2.6. *Let (X,S) be a complete cone S_b-metric space with the coefficient $b \geq 1$ and P be a normal cone with normal constant K such that for positive integer n, T^n satisfies the contraction condition (2.1) for all $u,v \in X$, where $h_1,h_2,h_3,h_4 > 0$ are constants such that $2h_1 + b(b+2)h_2 + h_3 + (b+1)^2 h_4 < 1$. Then T has a unique fixed point in X.*

Proof. From Theorem 2.1, let p_0 be the unique fixed point of T^n. Then

$$T(T^n p_0) = Tp_0 \quad \text{or} \quad T^n(Tp_0) = Tp_0.$$

This gives $Tp_0 = p_0$. This shows that p_0 is a unique fixed point of T. This completes the proof. □

Theorem 2.7. *Let (X,S) be a complete cone S_b-metric space with the coefficient $b \geq 1$ and P be a normal cone with normal constant K. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$\begin{aligned} S(Tu,Tu,Tv) \;&\leq\; \lambda \Big[S(u,u,v) + S(u,u,Tu) + S(v,v,Tv)\Big] \\ &\quad + \mu \Big[S(u,u,Tv) + S(v,v,Tu)\Big] \end{aligned} \tag{2.12}$$

for all $u,v \in X$, where $\lambda,\mu > 0$ are constants such that $3\lambda + b(b+2)\mu < 1$. Then T has a unique fixed point w in X and we have $\lim_{n\to\infty} T^n(u) = w$, for each $u \in X$.

Proof. Let $u_0 \in X$ and a sequence $\{u_n\}$ be defined by $T^n(u_0) = u_n$. Suppose that $u_n \neq u_{n+1}$ for all n. Using the inequality (2.12) and the condition (CS_bM_3) of definition 1.22, we get

$$\begin{aligned} S(u_n,u_n,u_{n+1}) &= S(Tu_{n-1},Tu_{n-1},Tu_n) \\ &\leq \lambda \Big[S(u_{n-1},u_{n-1},u_n) + S(u_{n-1},u_{n-1},Tu_{n-1}) + S(u_n,u_n,Tu_n)\Big] \\ &\quad + \mu \Big[S(u_{n-1},u_{n-1},Tu_n) + S(u_n,u_n,Tu_{n-1})\Big] \\ &= \lambda \Big[S(u_{n-1},u_{n-1},u_n) + S(u_{n-1},u_{n-1},u_n) + S(u_n,u_n,u_{n+1})\Big] \\ &\quad + \mu \Big[S(u_{n-1},u_{n-1},u_{n+1}) + S(u_n,u_n,u_n)\Big] \\ &= 2\lambda\, S(u_{n-1},u_{n-1},u_n) + \lambda\, S(u_n,u_n,u_{n+1}) \\ &\quad + \mu\, S(u_{n-1},u_{n-1},u_{n+1}) \\ &\leq 2\lambda\, S(u_{n-1},u_{n-1},u_n) + \lambda\, S(u_n,u_n,u_{n+1}) \\ &\quad + 2b\mu\, S(u_{n-1},u_{n-1},u_n) + b^2\mu\, S(u_n,u_n,u_{n+1}) \\ &= 2(\lambda + b\mu)\, S(u_{n-1},u_{n-1},u_n) + (\lambda + b^2\mu)\, S(u_n,u_n,u_{n+1}). \end{aligned} \tag{2.13}$$

From equation (2.13), we obtain

$$\begin{aligned} S(u_n,u_n,u_{n+1}) &\leq \Big(\frac{2(\lambda + b\mu)}{1 - \lambda - b^2\mu}\Big) S(u_{n-1},u_{n-1},u_n) \\ &= \rho\, S(u_{n-1},u_{n-1},u_n), \end{aligned} \tag{2.14}$$

where $\rho = \Big(\frac{2(\lambda+b\mu)}{1-\lambda-b^2\mu}\Big) < 1$ as $3\lambda + b(b+2)\mu < 1$.

Let $D_n = S(u_n,u_n,u_{n+1})$ and $D_{n-1} = S(u_{n-1},u_{n-1},u_n)$, then from equation (2.14), we get

$$D_n \leq \rho\, D_{n-1}. \tag{2.15}$$

Using equation (2.15), we obtain

$$D_n \leq \rho\, D_{n-1} \leq \rho^2\, D_{n-2} \leq \rho^3\, D_{n-3} \leq \cdots \leq \rho^n\, D_0. \tag{2.16}$$

Now for $m,n \geq 1$ and $m > n$, we have

$$\begin{aligned} S(u_n,u_n,u_m) &\leq b[2S(u_n,u_n,u_{n+1}) + S(u_m,u_m,u_{n+1})] \\ &\leq 2bS(u_n,u_n,u_{n+1}) + b^2S(u_m,u_m,u_{n+1}) \\ &\leq 2bS(u_n,u_n,u_{n+1}) + 2b^3S(u_{n+1},u_{n+1},u_{n+2}) \\ &\quad + b^4S(u_{n+2},u_{n+2},u_m) \\ &\leq 2bS(u_n,u_n,u_{n+1}) + 2b^3S(u_{n+1},u_{n+1},u_{n+2}) \\ &\quad + 2b^5S(u_{n+2},u_{n+2},u_{n+3}) + \dots \\ &\quad + 2b^{2(m-n-1)}S(u_{m-1},u_{m-1},u_m) \\ &< 2b\Big\{S(u_n,u_n,u_{n+1}) + b^2S(u_{n+1},u_{n+1},u_{n+2}) \\ &\quad + b^4S(u_{n+2},u_{n+2},u_{n+3}) + \dots \\ &\quad + b^{2(m-n-1)}S(u_{m-1},u_{m-1},u_m)\Big\} \\ &\leq 2b\Big\{\rho^n + b^2\rho^{n+1} + b^4\rho^{n+2} + \cdots + b^{2(m-n-1)}\rho^{m-1}\Big\}S(u_0,u_0,u_1) \\ &= 2b\rho^n\Big\{1 + b^2\rho + (b^2\rho)^2 + \cdots + (b^2\rho)^{m-n-1}\Big\}D_0 \\ &\leq \Big(\frac{2b\rho^n}{1-b^2\rho}\Big)D_0. \end{aligned}$$

This implies that

$$\|S(u_n,u_n,u_m)\| \leq \Big(\frac{2b\rho^nK}{1-b^2\rho}\Big)\|D_0\|.$$

Taking limit as $n,m \to \infty$, we get

$$\|S(u_n,u_n,u_m)\| \to 0,$$

since $0 < \rho < 1$. Thus, we have $S(u_n,u_n,u_m) \to 0$ as $n,m \to \infty$.

Therefore, the sequence $\{u_n\}$ is a Cauchy sequence in X. By the completeness of the space, there exists $w \in X$ such that $\lim_{n\to\infty} u_n = w$, i.e., $\lim_{n\to\infty} T^nu_0 = w$. Rest of the proof follows from Theorem 2.1. This completes the proof. □

Remark 2.8. If we take $\lambda = 0$ and $\mu = h$ in Theorem 2.7, then we obtain Theorem 2.3 of [18] as corollary.

Theorem 2.9. *Let (X,S) be a complete cone S_b-metric space with the coefficient $b \geq 1$ and P be a normal cone with normal constant K such that for positive integer n, T^n satisfies the contraction condition (2.12) for all $u,v \in X$, where $\lambda,\mu > 0$ are constants such that $3\lambda + b(b+2)\mu < 1$. Then T has a unique fixed point in X.*

Proof. The proof of Theorem 2.9 follows from Theorem 2.6. □

Example 2.10. Let $E = \mathbb{R}^2$, the Euclidean plane, $P = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ a normal cone in E and $X = \mathbb{R}$. Then the function $S\colon X^3 \to E$ defined by $S(x,y,z) = \Big(|x-z|+|y-z|\Big)^2$ for all $x,y,z \in X$. Then (X,S) is a cone S_b-metric space with coefficient $b = 4$. Now, we consider the mapping $T\colon X \to X$ by $T(x) = \frac{x+1}{2}$. Then

$$\begin{aligned}
S(Tx, Tx, Ty) &= \Big[|Tx - Ty| + |Tx - Ty|\Big]^2 \\
&= 4|Tx - Ty|^2 = 4\Big|\Big(\frac{x+1}{2}\Big) - \Big(\frac{y+1}{2}\Big)\Big|^2 \\
&= |x - y|^2 \\
&\leq \frac{1}{3}[|x-1|^2 + |y-1|^2] \\
&= h\,[S(x, x, Tx) + S(y, y, Ty)]
\end{aligned}$$

where $h = \frac{1}{3} < \frac{1}{2}$. Thus T satisfies all the conditions of Corollary 2.2 and clearly $1 \in X$ is the unique fixed point of T.

Example 2.11. Let $E = \mathbb{R}^2$, the Euclidean plane, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ a normal cone in E and $X = \mathbb{R}$. Then the function $S\colon X^3 \to E$ defined by $S(x, y, z) = \Big(|x-z|+|y-z|\Big)^2$ for all $x, y, z \in X$. Then (X, S) is a cone S_b-metric space with coefficient $b = 4$. Now, we consider the mapping $T\colon X \to X$ by $T(x) = \frac{x+2}{3}$. Then

$$\begin{aligned}
S(Tx, Tx, Ty) &= \Big[|Tx - Ty| + |Tx - Ty|\Big]^2 \\
&= 4|Tx - Ty|^2 = 4\Big|\Big(\frac{x+2}{3}\Big) - \Big(\frac{y+2}{3}\Big)\Big|^2 \\
&= \frac{4}{9}|x - y|^2.
\end{aligned}$$

$$S(x, x, Ty) = 4|x - Ty|^2 = \frac{4}{9}|3x - y - 2|^2.$$

$$S(y, y, Tx) = 4|y - Tx|^2 = \frac{4}{9}|3y - x - 2|^2.$$

Now, we have

$$\begin{aligned}
S(Tx, Tx, Ty) &\leq \frac{1}{6}\Big[\frac{4}{9}|3x - y - 2|^2 + \frac{4}{9}|3y - x - 2|^2\Big]^2 \\
&= \frac{1}{6}[S(x, x, Ty) + S(y, y, Tx)] \\
&= h[S(x, x, Ty) + S(y, y, Tx)]
\end{aligned}$$

where $h = \frac{1}{6} < \frac{1}{2}$. Thus T satisfies all the conditions of Corollary 2.4 and clearly $1 \in X$ is the unique fixed point of T.

Open Question: Can we extend the results for rational contraction/ rational type contraction/ contraction involving rational expression?

3 Conclusion

In this paper, we establish some existence of fixed point theorems via contractive type conditions in the framework of cone S_b-metric spaces. Our results extend, unify and generalize several results given in the existing literature.

References

[1] M. Abbas and G. Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl.* **341**, 416–420 (2008).

[2] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, *Funct. Anal. Gos. Ped. Inst. Unianowsk* **30**, 26–37 (1989).

[3] S. Banach, Surles operation dans les ensembles abstraits et leur application aux equation integrals, *Fund. Math.* **3**, 133–181 (1922).

[4] S. K. Chatterjae, Fixed point theorems compactes, *Rend. Acad. Bulgare Sci.* **25**, 727–730 (1972).

[5] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. inf Univ. Ostraviensis* **1**, 5–11 (1993).

[6] D. Dhamodharan and R. Krishnakumar, Cone S-metric space and fixed point theorems of contractive mappings, *Annals of Pure Appl. Math.* **14(2)**, 237–243 (2017).

[7] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, (1985).

[8] L. -G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.* **332(2)**, 1468–1476 (2007).

[9] N. Hussain and MH. Shah, KKM mappings in cone b-metric spaces, *Comput. Math. Appl.* **62**, 1677–1684 (2011).

[10] D. Ilic and V. Rakocevic, Common fixed points for maps on cone metric space, *J. Math. Anal. Appl.* **341**, 876–882 (2008).

[11] R. Kannan, Some results on fixed point theorems, *Bull. Calcutta Math. Soc.* **60**, 71–78 (1969).

[12] R. Krishnakumar and D. Dhamodharan, Fixed point theorems in normal cone metric space, *Int. J. Math. Sci. Engg. Appl.* **10(III)**, 213–224 (2016).

[13] N. Yilmaz Ozgur and N. Tas, Some fixed point theorems on S-metric spaces, *Mat. Vesnik* **69(1)**, 39–52 (2017).

[14] Sh. Rezapour, *A review on topological properties of cone metric spaces*, in Proceedings of the International Conference on Analysis, Topology and Appl. (ATA 08), Vrinjacka Banja, Serbia, May-June (2008).

[15] Sh. Rezapour and R. Hamlbarani, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", *J. Math. Anal. Appl.* **345(2)**, 719–724 (2008).

[16] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, *Mat. Vesnik* **64(3)**, 258–266 (2012).

[17] S. Sedghi, A. Gholidahnch and K. P. R. Rao, Common fixed point of two R-weakly commuting mappings in S_b-metric spaces, *Math. Sci. Lett.* **6(3)**, 249–253 (2017).

[18] K. Anthony Singh and M. R. Singh, Some fixed point theorems of cone S_b-metric space, *J. Indian Acad. Math.* **40(2)**, 255-272 (2018).

[19] N. Souayah and N. Mlaiki, A fixed point theorem in S_b-metric spaces, *J. Math. Computer Sci.* **16**, 131–139 (2016).

[20] N. Tas and N. Yilmaz Ozgur, New generalized fixed point results on S_b-metric spaces, arxiv:1703.01868v2 [math.gn] 17 apr.

[21] J. Vandergraft, Newton method for convex operators in partially ordered spaces, *SIAM J. Numer. Anal.* **4**, 406-432 (1967).

[22] P. Vetro, Common fixed points in cone metric spaces, *Rend. Circ. Mat. Palermo*, (2) **56(3)**, 464–468 (2007).

[23] P. P. Zabrejko, K-metric and K-normed linear spaces: survey, *Collectanea Mathematica* **48(4-6)**, 825–859 (1997).

Author information

G. S. Saluja, Department of Mathematics, Govt. Kaktiya P. G. College Jagdalpur, Jagdalpur - 494001 (Chhattisgarh), India.
E-mail: `saluja1963@gmail.com`

Received: August 20, 2019.
Accepted: April 29, 2020.