Palestine Journal of Mathematics

Vol. 10(2)(2021) , 562–579

Fractional advection-dispersion equation described by the Caputo left generalized fractional derivative

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Communicated by Thabet Abdeljawad

MSC 2010 Classifications: Primary 35K57 ; Secondary 76R50, 76R99.

Keywords and phrases: Fractional advection-dispersion equations, Approximates solutions, Caputo left generalized fractional derivative.

Abstract. In this paper, we study the existence and the uniqueness of the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative. We propose the approximates solutions of the fractional advection-dispersion equations represented by the Caputo-Liouville generalized fractional derivative. We use the homotopy perturbation ρ-Laplace transforms method for getting the approximates solutions of the fractional advection-dispersion equations. We solve several examples of the fractional advection-dispersion equations under particular boundary conditions. We finish by analyzing the behaviors generated by the orders α and ρ in the diffusion processes.

1 Introduction

Fractional differential equations exist in many problems in fractional calculus [23, 32, 34, 39]. One of the interests of the fractional differential equations is the problem of existence and the uniqueness of the solutions. The second problem consists of solving the fractional differential equations. The problem of existence and uniqueness of the solutions of the differential equations is an old problem. There exist many techniques to prove the existence and uniqueness concepts. We can use Banach fixed theorem and the Schauer's fixed theorem. They are used to establish the existence and the uniqueness of the solutions of the fractional differential equations. After the existence and uniqueness problem, we have the problem consisting of getting the analytical solutions, the approximates solutions, or the numerical schemes of the differential equations. Many methods exist in the literature. We cite, the powers series solutions, the Fourier sine transformation [38], the separate variables, integral balance methods [17, 18] and others. All methods have advantages and inconveniences. It depends on the fractional differential equations and initial boundaries conditions. Recently, in fractional calculus, the fractional diffusion equation [17, 18, 38] have interested several authors. In this paper, the fractional differential equations under consideration are the fractional advection-dispersion equations [41, 42]. We note several works in the literature related to the fractional advection-dispersion equations described by the fractional-order derivatives.

We have many investigations related to the fractional advection-dispersion equation. In [41], Shen et al. have proposed novel numerical schemes for fractional space advection-dispersion equation. In [42], Singh et al. have proposed a numerical method based on the Galerkin approximation for solving the fractional advection-dispersion equation. In [28], Rubbab et al. have investigated the analytical solution of the fractional advection-diffusion equation described by the nonsingular fractional derivatives. In [26], Meerschaert et al. have proposed numerical schemes for the fractional advection-dispersion equation with variable coefficients on a finite domain described by the Riemann-Liouville fractional derivative. The fractional advection-dispersion equation known as the Fokker-Planck equation is investigated in [10], see also in Santoz [29]. In [10], the authors have proposed the numerical approximation of the Fokker-Planck equation described by the Caputo fractional derivative and the Riesz-Feller fractional derivative. They have proposed a novel method consisting of applying the Fourier transformation and the Laplace transformation. In [40], Schumer et al. have investigated the application of the fractional advection-dispersion equation for modeling transport at the earth surface. In

[45], Zhuang et al. have proposed the numerical schemes of the convection-diffusion equation described by the variable-order fractional derivative. Many other works related to the fractional advection-dispersion equations exist in the literature, see also in [12]. In the above investigations, we particularly note the studies associated with analytical and numerical schemes. In our paper, after proving the existence and the uniqueness of the solution, we propose the approximates solutions of the fractional advection-dispersion equations using the homotopy perturbation ρ-Laplace transforms method [35].

Recently, some new types of fractional derivatives appear in the literature. In comparison to integer-order derivative, the fractional operators play an essential rule in the diffusion processes. The introduction of the fractional derivatives generates many types of diffusion processes in physics: subdiffusion process, super-diffusion process, ballistic diffusion process, hyper-diffusion process, and Richardson process. Note, we obtain all these diffusion processes by specific values of the orders of the used fractional derivative. It proves the importance of the fractional derivatives in diffusion equations. The application of the fractional derivatives in diffusion processes exists in many papers, see Saad et al. in[30], Delgado et al. in [15], Cuahutenango et al. [11]. Ones are without singular kernels: the Atangana-Baleanu derivative introduced by Atangana and Baleanu in [6], and the Caputo Fabrizio fractional derivative introduced by Caputo et al. in [13]. These two fractional derivatives received many successes nowadays due to their applicability to solving many real-world problems: in physics, mechanics, sciences, and engineering, see in [3, 4, 7, 25, 31, 43, 44]. The generalization of the classical fractional derivatives, namely the Caputo fractional derivative [33] and the Riemann-Liouville fractional derivative, was proposed in the literature. Udita [22] introduced them. Thabet and Jarad have developed them in many works [1, 2, 5]. In our works, we use the Caputo generalized fractional derivative.

In [24] 2004, Liao has introduced a novel method for solving, in general, the differential equations. Liao has proposed the homotopy method. Many modified form of the original method exists in the literature, see in[8, 9, 14, 24, 27]. The method has begun with the use of the classical integral or the Riemann-Liouville fractional integral. Nowadays, we have introduced the Laplace transform in the homotopy method. In other words, the modification done on the original process make this method more accessible, comprehensive, and straightforward in applicability. Here in this paper, we bring a new extension of this method by using the Laplace transform of the Caputo generalized fractional derivative (which is a unique and novel Laplace transform). The proposed method called homotopy perturbation ρ-Laplace transforms [35] is applied to solve the fractional advection-diffusion equations described by the Caputo left generalized fractional derivative. In the literature, we have several methods for getting the solutions of the fractional diffusion equations. And many of them use physical concepts as the integral balance methods, see in [18]. In our mind, the existing mathematical approach is the Fourier sine transform. But the technique is not trivial for many classes of the fractional diffusion equations due to the form of the boundary conditions. In this paper, we introduce the homotopy perturbation ρ-Laplace transforms method. It is a mathematical concept to propose the solutions of the fractional advection-dispersion equations. Is it suitable for this class of the fractional diffusion equations? We will answer this question in the next sections.

In Section 2, we recall some necessaries definitions which we will use later. In Section 3, we describe the homotopy perturbation ρ-Laplace transforms method. In Section 4, we study the existence and the uniqueness of the solution of the fractional advection-dispersion equations. In Section 5, we give the approximate solution of the fractional advection-dispersion equations using the proposed method. We present the graphical representations and the interpretations of our results. In section 6, we give our conclusions and general remarks.

2 Background on the generalized fractional derivatives

This section proposes the generalized fractional derivatives necessaries for our investigations. To address these issues, we recall the left generalized fractional derivative and the Caputo-Liouville generalized fractional derivative. We recall the Laplace transform of the left generalized fractional derivative and the Caputo-Liouville generalized fractional derivative. We found all the definitions and lemmas in [19, 22]. We found the generalization of the definitions recalled in this section in the Fahd and Thabet recent works, see in [21].

Let's the function $u : [0,+\infty[\longrightarrow \mathbb{R}$, the Caputo-Liouville derivative of the function u of order α is expressed in the form

$$D_{\alpha}^{c}u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{u'(x(s),s)}{(t-s)^{\alpha}}ds, \tag{2.1}$$

for all $t>0$, where the order $\alpha\in(0,1)$ and $\Gamma(...)$ is the gamma function.

We consider the function $u : [0,+\infty[\longrightarrow \mathbb{R}$, its Riemann-Liouville derivative of order α is expressed by

$$D_{\alpha}^{RL}u(x,t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{0}^{t}\frac{u(x(s),s)}{(t-s)^{\alpha}}ds, \tag{2.2}$$

for all $t>0$, where the order $\alpha\in(0,1)$ and $\Gamma(...)$ is the gamma function.

We consider the function $u : [0,+\infty[\longrightarrow \mathbb{R}$, its Caputo-Liouville generalized derivative of order α is expressed in the form

$$D_{c}^{\alpha,\rho}u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{-\alpha}u'(x(s),s)ds, \tag{2.3}$$

for all $t>0$, where the order $\alpha\in(0,1)$ and $\Gamma(...)$ is the gamma function.

We consider the function $u : [0,+\infty[\longrightarrow \mathbb{R}$, its left generalized derivative of order α is expressed in the form

$$D^{\alpha,\rho}u(x,t)=\frac{1}{\Gamma(1-\alpha)}\left(\frac{d}{dt}\right)\int_{0}^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{-\alpha}u(x(s),s)\frac{ds}{s^{1-\rho}}, \tag{2.4}$$

for all $t>0$, where the order $\alpha\in(0,1)$ and $\Gamma(...)$ is the gamma function.

We define the Laplace transform of the Caputo-Liouville generalized fractional derivative. The ρ-Laplace transform was introduced in the literature by Fahd and Thabet in their works [19]. The ρ-Laplace transform of the Caputo left generalized fractional derivative is defined in the form

$$\mathcal{L}_{\rho}\left\{\left(D_{c}^{\alpha,\rho}u\right)(t)\right\}=s^{\alpha}\mathcal{L}_{\rho}\left\{u(x,t)\right\}-s^{\alpha-1}u(x,0). \tag{2.5}$$

The ρ-Laplace transform of the function u is given in the form

$$\mathcal{L}_{\rho}\left\{u(x,t)\right\}(s)=\int_{0}^{\infty}e^{-s\frac{t^{\rho}}{\rho}}u(x,t)\frac{dt}{t^{1-\rho}}. \tag{2.6}$$

The Mittag-Leffler function with two parameters is defined by the following series

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}, \tag{2.7}$$

where $\alpha>0$, $\beta\in\mathbb{R}$ and $z\in\mathbb{C}$. The classical exponential function is obtained with $\alpha=\beta=1$.

Note, the generalization of the Laplace transformation can be found in recent works [21]. This generalization open new doors in fractional calculus.

3 Homotopy perturbation ρ-Laplace transforms method

This section concerns the background of the homotopy perturbation method. We combine the technique with the use of the ρ-Laplace transform. The method is called homotopy perturbation ρ-Laplace transforms method [35]. We represent the fractional advection-dispersion equation under consideration as the following form

$${}_{t}D_{c}^{\alpha,\rho}u=\mu\frac{\partial^{2}u}{\partial x^{2}}-\nu\frac{\partial u}{\partial x}, \tag{3.1}$$

with initial boundary condition given by

$$\frac{\partial^{m}u(x,0)}{\partial t^{m}}=u_{m}(x). \tag{3.2}$$

According to Delgado et al in [16], we apply the ρ-Laplace transform to both sides of Eq. (3.1), it yields

$$\bar{u}(x,s) = \frac{1}{s^{\alpha}}\left[\mu\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}\right]\bar{u}(x,s) + \frac{1}{s^n}\left[s^{n-1}u_0(x) + ... + u_{n-1}(x)\right], \tag{3.3}$$

where $\bar{u}(x,s)$ denotes the Laplace transform of the function $u(x,t)$. We now use the homotopy perturbation method. We define the following homotopy for Eq. (3.3), it yields

$$\bar{u}(x,s) = \frac{p}{s^{\alpha}}\left[\mu\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}\right]\bar{u}(x,s) + \frac{1}{s^n}\left[s^{n-1}u_0(x) + ... + u_{n-1}(x)\right], \tag{3.4}$$

where $p \in [0,1]$ represents the homotopy parameter. The approximate solution of Eq. (3.4) according to the homotopy method, is given by

$$\bar{u}(x,s) = \sum_{i=0}^{\infty} p^i \bar{u}_i(x,s). \tag{3.5}$$

We replace Eq. (3.5) into Eq. (3.4), we obtain the following relationship

$$\sum_{i=0}^{\infty} p^i \bar{u}_i(x,s) = \frac{p}{s^{\alpha}}\left[\mu\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}\right]\sum_{i=0}^{\infty} p^i \bar{u}_i(x,s) + \frac{1}{s^n}\left[s^{n-1}u_0(x) + ... + u_{n-1}(x)\right]. \tag{3.6}$$

Finaly, under Eq. (3.6), we have to solve the following fractional iterative advection-dispersion equations described by

$$\begin{aligned}
p^0 &: \quad \bar{u}_0(x,s) = \frac{1}{s^n}\left[s^{n-1}u_0(x) + ... + u_{n-1}(x)\right];\\
p^1 &: \quad \bar{u}_1(x,s) = \frac{1}{s^{\alpha}}\left[\mu\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}\right]\bar{u}_0(x,s);\\
p^2 &: \quad \bar{u}_2(x,s) = \frac{1}{s^{\alpha}}\left[\mu\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}\right]\bar{u}_1(x,s);\\
p^3 &: \quad \bar{u}_3(x,s) = \frac{1}{s^{\alpha}}\left[\mu\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}\right]\bar{u}_2(x,s);\\
\vdots &: \quad \vdots\\
p^{n+1} &: \quad \bar{u}_{n+1}(x,s) = \frac{1}{s^{\alpha}}\left[\mu\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x}\right]\bar{u}_n(x,s).
\end{aligned} \tag{3.7}$$

The solution of the Eq. (3.7), when p converge to 1 is given by

$$H_n(x,s) = \sum_{i=0}^{\infty} \bar{u}_i(x,s). \tag{3.8}$$

We obtain the approximate solution of the fractional advection-dispersion equation by applying the inverse of ρ-Laplace transform to both sides of Eq. (3.8)

$$u(x,t) = \mathcal{L}_{\rho}^{-1}\left(H_n(x,s)\right). \tag{3.9}$$

4 Existence and uniqueness of the solution

In this section, we discuss the existence, and the uniqueness of the solution of the fractional advection-dispersion equation described by the Caputo left generalized fractional derivative. Let's the function defined by

$$\psi(x,u) = \mu\frac{\partial^2 u}{\partial x^2} - \nu\frac{\partial u}{\partial x}. \tag{4.1}$$

Let's $\mathcal{C}^n\left([a,b],\mathbb{C}\right)$ be the Banach space of all continuous and differentiable functions from $[a,b]$ to $\mathbb{C}$, we set

$$\mathcal{C}^n_\gamma\left[a,b\right]=\left\{\psi:[a,b]\to\mathbb{C}\text{ s.t }\gamma^n\psi\in\mathcal{C}\left[a,b\right]\right\}, \tag{4.2}$$

and

$$\mathcal{C}_{\epsilon,\rho}\left[a,b\right]=\left\{\psi:\left(\frac{t^\rho-a^\rho}{\rho}\right)^\epsilon\psi\in\mathcal{C}\left[a,b\right]\right\}, \tag{4.3}$$

and we use the following norm

$$\|\psi\|_{\mathcal{C}_{\epsilon,\rho}}=\max_{t\in[a,b]}\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^\epsilon\psi(t)\right|. \tag{4.4}$$

We use the Banach fixed point theorem. It is classically used to prove the existence and the uniqueness of the differential equations.

Firstly, we prove the function ψ is Lipschitz continuous with a Lipchitz constant. We use the norm defined above; we have the following relationships

$$\begin{aligned}\left\|\psi\left(x,u\right)-\psi\left(x,v\right)\right\|_{\mathcal{C}_{\epsilon,\rho}} &= \left\|\mu\frac{\partial^2u}{\partial x^2}-\nu\frac{\partial u}{\partial x}-\mu\frac{\partial^2v}{\partial x^2}+\nu\frac{\partial v}{\partial x}\right\|_{\mathcal{C}_{\epsilon,\rho}},\\ &= \left\|\mu\frac{\partial^2u}{\partial x^2}-\mu\frac{\partial^2v}{\partial x^2}-\nu\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial x}\right\|_{\mathcal{C}_{\epsilon,\rho}}.\end{aligned} \tag{4.5}$$

Using the triangular inequality we have the following relationship

$$\left\|\psi\left(x,u\right)-\psi\left(x,v\right)\right\|_{\mathcal{C}_{\epsilon,\rho}} \leq \left\|\mu\frac{\partial^2u}{\partial x^2}-\mu\frac{\partial^2v}{\partial x^2}\right\|_{\mathcal{C}_{\epsilon,\rho}}+\left\|\nu\frac{\partial u}{\partial x}-\nu\frac{\partial v}{\partial x}\right\|_{\mathcal{C}_{\epsilon,\rho}}. \tag{4.6}$$

Using the assumptions the functions $u\leq b_1$ and $v\leq b_2$ (where b_1 and b_2 are positive constant) are bounded, we have the following relationships

$$\begin{aligned}\left\|\psi\left(x,u\right)-\psi\left(x,v\right)\right\|_{\mathcal{C}_{\epsilon,\rho}} &\leq \left\|\mu\frac{\partial^2u}{\partial x^2}-\mu\frac{\partial^2v}{\partial x^2}\right\|_{\mathcal{C}_{\epsilon,\rho}}+\left\|\nu\frac{\partial u}{\partial x}-\nu\frac{\partial v}{\partial x}\right\|_{\mathcal{C}_{\epsilon,\rho}},\\ \left\|\psi\left(x,u\right)-\psi\left(x,v\right)\right\|_{\mathcal{C}_{\epsilon,\rho}} &\leq \mu\left\|\frac{\partial^2u}{\partial x^2}-\frac{\partial^2v}{\partial x^2}\right\|_{\mathcal{C}_{\epsilon,\rho}}+\nu\left\|\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}\right\|_{\mathcal{C}_{\epsilon,\rho}},\\ \left\|\psi\left(x,u\right)-\psi\left(x,v\right)\right\|_{\mathcal{C}_{\epsilon,\rho}} &\leq \mu b_1\left\|u-v\right\|_{\mathcal{C}_{\epsilon,\rho}}+\nu b_2\left\|u-v\right\|_{\mathcal{C}_{\epsilon,\rho}},\\ \left\|\psi\left(x,u\right)-\psi\left(x,v\right)\right\|_{\mathcal{C}_{\epsilon,\rho}} &\leq \left(\mu b_1+\nu b_2\right)\left\|u-v\right\|_{\mathcal{C}_{\epsilon,\rho}}.\end{aligned} \tag{4.7}$$

Let's $k=(\mu b_1+\nu b_2)$, finally, the function ψ is Lipschitz continuous with a Lipchitz constant k, that is

$$\left\|\psi\left(x,u\right)-\psi\left(x,v\right)\right\|_{\mathcal{C}_{\epsilon,\rho}}\leq k\left\|u-v\right\|_{\mathcal{C}_{\epsilon,\rho}}. \tag{4.8}$$

Applying the generalized fractional integral to both sides of Eq. (3.1), thus, the fractional advection-dispersion equation satisfies the following relation

$$u(x,t)-u(x,0)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\psi\left(x(s),u\right)\frac{ds}{s^{1-\rho}}. \tag{4.9}$$

Let's the operator $\Gamma u:H\to H$ defined by the following expression

$$\Gamma u(x,t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\psi\left(x(s),u\right)\frac{ds}{s^{1-\rho}}. \tag{4.10}$$

We will prove the function Γ is well defined. Using the norm defined in this section, we have the following relationships

$$
\begin{aligned}
\|\Gamma u(x,t)-u(x,0)\|_{\mathcal{C}_{\epsilon,\rho}} &= \left\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\psi\left(x(s),u\right)\frac{ds}{s^{1-\rho}}\right\|_{\mathcal{C}_{\epsilon,\rho}},\\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\left\|\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\psi\left(x(s),u\right)\right\|_{\mathcal{C}_{\epsilon,\rho}}\frac{ds}{s^{1-\rho}},\\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left\|\psi\left(x(s),u\right)\right\|_{\mathcal{C}_{\epsilon,\rho}}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\frac{ds}{s^{1-\rho}}. \qquad (4.11)
\end{aligned}
$$

From the fact $\|\psi\left(x,u\right)\|\leq M$ and $t\leq T$, Eq. (4.11) can be expressed in the following form

$$
\|\Gamma u(x,t)-u(x,0)\|_{\mathcal{C}_{\epsilon,\rho}} \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^{\alpha}M. \qquad (4.12)
$$

Thus, its prove the function Γ is well defined. Let's provided a condition under which the operator Γ is an contraction. We have the following relationships

$$
\begin{aligned}
\|\Gamma u(x,t)-\Gamma v(x,t)\|_{\mathcal{C}_{\epsilon,\rho}} &= \left\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\left(\psi\left(x(s),u\right)-\psi\left(x(s),v\right)\right)\frac{ds}{s^{1-\rho}}\right\|_{\mathcal{C}_{\epsilon,\rho}},\\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\left\|\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\psi\left(x(s),u\right)-\psi\left(x(s),v\right)\right\|_{\mathcal{C}_{\epsilon,\rho}}\frac{ds}{s^{1-\rho}},\\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left\|\psi\left(x(s),u\right)-\psi\left(x(s),v\right)\right\|_{\mathcal{C}_{\epsilon,\rho}}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\frac{ds}{s^{1-\rho}}.
\end{aligned}
$$

Under the fact the function ψ is lipschitz continuous with a Lipschitz constant k, we have the following

$$
\|\Gamma u(x,t)-\Gamma v(x,t)\|_{\mathcal{C}_{\epsilon,\rho}} \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha+1)}\left(\frac{T^\rho}{\rho}\right)^{\alpha}k\,\|u-v\|_{\mathcal{C}_{\epsilon,\rho}}. \qquad (4.13)
$$

Thus, the operator Γ is an contraction under the condition

$$
\frac{\rho^{1-\alpha}}{\Gamma(\alpha+1)}\left(\frac{T^\rho}{\rho}\right)^{\alpha}k<1. \qquad (4.14)
$$

In conclusion, using the Banach fixed Theorem, the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative has a unique solution. The solution of the fractional advection-dispersion equation represented by the Caputo left generalized fractional derivative exists and is unique. Thus, we can approximate its analytical solution. In the next section, we will investigate getting the approximate solution using the homotopy perturbation ρ-Laplace transform method.

5 Approximate solution with homototy perturbation method

In this section, we investigate the approximate solution of the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative under three different boundary conditions. The fractional differential equation under consideration is given by

$$
{}_tD_c^{\alpha,\rho}u=\mu\frac{\partial^2 u}{\partial x^2}-\nu\frac{\partial u}{\partial x}. \qquad (5.1)
$$

5.1 Boundary condition on exponential form

In this section, we highlight the form of the solution of the Eq. (5.1) under exponential boundary condition. The fractional advection-dispersion equation is described by

$$
{}_tD_c^{\alpha,\rho}u=\mu\frac{\partial^2 u}{\partial x^2}-\nu\frac{\partial u}{\partial x}, \qquad (5.2)
$$

under Dirichlet boundary condition defined by

$$u(x,0) = e^x - 1. \tag{5.3}$$

The use of the Fourier sine transform is not possible due to the form of the boundary condition. Note that the Fourier sine transform offers a possible analytical solution. Due to this fact, its application is not possible. In this section, we propose the homotopy perturbation ρ-Laplace transform method. The mains objective of this section is to analyze the effect of the order ρ in the diffusion processes. Before examining the impact, let's apply the homotopy perturbation ρ-Laplace transform method of getting the approximate solution of the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative. We begin the resolution according to the homotopy perturbation ρ-Laplace transform method. In the first iteration, we have to solve the fractional differential equation given by

$$D^{\alpha,\rho} u_0(x,t) = 0. \tag{5.4}$$

We consider the initial condition expressed as $u_0(x,0) = e^x - 1$. Applying the ρ-Laplace transform on Eq. (5.4), we have the following relationships

$$\begin{aligned} s^{\alpha} u_0(x,s) - s^{\alpha-1} u_0(x,0) &= 0, \\ u_0(x,s) &= \frac{e^x - 1}{s}. \end{aligned} \tag{5.5}$$

Applying the inverse of ρ-Laplace transform on Eq. (5.5), we recover the solution of the Eq. (5.4) expressed as the following form

$$u_0(x,t) = e^x - 1. \tag{5.6}$$

In the second iteration, we have to solve the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative given by (using the previous solution)

$$\begin{aligned} D^{\alpha,\rho} u_1(x,t) &= \mu \frac{\partial^2 u_0(x,t)}{\partial x^2} - \nu \frac{\partial u_0(x,t)}{\partial x}, \\ &= e^x \left(\mu - \nu\right). \end{aligned} \tag{5.7}$$

In this step, the initial condition is defined by $u_1(x,0) = 0$. Applying the ρ-Laplace transform on Eq. (5.7), we have the following relationships

$$\begin{aligned} s^{\alpha} u_1(x,s) - s^{\alpha-1} u_1(x,0) &= \frac{e^x \left(\mu - \nu\right)}{s}, \\ u_1(x,s) &= \frac{e^x \left(\mu - \nu\right)}{s^{1+\alpha}}. \end{aligned} \tag{5.8}$$

Applying the inverse of ρ-Laplace transform on Eq. (5.8), we recover the solution of Eq. (5.7) expressed as the following form

$$u_1(x,t) = \frac{e^x \left(\mu - \nu\right)}{\Gamma\left(1+\alpha\right)} \left(\frac{t^{\rho}}{\rho}\right)^{\alpha}. \tag{5.9}$$

In the third iteration, we have to solve the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative given by (using the previous solution)

$$\begin{aligned} D^{\alpha,\rho} u_2(x,t) &= \mu \frac{\partial^2 u_1(x,t)}{\partial x^2} - \nu \frac{\partial u_1(x,t)}{\partial x}, \\ &= \frac{e^x \left(\mu - \nu\right)^2}{\Gamma\left(1+\alpha\right)} \left(\frac{t^{\rho}}{\rho}\right)^{\alpha}. \end{aligned} \tag{5.10}$$

In this step, the initial condition is given by $u_2(x,0)=0$. Applying the ρ-Laplace transform on Eq. (5.10), we have the following relationships

$$\begin{aligned} s^{\alpha}u_1(x,s)-s^{\alpha-1}u_1(x,0) &= \frac{e^x(\mu-\nu)^2}{s^{1+\alpha}}, \\ u_1(x,s) &= \frac{e^x(\mu-\nu)^2}{s^{1+2\alpha}}. \end{aligned} \tag{5.11}$$

Applying the inverse of ρ-Laplace transform on Eq. (5.11), we recover the solution of Eq. (5.10) expressed as the following form

$$u_2(x,t)=\frac{e^x(\mu-\nu)^2}{\Gamma(1+2\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{2\alpha}. \tag{5.12}$$

In the fourth iteration, we have to solve the fractional advection-dispersion equation described by the Caputo generalized fractional derivative given by (using the previous solution)

$$\begin{aligned} D^{\alpha,\rho}u_3(x,t) &= \mu\frac{\partial^2 u_2(x,t)}{\partial x^2}-\nu\frac{\partial u_2(x,t)}{\partial x}, \\ &= \frac{e^x(\mu-\nu)^3}{\Gamma(1+2\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}. \end{aligned} \tag{5.13}$$

In this step, the initial condition is given by $u_3(x,0)=0$. Applying the ρ-Laplace transform on Eq. (5.13), we have the following relationships

$$\begin{aligned} s^{\alpha}u_1(x,s)-s^{\alpha-1}u_1(x,0) &= \frac{e^x(\mu-\nu)^3}{s^{1+2\alpha}}, \\ u_1(x,s) &= \frac{e^x(\mu-\nu)^3}{s^{1+3\alpha}}. \end{aligned} \tag{5.14}$$

Applying the inverse of ρ-Laplace transform on Eq. (5.14), we recover the solution of Eq. (5.13) expressed as the following form

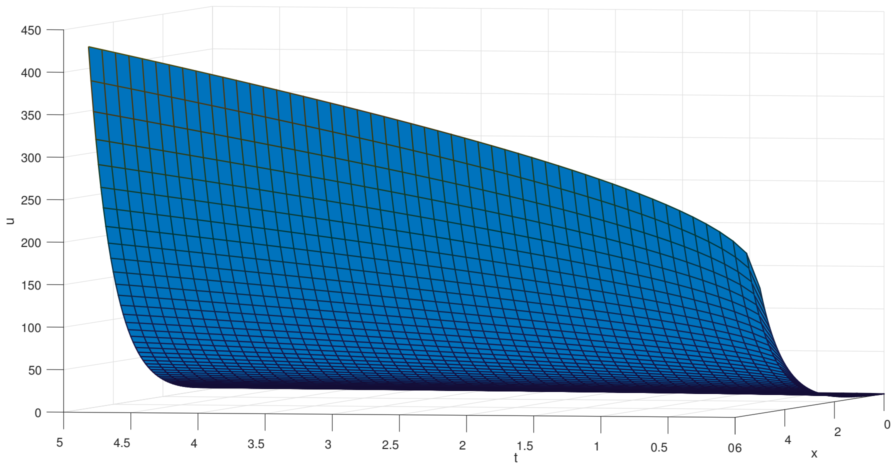
$$u_2(x,t)=\frac{e^x(\mu-\nu)^3}{\Gamma(1+3\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{3\alpha}. \tag{5.15}$$

The approximate solution of the fractional advection dispersion equation (5.2) described by the Caputo-Liouville generalized fractional derivative under boundary condition (5.3), is obtained respecting the following procedure

$$\begin{aligned} u(x,t) &= \lim_{p\to 1}\sum_{i=0}^{n}p^i u_i(x,t); \\ u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+...; \\ u(x,t) &= e^x-1+\frac{e^x(\mu-\nu)}{\Gamma(1+\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}+\frac{e^x(\mu-\nu)^2}{\Gamma(1+2\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{2\alpha}+...; \\ u(x,t) &= -1+e^x\left[1+\frac{(\mu-\nu)}{\Gamma(1+\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}+\frac{(\mu-\nu)^2}{\Gamma(1+2\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{2\alpha}+...\right]; \\ u(x,t) &= -1+e^x E_{\alpha}\left((\mu-\nu)\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right). \end{aligned} \tag{5.16}$$

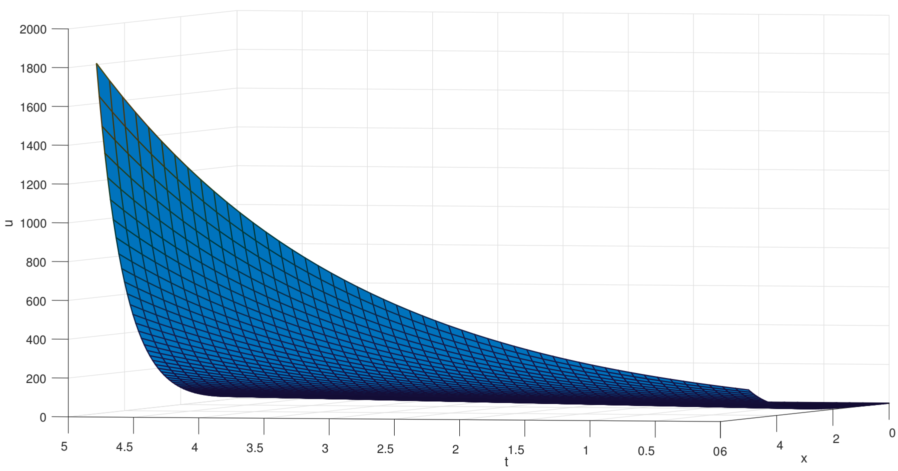
We depict in Figure 1a, the approximate solution of the fractional advection dispersion equation described by the Caputo-Liouville generalized fractional derivative operator for arbitrary values of the orders $\alpha=0.5$ and$\rho=0.75$, with $\mu=1$, and $\nu=0.5$.

We depict in Figure 2a, the approximate solution of the fractional advection diffusion equation described by the Caputo left generalized fractional derivative operator for arbitrary values of the orders $\alpha=1$ and $\rho=1$, with $\mu=1$, and $\nu=0.5$.



(a)

Figure 1: Approximate solutions of the fractional advection-dispersion equation, $\alpha = 0.5$, $\rho = 0.75$.

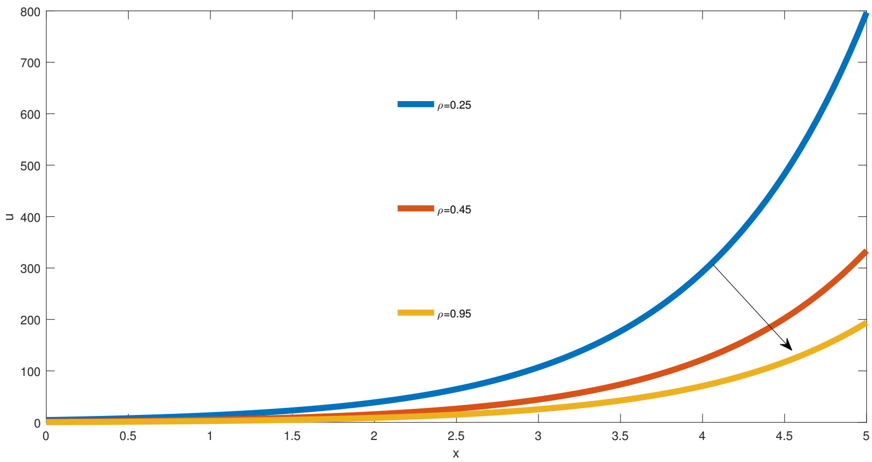


(a)

Figure 2: Approximate solutions of the fractional advection-dispersion equation, $\alpha = 1$, $\rho = 1$.

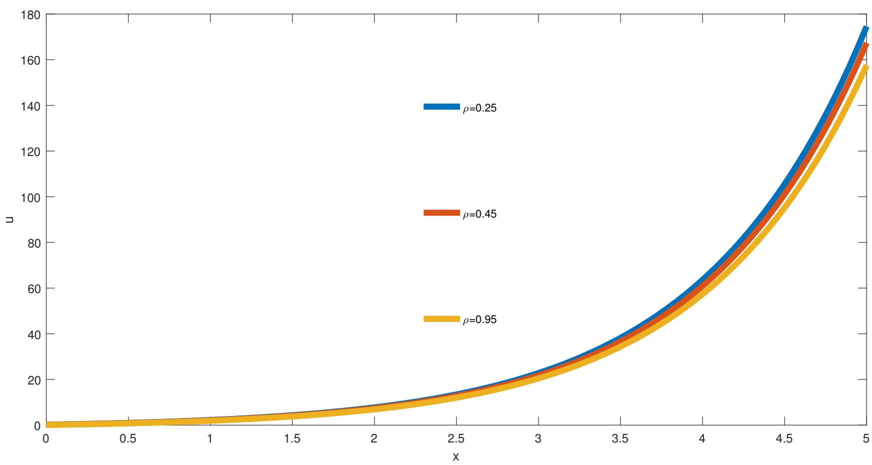
We depict in Figure 3a, the approximate solution of the fractional advection dispersion equation described by the Caputo-Liouville generalized fractional derivative operator when we fix $\alpha = 1$ and $\rho < 1$ take different values. In Figure 3a, we fix $t = 0.7.$, and with $\mu = 1$, and $\nu = 0.5$.

We depict in Figure 4a, the approximate solution of the fractional advection dispersion equation described by the Caputo-Liouville generalized fractional derivative operator when we fix $\alpha = 1$ and $\rho > 1$ take different values. In Figure 4a, we fix $t = 0.7.$ with $\mu = 1$, and $\nu = 0.5$. Following the direction of the arrows, when we fix $\alpha = 1$, we note an acceleration effect with convergence to zero following the increase of the orders $\rho < 1$. We note the same effect when we fix the orders $\alpha = 1$ and $\rho > 1$. In general, for certain values of the orders of the Caputo-Liouville generalized fractional derivative, we note the same diffusion processes as the fractional diffusion equation described by the Caputo-Liouville generalized fractional derivative presented by Sene et al. in [35].



(a)

Figure 3: Approximate solutions of the fractional advection-dispersion equation, $\alpha = 1$, $\rho < 1$.



(a)

Figure 4: Approximate solutions of the fractional advection-dispersion equation, $\alpha = 1$, $\rho < 1$.

5.2 Boundary condition on sinusoidal form

In this section, we propose the approximate solution of the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative under boundary condition in the form $u(x, 0) = sinx$. We represent the fractional differential equation under consideration in the following form

$$ {}_tD_c^{\alpha,\rho} u = \mu \frac{\partial^2 u}{\partial x^2} - \nu \frac{\partial u}{\partial x}. \tag{5.17}$$

The Fourier sine transform cannot be applied due to the form of the boundary condition. We use the homotopy perturbation ρ-Laplace transforms method. The first iteration, we consider the fractional differential equation described by

$$ D^{\alpha,\rho} u_0(x,t) = 0. \tag{5.18}$$

Let's the initial condition expressed as $u_0(x, 0) = sinx$. Applying the ρ-Laplace transform on

Eq. (48), we have the following relationships

$$
\begin{aligned}
s^{\alpha}u_0(x,s) - s^{\alpha-1}u_0(x,0) &= 0, \\
u_0(x,s) &= \frac{sinx}{s}.
\end{aligned} \tag{5.19}
$$

Applying the inverse of ρ-Laplace transform on Eq. (5.19), we recover the solution of Eq. (5.18) expressed as the following form

$$
u_0(x,t) = sinx. \tag{5.20}
$$

In the second iteration, we use the previous solution. We consider the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative given by

$$
\begin{aligned}
D^{\alpha,\rho}u_1(x,t) &= \mu\frac{\partial^2 u_0(x,t)}{\partial x^2} - \nu\frac{\partial u_0(x,t)}{\partial x}, \\
&= -\mu\sin x - \nu\cos x.
\end{aligned} \tag{5.21}
$$

In this step, the initial condition is given by $u_1(x,0) = 0$. Applying the ρ-Laplace transform on Eq. (5.21), we have the following relationships

$$
\begin{aligned}
s^{\alpha}u_1(x,s) - s^{\alpha-1}u_1(x,0) &= -\frac{\mu\sin x}{s} - \frac{\nu\cos x}{s}, \\
u_1(x,s) &= -\frac{\mu\sin x}{s^{1+\alpha}} - \frac{\nu\cos x}{s^{1+\alpha}}.
\end{aligned} \tag{5.22}
$$

Applying the inverse of ρ-Laplace transform on Eq. (5.22), we recover the solution of the advection-dispersion equation (5.21) expressed as the following form

$$
u_1(x,t) = -\frac{\mu\sin x}{\Gamma(1+\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha} - \frac{\nu\cos x}{\Gamma(1+\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}. \tag{5.23}
$$

In the fourth iteration, we use the previous solution. Let's the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative given by

$$
\begin{aligned}
D^{\alpha,\rho}u_2(x,t) &= \mu\frac{\partial^2 u_1(x,t)}{\partial x^2} - \nu\frac{\partial u_1(x,t)}{\partial x}, \\
&= \frac{(\mu^2-\nu^2)\sin x}{\Gamma(1+\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha} + \frac{\mu\nu\cos x}{\Gamma(1+\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}.
\end{aligned} \tag{5.24}
$$

In this step, the initial condition is given by $u_2(x,0) = 0$. Applying the ρ-Laplace transform on Eq. (5.24), we have the following relationships

$$
\begin{aligned}
s^{\alpha}u_1(x,s) - s^{\alpha-1}u_1(x,0) &= \frac{(\mu^2-\nu^2)\sin x}{s^{1+\alpha}} + \frac{2\mu\nu\cos x}{s^{1+\alpha}}, \\
u_1(x,s) &= \frac{(\mu^2-\nu^2)\sin x}{s^{1+2\alpha}} + \frac{2\mu\nu\cos x}{s^{1+2\alpha}}.
\end{aligned} \tag{5.25}
$$

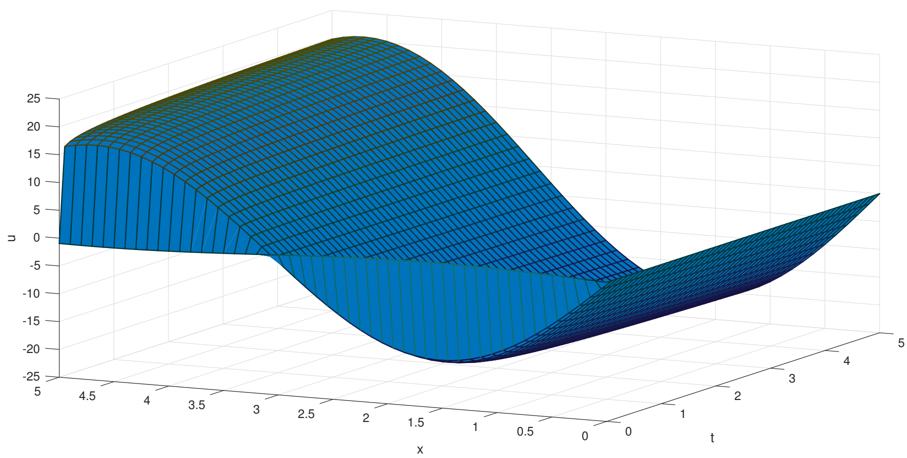
Applying the inverse of ρ-Laplace transform on Eq. (5.25), we recover the solution of the Eq. (5.24) expressed as the following form

$$
u_2(x,t) = \frac{(\mu^2-\nu^2)\sin x}{\Gamma(1+2\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{2\alpha} + \frac{2\mu\nu\cos x}{\Gamma(1+2\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{2\alpha}. \tag{5.26}
$$

For the other iterations, we repeat the same procedure as in the previous iterations. The approximate solution of the fractional advection-dispersion equation (5.17) respecting the following procedure

$$
\begin{aligned}
u(x,t) &= \lim_{p\to 1}\sum_{i=0}^{n} p^i u_i(x,t);\\
u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+...;\\
u(x,t) &= sinx-\frac{\mu\sin x}{\Gamma(1+\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\alpha}-\frac{\nu\cos x}{\Gamma(1+\alpha)}\left(\frac{t^\rho}{\rho}\right)^{\alpha}+\frac{(\mu^2-\nu^2)\sin x}{\Gamma(1+2\alpha)}\left(\frac{t^\rho}{\rho}\right)^{2\alpha};\\
&\quad +\frac{2\mu\nu\cos x}{\Gamma(1+2\alpha)}\left(\frac{t^\rho}{\rho}\right)^{2\alpha}+... \qquad (5.27)
\end{aligned}
$$

In the graphical representations, we consider the solution after two iterations. We depict in Figure 5a, the approximate solution of the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative operator for arbitrary values of the orders $\alpha = 1$ and $\rho = 0.75$, with $\mu = \nu = 1$.



(a)

Figure 5: Approximate solutions of the fractional avection-dispersion equation, $\alpha = 1$, $\rho = 0.75$.

We depict in Figure 6a, the approximate solution of the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative operator when we fix $\alpha = 1$ and $\rho < 1$, with $\mu = \nu = 1$. In Figure 6a, we fix $t = 0.7$.

We depict in Figure 6a, the approximate solution of the fractional advection-diffusion equation described by the Caputo-Liouville generalized fractional derivative operator when we fix $\alpha = 1$ and we take the different values for the order $\rho > 1$, with $\mu = \nu = 1$. In Figure 6a, we fix $t = 0.7$.

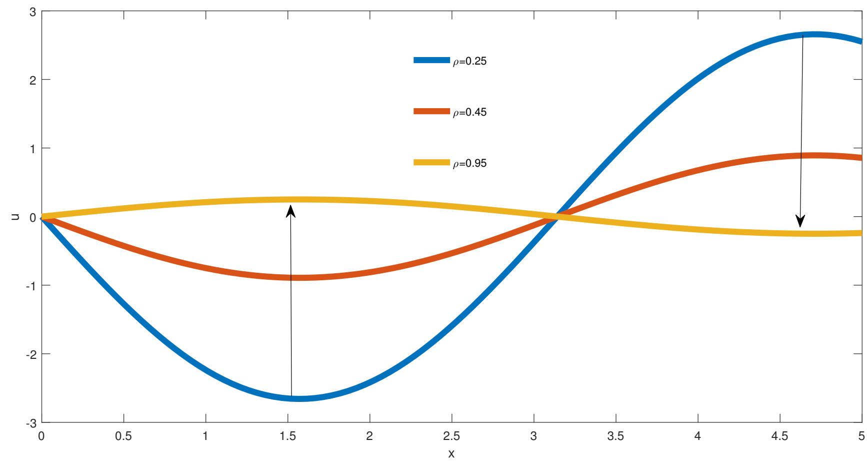
The effect of the orders $\alpha = 1$ and $\rho < 1$ in the behaviors of the approximate solutions are indicated by the directions of the arrows. The impact of the orders $\alpha = 1$ and $\rho > 1$ in the behavior of the approximate solution are indicated by the directions of the arrows.

5.3 Boundary condition on polynomial form

In this section, we propose the approximate solution of the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative under boundary condition defined by $u(x,0) = x$. The fractional differential equation under consideration is in the following form

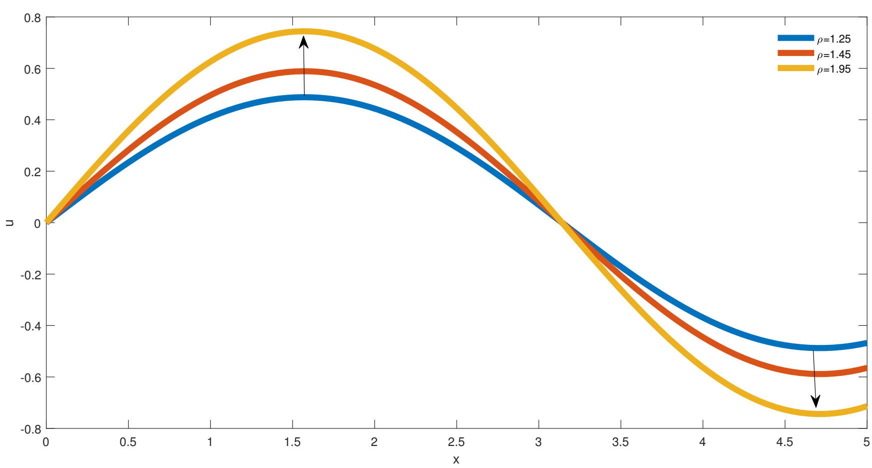
$$
{}_tD_c^{\alpha,\rho}u = \mu\frac{\partial^2 u}{\partial x^2}-\nu\frac{\partial u}{\partial x}. \qquad (5.28)
$$

The Fourier sine transform can be applied but not trivial due to the form of the boundary condition. We use the homotopy perturbation ρ-Laplace transforms method. The first iteration,



(a)

Figure 6: Approximate solutions of the fractional avection-dispersion equation, $\alpha = 1$ and $\rho < 1$.



(a)

Figure 7: Approximate solutions of the fractional avection-dispersion equation, $\alpha = 1$ and $\rho > 1$.

we consider the fractional differential equation described by

$$D^{\alpha,\rho} u_0(x,t) = 0. \tag{5.29}$$

Let's the initial condition expressed as $u_0(x,0) = x$. Applying the ρ-Laplace transform on Eq. (5.29), we have the following relationships

$$\begin{aligned} s^\alpha u_0(x,s) - s^{\alpha-1} u_0(x,0) &= 0, \\ u_0(x,s) &= \frac{x}{s}. \end{aligned} \tag{5.30}$$

Applying the inverse of ρ-Laplace transform on Eq. (5.30), we recover the solution of the equation (5.29) expressed as the following form

$$u_0(x,t) = x. \tag{5.31}$$

In the second iteration, we use the previous solution. Let's the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative given by

$$\begin{aligned} D^{\alpha,\rho}u_1(x,t) &= \mu\frac{\partial^2 u_0(x,t)}{\partial x^2} - \nu\frac{\partial u_0(x,t)}{\partial x}, \\ &= -\nu. \end{aligned} \tag{5.32}$$

In this step, the initial condition is given by $u_1(x,0) = 0$. Applying the ρ-Laplace transform on Eq. (5.32), we have the following relationships

$$\begin{aligned} s^{\alpha}u_1(x,s) - s^{\alpha-1}u_1(x,0) &= -\frac{\nu}{s}, \\ u_1(x,s) &= -\frac{\nu}{s^{1+\alpha}}. \end{aligned} \tag{5.33}$$

Applying the inverse of ρ-Laplace transform on Eq. (5.33), we recover the solution of the advection-dispersion equation (5.32) expressed as the following form

$$u_1(x,t) = -\frac{\nu}{\Gamma(1+\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}. \tag{5.34}$$

In the third iteration, we use the previous solution. Let's the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative given by

$$\begin{aligned} D^{\alpha,\rho}u_2(x,t) &= \mu\frac{\partial^2 u_1(x,t)}{\partial x^2} - \nu\frac{\partial u_1(x,t)}{\partial x}, \\ &= 0. \end{aligned} \tag{5.35}$$

In this step, the initial condition is given by $u_2(x,0) = 0$. Applying the ρ-Laplace transform on Eq. (5.35), we have the following relationships

$$\begin{aligned} s^{\alpha}u_1(x,s) - s^{\alpha-1}u_1(x,0) &= 0, \\ u_1(x,s) &= 0. \end{aligned} \tag{5.36}$$

Applying the inverse of ρ-Laplace transform on Eq. (5.36), we recover the solution of the Eq. (5.35) expressed as the following form

$$u_2(x,t) = 0. \tag{5.37}$$

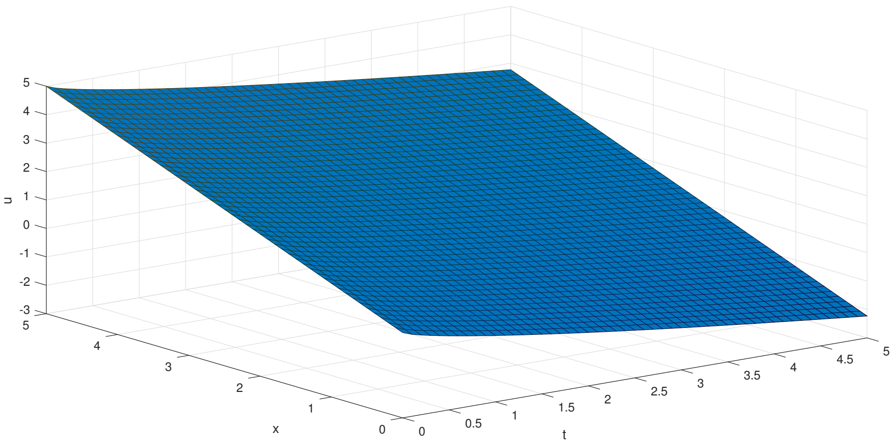
For the other iterations, we repeat the same procedure as in the previous iterations. The approximate solution of the fractional advection-dispersion equation (5.28) respecting the following procedure

$$\begin{aligned} u(x,t) &= \lim_{p\to 1}\sum_{i=0}^{n} p^i u_i(x,t); \\ u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ...; \\ u(x,t) &= x - \frac{\nu}{\Gamma(1+\alpha)}\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}. \end{aligned} \tag{5.38}$$

In the graphical representations, we consider the solution with two iterations. We depict in Figure 8a, the approximate solution of the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative operator for the values of the orders $\alpha = 1$ and $\rho = 0.75$ with $\nu = 0.5$.

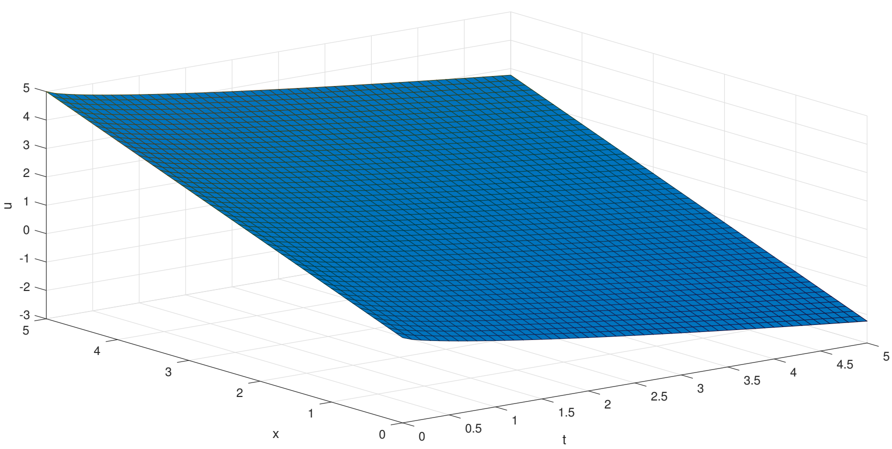
We depict in Figure 9a, the approximate solution of the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative operator for arbitrary values of the orders $\alpha = 1$ and $\rho = 1$.

We depict in Figure 10a, the approximate solution of the fractional advection-dispersion equation described by the Caputo-Liouville generalized fractional derivative operator when we fix $\alpha = 1$ and the order $\rho > 1$ take different values. In Figure 10a, we fix $t = 0.7$.



(a)

Figure 8: Approximate solutions of the fractional avection-dispersion equation, $\alpha = 1$, $\rho = 0.75$.



(a)

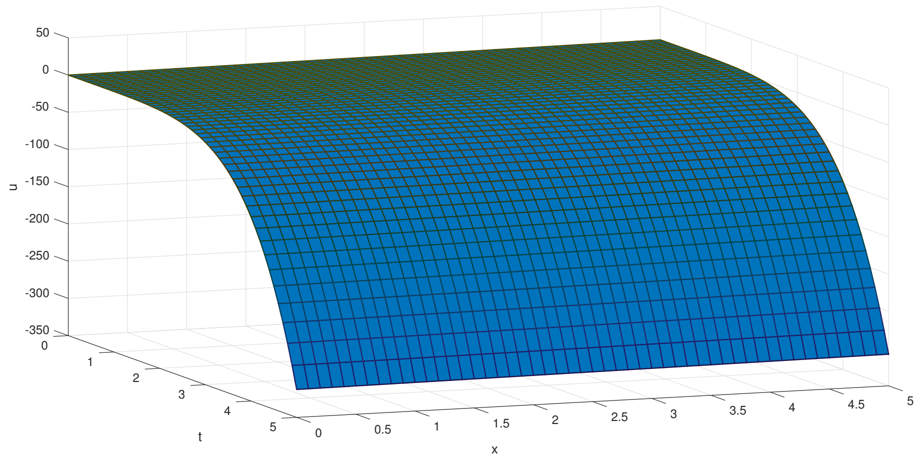
Figure 9: Approximate solutions of the fractional avection-dispersion equation, $\alpha = 1$, $\rho = 1$.

We depict in Figure 11a, the approximate solution of the fractional advection diffusion equation described by the Caputo-Liouville generalized fractional derivative operator when we fix $\alpha = 1$ and the order $\rho < 1$ take different values.

In conclusion, we note the behavior of the approximate solution depends strongly for the order of the fractional derivative. Thus the responses of the diffusion processes are generated by the chosen order.

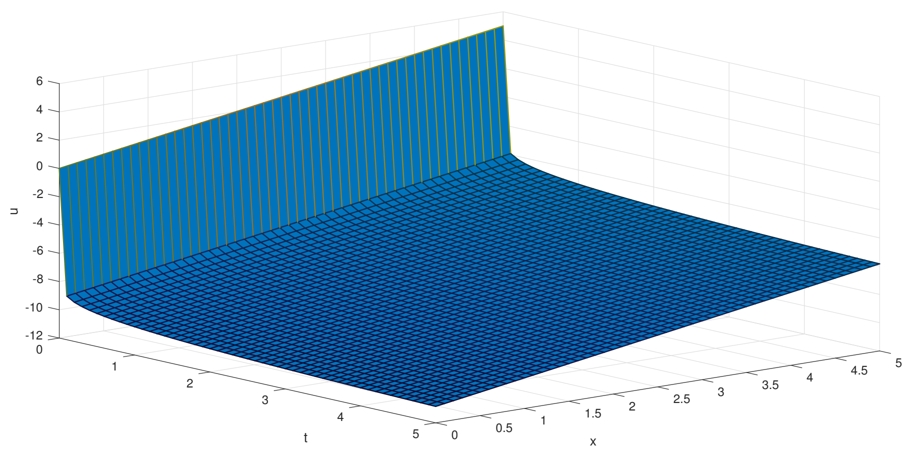
6 Conclusion

This paper offers the graphical representations of the behaviors of the approximate solutions of the fractional advection-dispersion equations under some particulars boundary conditions. We note, in general, the orders of the Caputo-Liouville generalized fractional derivative determine the behavior of the solutions. The new homotopy perturbation method is known as the homotopy perturbation ρ-Laplace transform. It was performed to get the approximate solutions. The paper also offers some approximates solutions which are in agreement with the exact analytical solutions. The used method opens a new door of getting the approximates solutions of the fractional



(a)

Figure 10: Approximate solutions of the fractional avection-dispersion equation, $\alpha = 1$, $\rho > 1$.



(a)

Figure 11: Approximate solutions of the fractional avection-dispersion equation, $\alpha = 1$, $\rho < 1$.

diffusion equations equation described by the fractional-order derivatives.

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Received: September 19, 2019.
Accepted: November 30, 2019.