SOME FIXED POINT RESULTS UNDER CONTRACTIVE TYPE MAPPINGS IN CONE $S_b$-METRIC SPACES

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Abstract. In this paper, we establish fixed point results using some contractive type mappings in the setting of cone $S_b$-metric spaces. Our results extend, unify and generalize several results given in the current existing literature.

1 Introduction and Preliminaries

The Banach contraction principle [3] is a basic tool in studying the existence of solutions to many problems in mathematics and many different fields. In recent times, the contraction principle has been extended in various directions. The Banach fixed point theorem (or Banach contraction principle) is stated as follows.

Theorem 1.1. (Banach contraction principle) ([3]) Let $(X, d)$ be a complete metric space and $T$ be a mapping of $X$ into itself satisfying:

$$d(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in X,$$

(1.1)

where $\lambda$ is a constant in $(0, 1)$. Then $T$ has a fixed point $v \in X$.

Since then, fixed point theory has had a rapid development. There is a great number of generalization of the Banach contraction principle. The underlying metric space can be generalized in many ways. In addition to the improvement of Banach’s contractive condition, more and more attention is devoted itself to the generalization of metric spaces such as 2-metric spaces, $D_*$-metric spaces, partial metric spaces, cone metric spaces, $b$-metric spaces, $G$-metric spaces and cone $b$-metric spaces etc.

The notion of a $b$-metric space was introduced by Bakhtin [2] and then extensively used by Czerwik in [5]. Since then $b$-metric fixed point theory grew up in the classical metric fixed point theory to obtain a generalization of some known metric version of fixed point results. On the other hand, some authors are interested and have tried to give generalizations of metric spaces in different way.

In 2007, Huang and Zhang [8] introduced the concept of cone metric spaces as a generalization of metric spaces by replacing the set of real numbers by a general Banach space $E$ which is partially ordered with respect to a cone $P \subset E$ and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [1, 10, 15, 22] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space.

In 2011, Hussain and Shah [9] introduced the concept of cone $b$-metric space as a generalization of $b$-metric space and cone metric spaces. They established some topological properties in such spaces and improved some recent results about $KKM$ mappings in the setting of a cone $b$-metric space.
In 2012, Sedghi et al. [16] introduced the concept of $S$-metric space which is different from other space and proved fixed point theorems in $S$-metric space. They also give some examples of $S$-metric space which shows that $S$-metric space is different from other spaces.

In 2016, Souayah and Mlaiki [19] introduced the concept of $S_0$-metric space. The concept of $S_0$-metric space is further used in many other research papers. In 2017, Dhamdharan and Krishnakumar [6] further extended the notion of $S$-metric spaces to cone $S$-metric spaces and proved fixed point results using different contractive type mappings.

Recently, K. A. Singh and M. R. Singh [18] have introduced the concept of cone $S_0$-metric space and proved some fixed point results via different contractive conditions which extend various results in the existing literature.

The importance of cone $S_0$-metric space is to further study of topological properties like continuity and compactness and to extend and generalise several standard results, for example, Banach contraction principle, Kannan contraction, Chatterjae contraction etc. in the said space.

The main interest of this paper is to establish some existence of fixed point theorems under various contractive type conditions in the setting of cone $S_0$-metric spaces. Our results extend, generalize and unify several results from the existing literature.

We need the following definitions and properties in the sequel.

**Definition 1.2.** ([8]) Let $E$ be a real Banach space. A subset $P$ of $E$ is called a cone whenever the following conditions hold:

$(c_1)$ $P$ is closed, nonempty and $P \neq \{0\}$;

$(c_2)$ $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;

$(c_3)$ $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ in $E$ with respect to $P$ by $x \preceq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where $P^0$ stands for the interior of $P$. If $P^0 \neq \emptyset$ then $P$ is called a solid cone (see [21]).

There exist two kinds of cones- normal (with the normal constant $K$) and non-normal ones ([7]).

Let $E$ be a real Banach space, $P \subset E$ a cone and $\preceq$ partial ordering defined by $P$. Then $P$ is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \preceq y \implies \|x\| \leq K\|y\|,$$

(1.2)

or equivalently, if $(\forall n) \ x_n \preceq y_n \preceq z_n$ and

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n \quad \implies \quad \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = x.$$

(1.3)

The least positive number $K$ satisfying (1.2) is called the normal constant of $P$.

The cone $P$ is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone $P$ is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose $E$ is a Banach space, $P$ is a cone in $E$ with $\text{int}(P) \neq \emptyset$ and $\preceq$ is partial ordering in $E$ with respect to $P$.

**Example 1.3.** ([12]) Let $K > 1$ be given. Consider the real vector space

$$E = \left\{ ax + b : a, b \in R; x \in \left[ 1 - \frac{1}{K}, 1 \right] \right\}$$

with supremum norm and the cone

$$P = \{ ax + b \in E : a \geq 0, b \geq 0 \}$$

in $E$. The cone $P$ is regular and so normal.

**Definition 1.4.** ([8, 23]) Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:
(CM₁) \(0 \leq d(x, y)\) for all \(x, y \in X\) with \(x \neq y\) and \(d(x, y) = 0 \iff x = y\);

(CM₂) \(d(x, y) = d(y, x)\) for all \(x, y \in X\);

(CM₃) \(d(x, y) \leq d(x, z) + d(z, y)\) for all \(x, y, z \in X\).

Then \(d\) is called a cone metric [8] on \(X\) and \((X, d)\) is called a cone metric space [8] or simply CMS.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where \(E = \mathbb{R}\) and \(P = [0, +\infty)\).

**Lemma 1.5.** [15] Every regular cone is normal.

**Example 1.6.** ([8]) Let \(E = \mathbb{R}^2, P = \{ (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0 \}\), \(X = \mathbb{R}\) and \(d: X \times X \to E\) defined by \(d(x, y) = (|x - y|, \alpha|x - y|)\), where \(\alpha \geq 0\) is a constant. Then \((X, d)\) is a cone metric space with normal cone \(P\) where \(K = 1\).

**Example 1.7.** ([14]) Let \(E = \mathbb{R}^2, P = \{ (x_n)_{n \geq 1} \in E : x_n \geq 0, \text{ for all } n \}\), \((X, \rho)\) a metric space, and \(d: X \times X \to E\) defined by \(d(x, y) = (\rho(x, y)/2^n)_{n \geq 1}\). Then \((X, d)\) is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

**Definition 1.8.** ([16, 13]) Let \(X\) be a nonempty set and \(S: X^3 \to [0, \infty)\) be a function satisfying the following conditions for all \(x, y, z, t \in X\):

\[(SM₁) \ S(x, y, z) \geq 0;\]

\[(SM₂) \ S(x, y, z) = 0 \text{ if and only if } x = y = z;\]

\[(SM₃) \ S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t).\]

Then the function \(S\) is called an \(S\)-metric on \(X\) and the pair \((X, S)\) is called an \(S\)-metric space or simply SMS.

**Example 1.9.** ([20]) Let \(X\) be a nonempty set and \(d\) be the ordinary metric on \(X\). Then \(S(x, y, z) = d(x, z) + d(y, z)\) is an \(S\)-metric on \(X\).

**Example 1.10.** ([16]) Let \(X = \mathbb{R}^n\) and \(\|\cdot\|\) a norm on \(X\), then \(S(x, y, z) = \|y + z - 2x\| + \|y - z\|\) is an \(S\)-metric on \(X\).

**Example 1.11.** ([16]) Let \(X = \mathbb{R}^n\) and \(\|\cdot\|\) a norm on \(X\), then \(S(x, y, z) = \|x - z\| + \|y - z\|\) is an \(S\)-metric on \(X\).

**Definition 1.12.** ([19]) Let \(X\) be a nonempty set and \(b \geq 1\) be a given real number. A function \(S_b: X^3 \to [0, \infty)\) is said to be \(S_b\)-metric if and only if for all \(x, y, z, t \in X\) the following conditions are satisfied:

\[(S_bM₁) \ S_b(x, y, z) = 0 \text{ if and only if } x = y = z;\]

\[(S_bM₂) \ S_b(x, y, z) \leq b[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)].\]

The pair \((X, S_b)\) is called an \(S_b\)-metric space.

**Remark 1.13.** Note that the class of \(S_b\)-metric spaces is larger than the class of \(S\)-metric spaces. Indeed every \(S\)-metric space is an \(S_b\)-metric space with \(b = 1\). However, the converse is not always true.

**Example 1.14.** ([19]) Let \(X\) be a non-empty set and \(\text{card}(X) \geq 5\). Suppose \(X = X₁ \cup X₂\) a partition of \(X\) such that \(\text{card}(X₁) \geq 4\). Let \(s \geq 1\). Then

\[
S_b(x, y, z) = \begin{cases} 
0 & \text{if } x = y = z, \\
3s & \text{if } (x, y, z) \in X₁^3, \\
1 & \text{if } (x, y, z) \notin X₁^3,
\end{cases}
\]

for all \(x, y, z \in X\), \(S_b\) is a \(S_b\)-metric on \(X\) with the coefficient \(s \geq 1\).
Proof. (i) If \( x = y = z \) then \( S_b(x, y, z) = 0 \). Thus the first assertion of the definition of \( S_b \)-metric space is satisfied.

(ii) Now to prove the triangle inequality:

\[
S_b(x, y, z) \leq s[S_b(x, t) + S_b(y, t) + S_b(z, t)] \quad (\ast).
\]

(a) Case 1. If \((x, y, z) \notin X^3\), we have \( S_b(x, y, z) = 1 \), \( S_b(x, t) \geq 1 \), \( S_b(y, t) \geq 1 \) and \( S_b(z, t) \geq 1 \) for all \( t \in X \). Thus \((\ast)\) holds \((1 \leq 3s)\).

(b) Case 2. If \((x, y, z) \in X^3\), we have the following two sub-cases:

(i) If \( t \in X_1 \), \((\ast)\) is satisfied since \( S_b(x, y, z) = S_b(x, t) = S_b(y, t) = S_b(z, t) = 3s \).

(ii) If \( t \notin X_1 \), we have \( S_b(x, x, t) = S_b(y, y, t) = S_b(z, z, t) = 1 \) and \( S_b(x, y, z) = 3s \). Then \((\ast)\) holds. \(\square\)

Example 1.15. ([17]) Let \((X, S)\) be an \( S \)-metric space and \( S_\alpha(x, y, z) = \left\{ S(x, y, z) \right\}^\alpha \), where \( p > 1 \) is a real number. Then \( S_\alpha \) is an \( S_b \)-metric on \( X \) with \( b = 2^{2(p-1)} \).

Example 1.16. ([18]) Let \( X = \mathbb{R} \) and let the function \( S : X^3 \to \mathbb{R} \) be defined as \( S(x, y, z) = |x - z| + |y - z| \). Then \( S \) is an \( S \)-metric on \( X \). Therefore, the function \( S_b(x, y, z) = \left\{ S(x, y, z) \right\}^2 = \left\{ |x - z| + |y - z| \right\}^2 \) is an \( S_b \)-metric on \( X \) with \( b = 2^{2(2-1)} = 4 \).

Definition 1.17. ([6]) Suppose that \( E \) is a real Banach space, \( P \) is a cone in \( E \) with \( \text{int} \ P \neq \emptyset \) and \( \leq \) is partial ordering with respect to \( P \). Let \( X \) be a nonempty set and let the function \( S : X^3 \to E \) satisfy the following conditions:

1. \( S(x, y, z) \geq 0 \);
2. \( S(x, y, z) = 0 \) if and only if \( x = y = z \);
3. \( S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a) \) for each \( \alpha \in X, y, z, a \in X \).

Then the function \( S \) is called a cone \( S \)-metric on \( X \) and the pair \((X, S)\) is called a cone \( S \)-metric space or simply CSMS.

Example 1.18. ([6]) Let \( E = \mathbb{R}^2 \), \( P = \{ (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0 \} \), \( X = \mathbb{R} \) and \( d \) be the ordinary metric on \( X \). Then the function \( S : X^3 \to E \) defined by \( S(x, y, z) = \left( d(x, z) + d(y, z), \alpha(d(x, z) + d(y, z)) \right) \), where \( \alpha > 0 \) is a constant, is a cone \( S \)-metric on \( X \).

Lemma 1.19. ([6]) Let \((X, S)\) be a cone \( S \)-metric space. Then we have \( S(x, y, z) = S(y, x, y) \).

Definition 1.20. ([6]) Let \((X, S)\) be a cone \( S \)-metric space.

(i) A sequence \( \{u_n\} \) in \( X \) converges to \( u \) if and only if \( S(u_n, u_n, u) \to 0 \) as \( n \to \infty \), that is, there exists \( n_0 \in N \) such that for all \( n \geq n_0 \), \( S(u_n, u_n, u) \ll c \) for each \( c \in E \), \( 0 \ll c \). We denote this by \( \lim_{n \to \infty} u_n = u \) or \( \lim_{n \to \infty} S(u_n, u_n, u) = 0 \).

(ii) A sequence \( \{u_n\} \) in \( X \) is called a Cauchy sequence if \( S(u_n, u_n, u_m) \to 0 \) as \( n, m \to \infty \), that is, there exists \( n_0 \in N \) such that for all \( n, m \geq n_0 \), \( S(u_n, u_n, u_m) \ll c \) for each \( c \in E \), \( 0 \ll c \).

(iii) The cone \( S \)-metric space \((X, S)\) is called complete if every Cauchy sequence in \( X \) is convergent in \( X \).

In the following lemma, we see the relationship between a cone metric and a cone \( S \)-metric.

Lemma 1.21. ([6]) Let \((X, S)\) be a cone \( S \)-metric space. Then, the following properties are satisfied:

1. \( S(u, v, z) = d(u, z) + d(v, z) \) for all \( u, v, z \in X \), is a cone \( S \)-metric on \( X \).
2. \( u_n \to u \) in \((X, d)\) if and only if \( u_n \to u \) in \((X, S_d)\).
3. \( \{u_n\} \) is Cauchy in \((X, d)\) if and only if \( \{u_n\} \) is Cauchy in \((X, S_d)\).
4. \((X, d)\) is complete if and only if \((X, S_d)\) is complete.

The notion of cone \( S_b \)-metric space is introduced by K. Anthony Singh and M. R. Singh [18] in 2018 as follows:
Lemma 1.27. ([18]) Let $K$ be a normal cone with normal constant $b > 1$ in the setting of cone metric spaces. In this section, we shall prove some existence of fixed point results under contractive type conditions in the setting of cone $S_b$-metric space or simply $CS_bMS$.

We note that cone $S_b$-metric spaces are generalizations of cone metric spaces since every cone metric space is a cone $S_b$-metric space with $b = 1$.

Example 1.23. ([18]) Let $E = \mathbb{R}^2$, the Euclidean plane and $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, a normal cone in $E$. Let $X = \mathbb{R}$ and $S: X^3 \to E$ be such that $S(x, y, z) = \left(\alpha S_\nu(x, y, z), \beta S_\nu(x, y, z)\right)$, where $\alpha, \beta > 0$ are constants and $S_\nu$ is an $S_b$-metric on $X$. Then $S$ is a cone $S_b$-metric on $X$.

In particular, the function $S_\nu(x, y, z) = \left\{(|x| + |y|)^2\right\}$ for all $x, y, z \in X$ is an $S_b$-metric on $X$ with $b = 4$.

Therefore, the function $S(x, y, z) = \left\{||x| + |y||^2, \frac{1}{4}\right\}$ for all $x, y, z \in X$ is an $S_b$-metric on $X$ with $b = 4$.

Definition 1.24. ([18]) Let $(X, S)$ be a cone $S_b$-metric space.

1. A sequence $\{u_n\}$ in $X$ converges to $u$ if and only if $S(u_n, u_n, u) \to 0$ as $n \to \infty$, that is, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(u_n, u_n, u) \leq \epsilon$ for each $\epsilon \in E$, $0 < \epsilon$. We denote this by $\lim_{n \to \infty} u_n = u$ or $u_n \to u$ as $n \to \infty$.

2. A sequence $\{u_n\}$ in $X$ is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $m, n \to \infty$, that is, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $S(u_n, u_n, u_m) \leq \epsilon$ for each $\epsilon \in E$, $0 < \epsilon$.

3. The cone $S_b$-metric space $(X, S)$ is called complete if every Cauchy sequence in $X$ is convergent in $X$.

Lemma 1.25. ([18]) Let $(X, S)$ be a cone $S_b$-metric space, $P$ be a normal cone with normal constant $K$. Then a sequence $\{u_n\}$ in $X$ converges to $u$ if and only if $S(u_n, u_n, u) \to 0$ as $n \to \infty$.

Lemma 1.26. ([18]) Let $(X, S)$ be a cone $S_b$-metric space, $P$ be a normal cone with normal constant $K$. Let $\{u_n\}$ be a sequence in $X$. If the sequence $\{u_n\}$ converges to $u_1$ and $u_2$, then $u_1 = u_2$, that is, the limit of a convergent sequence is unique.

Lemma 1.27. ([18]) Let $(X, S)$ be a cone $S_b$-metric space, $P$ be a normal cone with normal constant $K$. Then a sequence $\{u_n\}$ in $X$ is a Cauchy sequence if and only if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$.

Lemma 1.28. ([18]) Let $(X, S)$ be a cone $S_b$-metric space, $P$ be a normal cone with normal constant $K$. Let $\{u_n\}$ be a sequence in $X$. If the sequence $\{u_n\}$ converges to $u$, then $\{u_n\}$ is a Cauchy sequence, that is, every convergent sequence is a Cauchy sequence.

2 Main Results

In this section, we shall prove some existence of fixed point results under contractive type conditions in the setting of cone $S_b$-metric spaces.

Theorem 2.1. Let $(X, S)$ be a complete cone $S_b$-metric space with the coefficient $b \geq 1$ and $P$ be a normal cone with normal constant $K$. Suppose that the mapping $T: X \to X$ satisfies the following condition:

$$S(Tu, Tu, Tv) \leq h_1 |S(u, u, Tu) + S(v, v, Tv)| + h_2 |S(u, u, Tv) + S(v, v, Tu)|$$

$$+ h_3 \max \left\{S(u, u, Tu), S(v, v, Tv), S(v, v, Tu)\right\}$$

$$+ h_4 |S(u, u, v) + S(u, u, Tv)|$$

(2.1)
for all \( u, v \in X \), where \( h_1, h_2, h_3, h_4 > 0 \) are constants such that \( 2h_1 + b(b + 2)h_2 + h_3 + (b + 1)^2h_4 < 1 \). Then \( T \) has a unique fixed point \( w \) in \( X \) and we have \( \lim_{n \to \infty} T^n(u) = w \), for each \( u \in X \).

**Proof.** Let \( u_0 \in X \) and a sequence \( \{u_n\} \) be defined by \( T^n(u_0) = u_n \). Suppose that \( u_n \neq u_{n+1} \) for all \( n \). Using the inequality (2.1) and the condition \((CS_1M_3)\) of definition 1.22, we get

\[
S(u_n, u_n, u_{n+1}) = S(Tu_{n-1}, Tu_{n-1}, Tu_n) 
\leq h_1 [S(u_{n-1}, u_{n-1}, Tu_{n-1}) + S(u_n, u_n, Tu_n)] 
+ h_2 [S(u_{n-1}, u_{n-1} + Tu_{n-1}) + S(u_n, u_n, Tu_n)] 
+ h_3 \max \left\{ S(u_{n-1}, u_{n-1}, Tu_{n-1}), S(u_n, u_n, Tu_n) \right\} 
+ h_4 [S(u_{n-1}, u_{n-1}, u_n) + S(u_{n-1}, u_{n-1}, Tu_n)] 
= h_1 [S(u_{n-1}, u_{n-1}, u_n) + S(u_n, u_n, u_{n+1})] 
+ h_2 [S(u_{n-1}, u_{n-1}, u_n) + S(u_n, u_n, u_{n+1})] 
+ h_3 \max \left\{ S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1}) \right\} 
+ h_4 [S(u_{n-1}, u_{n-1}, u_n) + S(u_{n-1}, u_{n-1}, u_{n+1})] 
\leq (h_1 + h_4) S(u_{n-1}, u_{n-1}, u_n) 
+ h_1 S(u_n, u_n, u_{n+1}) 
\leq (h_1 + h_4) S(u_{n-1}, u_{n-1}, u_n) 
+ h_1 S(u_n, u_n, u_{n+1}) 
\leq (h_1 + h_4) S(u_{n-1}, u_{n-1}, u_n) 
+ 2b(h_2 + h_4) S(u_{n-1}, u_{n-1}, u_n) 
+ b^2(h_2 + h_4) S(u_n, u_n, u_{n+1}) 
\leq [h_1 + 2bh_2 + (2b + 1)h_4] S(u_{n-1}, u_{n-1}, u_n) 
+ [h_1 + b^2(h_2 + h_4)] S(u_n, u_n, u_{n+1}) 
\leq [h_1 + 2bh_2 + h_3 + (2b + 1)h_4] S(u_{n-1}, u_{n-1}, u_n) 
+ [h_1 + b^2(h_2 + h_4)] S(u_n, u_n, u_{n+1}) 
\leq [h_1 + 2bh_2 + h_3 + (2b + 1)h_4] \times 
S(u_{n-1}, u_{n-1}, u_n) 
\times \frac{1 - h_1 - b^2(h_2 + h_4)}{1 - h_1 - b^2(h_2 + h_4)} 
\times S(u_n, u_n, u_{n+1}) 
= t S(u_{n-1}, u_{n-1}, u_n), 
\tag{2.2}
\]

Here we consider the following cases.

**Case I:** If \( \max \left\{ S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1}) \right\} = S(u_{n-1}, u_{n-1}, u_n) \), then from equation (2.2), we obtain

\[
S(u_n, u_n, u_{n+1}) \leq [h_1 + 2bh_2 + h_3 + (2b + 1)h_4] S(u_{n-1}, u_{n-1}, u_n) 
+ [h_1 + b^2(h_2 + h_4)] S(u_n, u_n, u_{n+1}).
\]

or

\[
[1 - h_1 - b^2(h_2 + h_4)] S(u_n, u_n, u_{n+1}) \leq [h_1 + 2bh_2 + h_3 + (2b + 1)h_4] \times 
S(u_{n-1}, u_{n-1}, u_n) 
\times \frac{1 - h_1 - b^2(h_2 + h_4)}{1 - h_1 - b^2(h_2 + h_4)} \times 
S(u_n, u_n, u_{n+1}) 
= t S(u_{n-1}, u_{n-1}, u_n),
\tag{2.3}
\]

where \( t = \frac{h_1 + 2bh_2 + h_3 + (2b + 1)h_4}{1 - h_1 - b^2(h_2 + h_4)} \) < 1 as \( 2h_1 + b(b + 2)h_2 + h_3 + (b + 1)^2h_4 < 1 \).
Using equation (2.6), we obtain

\[ m,n<\mu< \]

where 0

Combining above two cases, we have

\[ V_n \leq \mu V_{n-1}. \]  \hspace{1cm} (2.6)

Using equation (2.6), we obtain

\[ V_n \leq \mu V_{n-1} \leq \mu^2 V_{n-2} \leq \mu^3 V_{n-3} \leq \cdots \leq \mu^n V_0. \]  \hspace{1cm} (2.7)

Now for \( m, n \geq 1 \) and \( m > n \), we have

\[ S(u_n, u_n, u_m) \leq b[2S(u_n, u_n, u_{n+1}) + S(u_m, u_m, u_{n+1})] \]

\[ \leq 2bS(u_n, u_n, u_{n+1}) + b^2 S(u_m, u_m, u_{n+1}) \]

\[ \leq 2bS(u_n, u_n, u_{n+1}) + 2b^2 S(u_{n+1}, u_{n+1}, u_{n+2}) \]

\[ + b^3 S(u_{n+2}, u_{n+2}, u_m) \]

\[ \leq 2bS(u_n, u_n, u_{n+1}) + 2b^2 S(u_{n+1}, u_{n+1}, u_{n+2}) \]

\[ + b^2 S(u_{n+2}, u_{n+2}, u_{n+3}) + \cdots \]

\[ + b^{2(m-n-1)} S(u_{m-1}, u_{m-1}, u_m) \]

\[ \leq 2b \left\{ \mu^n + b^2 \mu^{n+1} + b^4 \mu^{n+2} + \cdots + b^{2(m-n-1)} \mu^{m-1} \right\} S(u_0, u_0, u_1) \]

\[ \leq 2b \mu^n \left\{ 1 + b^2 \mu + (b^2 \mu)^2 + \cdots + (b^2 \mu)^{m-n-1} \right\} V_0 \]

\[ \leq \left( \frac{2b \mu^n}{1 - b^2 \mu} \right) V_0. \]

This implies that

\[ \| S(u_n, u_n, u_m) \| \leq \left( \frac{2b \mu^n K}{1 - b^2 \mu} \right) \| V_0 \|. \]
Taking limit as $n, m \to \infty$, we get

$$||S(u_n, u_n, u_m)|| \to 0,$$

since $0 < \mu < 1$. Thus, we have $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$.

Therefore, the sequence $\{u_n\}$ is a Cauchy sequence in $X$. By the completeness of the space, there exists $w \in X$ such that $\lim_{n \to \infty} u_n = w$, i.e., $\lim_{n \to \infty} T^nu_0 = w$.

Also, we have

$$S(Tw, Tw, w) \leq 2bS(Tw, Tw, Tu_n) + bS(w, w, Tu_n)$$

$$\leq 2b\bigg\{h_1[S(w, w, Tw) + S(u_n, u_n, Tu_n)]$$

$$+ h_2[S(u_n, u_n, Tw) + S(u_n, u_n, TTw)]$$

$$+ h_3 \max\{S(w, w, Tw), S(u_n, u_n, Tw), S(u_n, u_n, Tu_n)\}$$

$$+ h_4[S(w, w, u_n) + S(w, w, u_n)]\bigg\}$$

$$+ bS(w, w, Tu_n)$$

$$= 2b\bigg\{h_1[S(w, w, Tw) + S(u_n, u_n, u_n+1)]$$

$$+ h_2[S(w, w, u_n+1) + S(u_n, u_n, Tw)]$$

$$+ h_3 \max\{S(w, w, Tw), S(u_n, u_n, u_n+1), S(u_n, u_n, Tw)\}$$

$$+ h_4[S(w, w, u_n) + S(w, w, u_n+1)]\bigg\}$$

$$+ bS(w, w, u_n+1)$$

$$\leq 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_n+1)$$

$$+ b^2(2h_2 + 2h_4 + 1)S(u_n+1, u_n+1, w)$$

$$+ 2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w)$$

$$+ 2bh_3S(w, w, Tw)$$

$$\leq 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_n+1)$$

$$+ b^2(2h_2 + 2h_4 + 1)S(u_n+1, u_n+1, w)$$

$$+ 2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w)$$

$$+ 2b^2h_3S(Tw, Tw, w)$$

$$= 2b^2(h_1 + h_3)S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_n+1)$$

$$+ b^2(2h_2 + 2h_4 + 1)S(u_n+1, u_n+1, w)$$

$$+ 2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w).$$

We consider the following cases.

Case I: If $\max\{S(w, w, Tw), S(u_n, u_n, u_n+1), S(u_n, u_n, Tw)\} = S(w, w, Tw)$, then from equation (2.8), we obtain

$$S(Tw, Tw, w) \leq 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_n+1)$$

$$+ b^2(2h_2 + 2h_4 + 1)S(u_n+1, u_n+1, w)$$

$$+ 2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w)$$

$$+ 2b^2h_3S(Tw, Tw, w)$$

$$\leq 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_n+1)$$

$$+ b^2(2h_2 + 2h_4 + 1)S(u_n+1, u_n+1, w)$$

$$+ 2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w)$$

$$+ 2b^2h_3S(Tw, Tw, w)$$

$$= 2b^2(h_1 + h_3)S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_n+1)$$

$$+ b^2(2h_2 + 2h_4 + 1)S(u_n+1, u_n+1, w)$$

$$+ 2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w).$$

(2.9)
From equation (2.9), we obtain
\[
S(Tw, Tw, w) \leq \frac{1}{1 - 2b^2(h_1 + h_3)} \left[ 2bh_1 S(u_n, u_n, u_{n+1}) + b^2(2h_2 + 2h_4 + 1) \times 
S(u_{n+1}, u_{n+1}, w) + b^2(2h_2 + 2h_4 + 1) S(u_{n+1}, u_{n+1}, w) 
+ 2b^2h_2 S(Tw, Tw, u_n) + 2b^2h_4 S(u_n, u_n, w) \right].
\]

This implies that
\[
\|S(Tw, Tw, w)\| \leq \frac{K}{1 - 2b^2(h_1 + h_3)} \left[ 2bh_1 \|S(u_n, u_n, u_{n+1})\| 
+ b^2(2h_2 + 2h_4 + 1) \|S(u_{n+1}, u_{n+1}, w)\| 
+ b^2(2h_2 + 2h_4 + 1) \|S(u_{n+1}, u_{n+1}, w)\| 
+ 2b^2h_2 \|S(Tw, Tw, u_n)\| + 2b^2h_4 \|S(u_n, u_n, w)\| \right].
\]

Taking the limit as \( n \to \infty \), we get
\[
\|S(Tw, Tw, w)\| \leq \frac{2b^2h_2K}{1 - 2b^2(h_1 + h_3)} \|S(Tw, Tw, w)\|,
\]
or
\[
\left(1 - \frac{2b^2h_2K}{1 - 2b^2(h_1 + h_3)}\right) \|S(Tw, Tw, w)\| \leq 0 \\
\Rightarrow \|S(Tw, Tw, w)\| \leq 0.
\]

Thus, we have \( S(Tw, Tw, w) = 0 \). Hence \( Tw = w \). This shows that \( w \) is a fixed point of \( T \).

Case II: If \( \max\{S(w, w, Tw), S(u_n, u_n, u_{n+1}), S(u_n, u_n, Tw)\} = S(u_n, u_n, u_{n+1}) \), then from equation (2.8), we obtain
\[
S(Tw, Tw, w) \leq 2b^2h_1 S(Tw, Tw, w) + 2bh_1 S(u_n, u_n, u_{n+1}) 
+ b^2(2h_2 + 2h_4 + 1) S(u_{n+1}, u_{n+1}, w) 
+ 2b^2h_2 S(Tw, Tw, u_n) + 2b^2h_4 S(u_n, u_n, w) 
+ 2b^3h_3 S(u_n, u_n, u_{n+1}) 
= 2b^2h_1 S(Tw, Tw, w) + 2b(h_1 + h_3) S(u_n, u_n, u_{n+1}) 
+ b^2(2h_2 + 2h_4 + 1) S(u_{n+1}, u_{n+1}, w) 
+ 2b^2h_2 S(Tw, Tw, u_n) + 2b^2h_4 S(u_n, u_n, w). \tag{2.10}
\]

From equation (2.10), we obtain
\[
S(Tw, Tw, w) \leq \frac{1}{1 - 2b^2h_1} \left[ 2b(h_1 + h_3) S(u_n, u_n, u_{n+1}) + b^2(2h_2 + 2h_4 + 1) \times 
S(u_{n+1}, u_{n+1}, w) + b^2(2h_2 + 2h_4 + 1) S(u_{n+1}, u_{n+1}, w) 
+ 2b^2h_2 S(Tw, Tw, u_n) + 2b^2h_4 S(u_n, u_n, w) \right].
\]

This implies that
\[
\|S(Tw, Tw, w)\| \leq \frac{K}{1 - 2b^2h_1} \left[ 2b(h_1 + h_3) \|S(u_n, u_n, u_{n+1})\| + b^2(2h_2 + 2h_4 + 1) \times 
\|S(u_{n+1}, u_{n+1}, w)\| + b^2(2h_2 + 2h_4 + 1) \|S(u_{n+1}, u_{n+1}, w)\| 
+ 2b^2h_2 \|S(Tw, Tw, u_n)\| + 2b^2h_4 \|S(u_n, u_n, w)\| \right].
\]
Taking the limit as \( n \to \infty \), we get

\[
\|S(Tw, Tw, w)\| \leq \frac{2b^2h_2K}{1 - 2b^2h_1} \|S(Tw, Tw, w)\|
\]

or

\[
\left(1 - \frac{2b^2h_2K}{1 - 2b^2h_1}\right)\|S(Tw, Tw, w)\| \leq 0
\]

\[
\Rightarrow \|S(Tw, Tw, w)\| \leq 0.
\]

Thus, we have \( S(Tw, Tw, w) = 0 \). Hence \( Tw = w \). This shows that \( w \) is a fixed point of \( T \).

Case III: If \( \max\{S(w, w, Tw), S(u_n, u_n, u_{n+1}), S(u_n, u_n, Tw)\} = S(u_n, u_n, Tw) \), then from equation (2.8), we obtain

\[
S(Tw, Tw, w) \leq 2b^2h_1S(Tw, Tw, w) + 2bh_1S(u_n, u_n, u_{n+1}) + b^2(2h_2 + 2h_4 + 1)S(u_{n+1}, u_{n+1}, w) + 2b^2h_2S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w) + 2b^2h_3S(Tw, Tw, u_n)
\]

(2.11)

From equation (2.11), we obtain

\[
S(Tw, Tw, w) \leq \frac{1}{1 - 2b^2h_1} \left[ 2bh_1S(u_n, u_n, u_{n+1}) + b^2(2h_2 + 2h_4 + 1) \times S(u_{n+1}, u_{n+1}, w) + b^2(2h_2 + 2h_4 + 1)S(u_{n+1}, u_{n+1}, w) + 2b^2(h_2 + h_3)S(Tw, Tw, u_n) + 2b^2h_4S(u_n, u_n, w) \right].
\]

This implies that

\[
\|S(Tw, Tw, w)\| \leq \frac{K}{1 - 2b^2h_1} \left[ 2bh_1\|S(u_n, u_n, u_{n+1})\| + b^2(2h_2 + 2h_4 + 1) \times \|S(u_{n+1}, u_{n+1}, w)\| + b^2(2h_2 + 2h_4 + 1)\|S(u_{n+1}, u_{n+1}, w)\| + 2b^2(h_2 + h_3)\|S(Tw, Tw, u_n)\| + 2b^2h_4\|S(u_n, u_n, w)\| \right].
\]

Taking the limit as \( n \to \infty \), we get

\[
\|S(Tw, Tw, w)\| \leq \frac{2b^2(h_2 + h_3)K}{1 - 2b^2h_1} \|S(Tw, Tw, w)\|
\]

or

\[
\left(1 - \frac{2b^2(h_2 + h_3)K}{1 - 2b^2h_1}\right)\|S(Tw, Tw, w)\| \leq 0
\]

\[
\Rightarrow \|S(Tw, Tw, w)\| \leq 0.
\]
Thus, we have \( S(Tw, Tw, w) = 0 \). Hence \( Tw = w \). This shows that \( w \) is a fixed point of \( T \).

From above three cases we see that \( w \) is a fixed point of \( T \).

Now to show that the fixed point of \( T \) is unique.

Let \( w_0 \) be another fixed point of \( T \) in \( X \) such that \( Tw_0 = w_0 \). Then,

\[
S(w, w, w_0) = S(Tw, Tw, Tw_0) \\
\leq h_1[S(w, w, Tw) + S(w_0, w_0, Tw_0)] \\
+ h_2[S(w, w, Tw_0) + S(w_0, w_0, Tw)] \\
+ h_3 \max\{S(w, w, Tw), S(w_0, w_0, Tw_0), S(w_0, w_0, Tw)\} \\
+ h_4[S(w, w, w_0) + S(w, w, Tw_0)] \\
= h_1[S(w, w, w) + S(w_0, w_0, w_0)] \\
+ h_2[S(w, w, w_0) + S(w_0, w_0, w)] \\
+ h_3 \max\{S(w, w, w), S(w_0, w_0, w_0), S(w_0, w_0, w)\} \\
+ h_4[S(w, w, w_0) + S(w, w, w_0)] \\
= (h_2 + 2h_4)S(w, w, w_0) + (h_2 + h_3)S(w_0, w_0, w) \\
\leq (h_2 + 2h_4)S(w, w, w_0) + b(h_2 + h_3)S(w, w, w_0) \\
= [(b + 1)h_2 + bh_3 + 2h_4]S(w, w, w_0),
\]

or

\[
[1 - (b + 1)h_2 - bh_3 - h_4]S(w, w, w_0) \leq 0 \\
\Rightarrow S(w, w, w_0) \leq 0,
\]

since \( 1 - (b + 1)h_2 - bh_3 - h_4 < 1 \). Therefore, we have \( S(w, w, w_0) = 0 \). Hence \( w = w_0 \). This shows that the fixed point of \( T \) is unique. This completes the proof.

If we take \( h_1 = h \) and \( h_2 = h_3 = h_4 = 0 \) in Theorem 2.1, then we have the following result as corollary.

**Corollary 2.2.** ([18], Theorem 2.2) Let \((X, S)\) be a complete cone \( S_p \)-metric space with the coefficient \( b \geq 1 \) and \( P \) be a normal cone with normal constant \( K \). Suppose that the mapping \( T: X \rightarrow X \) satisfies the following condition:

\[
S(Tu, Tu, Tv) \leq h [S(u, u, Tu) + S(v, v, Tv)]
\]

for all \( u, v \in X \), where \( 0 \leq h < \frac{1}{2} \), that is, \( h \in [0, \frac{1}{2}) \) is a constant. Then \( T \) has a unique fixed point \( w \) in \( X \) and we have \( \lim_{n \to \infty} T^n(u) = w \), for each \( u \in X \).

**Remark 2.3.** The contractive condition of Corollary 2.2 is similar with the well known Kannan [11] contraction type condition in the setting of complete cone \( S_0 \)-metric space.

If we take \( h_2 = h \) and \( h_1 = h_3 = h_4 = 0 \) in Theorem 2.1, then we have the following result as corollary.

**Corollary 2.4.** ([18], Theorem 2.3) Let \((X, S)\) be a complete cone \( S_p \)-metric space with the coefficient \( b \geq 1 \) and \( P \) be a normal cone with normal constant \( K \). Suppose that the mapping \( T: X \rightarrow X \) satisfies the following condition:

\[
S(Tu, Tu, Tv) \leq h [S(u, u, Tv) + S(v, v, Tu)]
\]

for all \( u, v \in X \), where \( 0 \leq h < \frac{1}{b(b+2)} < \frac{1}{2} \), that is, \( h \in [0, \frac{1}{b(b+2)}) \) is a constant. Then \( T \) has a unique fixed point \( w \) in \( X \) and we have \( \lim_{n \to \infty} T^n(u) = w \), for each \( u \in X \).

**Remark 2.5.** The contractive condition of Corollary 2.4 is similar with the well known Chatter-jae [4] contraction type condition in the setting of complete cone \( S_0 \)-metric space.
Theorem 2.6. Let \((X, S)\) be a complete cone \(S\)-metric space with the coefficient \(b \geq 1\) and \(P\) be a normal cone with normal constant \(K\) such that for positive integer \(n\), \(T^n\) satisfies the contraction condition (2.1) for all \(u, v \in X\), where \(h_1, h_2, h_3, h_4 > 0\) are constants such that \(2h_1 + b(b + 2)h_2 + h_3 + (b + 1)^2h_4 < 1\). Then \(T\) has a unique fixed point in \(X\).

Proof. From Theorem 2.1, let \(p_0\) be the unique fixed point of \(T^n\). Then
\[
T(T^n p_0) = T p_0 \quad \text{or} \quad T^n (T p_0) = T p_0.
\]
This gives \(T p_0 = p_0\). This shows that \(p_0\) is a unique fixed point of \(T\). This completes the proof. \(\square\)

Theorem 2.7. Let \((X, S)\) be a complete cone \(S\)-metric space with the coefficient \(b \geq 1\) and \(P\) be a normal cone with normal constant \(K\). Suppose that the mapping \(T : X \to X\) satisfies the following condition:
\[
S(T u, T u, T v) \leq \lambda \left[ S(u, u, v) + S(u, u, T u) + S(v, v, T v) \right] + \mu \left[ S(u, u, T v) + S(v, v, T u) \right]
\]
(2.12)
for all \(u, v \in X\), where \(\lambda, \mu > 0\) are constants such that \(3\lambda + b(b + 2)\mu < 1\). Then \(T\) has a unique fixed point \(w\) in \(X\) and we have \(\lim_{n \to \infty} T^n (u) = w\), for each \(u \in X\).

Proof. Let \(u_0 \in X\) and a sequence \(\{u_n\}\) be defined by \(T^n (u_0) = u_n\). Suppose that \(u_n \neq u_{n+1}\) for all \(n\). Using the inequality (2.12) and the condition \((CS_N, M)\) of definition 1.22, we get
\[
S(u_n, u_n, u_{n+1}) = S(T u_{n-1}, T u_{n-1}, T u_n)
\]
\[
\leq \lambda \left[ S(u_{n-1}, u_{n-1}, u_n) + S(u_{n-1}, u_{n-1}, T u_{n-1}) + S(u_n, u_n, T u_n) \right] + \mu \left[ S(u_{n-1}, u_{n-1}, T u_n) + S(u_n, u_n, T u_{n-1}) \right]
\]
\[
= \lambda \left[ S(u_{n-1}, u_{n-1}, u_n) + S(u_{n-1}, u_{n-1}, u_n) + S(u_n, u_n, u_{n+1}) \right] + \mu \left[ S(u_{n-1}, u_{n-1}, u_{n+1}) + S(u_n, u_n, u_{n+1}) \right]
\]
\[
= 2\lambda S(u_{n-1}, u_{n-1}, u_n) + \lambda S(u_n, u_n, u_{n+1}) + \mu S(u_{n-1}, u_{n-1}, u_{n+1})
\]
\[
\leq 2\lambda S(u_{n-1}, u_{n-1}, u_n) + \lambda S(u_n, u_n, u_{n+1}) + 2b\mu S(u_{n-1}, u_{n-1}, u_n)
\]
\[
+ \lambda S(u_n, u_n, u_{n+1}) + (\lambda + b^2\mu) S(u_n, u_n, u_{n+1})
\]
(2.13)
From equation (2.13), we obtain
\[
S(u_n, u_n, u_{n+1}) \leq \left( \frac{2(\lambda + b\mu)}{1 - \lambda - \beta^2\mu} \right) S(u_{n-1}, u_{n-1}, u_n)
\]
(2.14)
\[
= \rho S(u_{n-1}, u_{n-1}, u_n),
\]
where \(\rho = \left( \frac{2(\lambda + b\mu)}{1 - \lambda - \beta^2\mu} \right) < 1\) as \(3\lambda + b(b + 2)\mu < 1\).

Let \(D_n = S(u_n, u_n, u_{n+1})\) and \(D_{n-1} = S(u_{n-1}, u_{n-1}, u_n)\), then from equation (2.14), we get
\[
D_n \leq \rho D_{n-1}.
\]
(2.15)
Using equation (2.15), we obtain
\[
D_n \leq \rho D_{n-1} \leq \rho^2 D_{n-2} \leq \rho^3 D_{n-3} \leq \cdots \leq \rho^n D_0.
\]
(2.16)
Now for \( m, n \geq 1 \) and \( m > n \), we have

\[
S(u_n, u_n, u_m) \leq b[2S(u_n, u_n, u_{n+1}) + S(u_m, u_m, u_{n+1})]
\]
\[
\leq 2bS(u_n, u_n, u_{n+1}) + b^2S(u_m, u_m, u_{n+1})
\]
\[
\leq 2bS(u_n, u_n, u_{n+1}) + 2b^3S(u_{n+1}, u_{n+1}, u_{n+2})
\]
\[
+ b^4S(u_{n+2}, u_{n+2}, u_m)
\]
\[
\leq 2bS(u_n, u_n, u_{n+1}) + 2b^3S(u_{n+1}, u_{n+1}, u_{n+2})
\]
\[
+ 2b^5S(u_{n+2}, u_{n+2}, u_{n+3}) + \ldots
\]
\[
+ 2b^{2(m-n-1)}S(u_{m-1}, u_{m-1}, u_m)
\]
\[
< 2b\left\{S(u_n, u_n, u_{n+1}) + b^2S(u_{n+1}, u_{n+1}, u_{n+2})
\right.
\]
\[
+ b^4S(u_{n+2}, u_{n+2}, u_{n+3}) + \ldots
\]
\[
+ b^{2(m-n-1)}S(u_{m-1}, u_{m-1}, u_m)\right\}
\]
\[
\leq 2b\left\{\rho^n + b^2\rho^{n+1} + b^4\rho^{n+2} + \ldots + b^{2(m-n-1)}\rho^{m-1}\right\}S(u_0, u_0, u_1)
\]
\[
= 2b\rho^n\left\{1 + b^2\rho + (b^2\rho)^2 + \ldots + (b^2\rho)^{m-1}\right\}D_0
\]
\[
\leq \left(\frac{2b\rho^n}{1-b^2\rho}\right)D_0.
\]

This implies that

\[
\|S(u_n, u_n, u_m)\| \leq \left(\frac{2b\rho^n}{1-b^2\rho}\right)\|D_0\|.
\]

Taking limit as \( n, m \to \infty \), we get

\[
\|S(u_n, u_n, u_m)\| \to 0,
\]

since \( 0 < \rho < 1 \). Thus, we have \( S(u_n, u_n, u_m) \to 0 \) as \( n, m \to \infty \).

Therefore, the sequence \( \{u_n\} \) is a Cauchy sequence in \( X \). By the completeness of the space, there exists \( w \in X \) such that \( \lim_{n \to \infty} u_n = w \), i.e., \( \lim_{n \to \infty} T^n u_0 = w \). Rest of the proof follows from Theorem 2.1. This completes the proof. \( \square \)

**Remark 2.8.** If we take \( \lambda = 0 \) and \( \mu = h \) in Theorem 2.7, then we obtain Theorem 2.3 of [18] as corollary.

**Theorem 2.9.** Let \( (X, S) \) be a complete cone \( S_0 \)-metric space with the coefficient \( b \geq 1 \) and \( P \) be a normal cone with normal constant \( K \) such that for positive integer \( n \), \( T^n \) satisfies the contraction condition (2.12) for all \( u, v \in X \), where \( \lambda, \mu > 0 \) are constants such that \( 3\lambda + b(b + 2)\mu < 1 \). Then \( T \) has a unique fixed point in \( X \).

**Proof.** The proof of Theorem 2.9 follows from Theorem 2.6. \( \square \)

**Example 2.10.** Let \( E = \mathbb{R}^2 \), the Euclidean plane, \( P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\} \) a normal cone in \( E \) and \( X = \mathbb{R} \). Then the function \( S : X^3 \to E \) defined by \( S(x, y, z) = (|x-z| + |y-z|)^2 \) for all \( x, y, z \in X \). Then \( (X, S) \) is a cone \( S_0 \)-metric space with coefficient \( b = 4 \). Now, we
consider the mapping $T: X \to X$ by $T(x) = \frac{x+1}{2}$. Then

\[
S(Tx, Tx, Ty) = \left[ |Tx - Ty| + |Tx - Ty| \right]^2 \\
= 4|Tx - Ty|^2 = 4\left( \left( \frac{x + 1}{2} \right) - \left( \frac{y + 1}{2} \right) \right)^2 \\
= |x - y|^2 \\
\leq \frac{1}{3}(|x - 1|^2 + |y - 1|^2) \\
= h \left[ S(x, x, Ty) + S(y, y, Tx) \right]
\]

where $h = \frac{1}{6} < \frac{1}{2}$. Thus $T$ satisfies all the conditions of Corollary 2.2 and clearly $1 \in X$ is the unique fixed point of $T$.

**Example 2.11.** Let $E = \mathbb{R}^2$, the Euclidean plane, $P = \{ (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0 \}$ a normal cone in $E$ and $X = \mathbb{R}$. Then the function $S: X^3 \to E$ defined by $S(x, y, z) = \left( |x - z| + |y - z| \right)^2$ for all $x, y, z \in X$. Then $(X, S)$ is a cone $S_b$-metric space with coefficient $b = 4$. Now, we consider the mapping $T: X \to X$ by $T(x) = \frac{x+2}{3}$. Then

\[
S(Tx, Tx, Ty) = \left[ |Tx - Ty| + |Tx - Ty| \right]^2 \\
= 4|Tx - Ty|^2 = 4\left( \left( \frac{x + 2}{3} \right) - \left( \frac{y + 2}{3} \right) \right)^2 \\
= \frac{4}{9} |x - y|^2.
\]

\[
S(x, x, Ty) = 4|x - Ty|^2 = \frac{4}{9} |3x - y - 2|^2.
\]

\[
S(y, y, Tx) = 4|y - Tx|^2 = \frac{4}{9} |3y - x - 2|^2.
\]

Now, we have

\[
S(Tx, Tx, Ty) \leq \frac{1}{6} \left[ \frac{4}{9} |3x - y - 2|^2 + \frac{4}{9} |3y - x - 2|^2 \right]^2 \\
= \frac{1}{6} \left[ S(x, x, Ty) + S(y, y, Tx) \right] \\
= h \left[ S(x, x, Ty) + S(y, y, Tx) \right]
\]

where $h = \frac{1}{6} < \frac{1}{2}$. Thus $T$ satisfies all the conditions of Corollary 2.4 and clearly $1 \in X$ is the unique fixed point of $T$.

**Open Question:** Can we extend the results for rational contraction/ rational type contraction/ contraction involving rational expression?

### 3 Conclusion

In this paper, we establish some existence of fixed point theorems via contractive type conditions in the framework of cone $S_b$-metric spaces. Our results extend, unify and generalize several results given in the existing literature.
References


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