Palestine Journal of Mathematics

Vol. 10(2)(2021) , 580–591

Fixed point algorithms for convex minimization problem with set-valued operator in real Hilbert spaces

T.M.M.Sow

Communicated by Thabet Abdeljawad

MSC 2010 Classifications: Primary 47J20, 65K05; Secondary 65K05.

Keywords and phrases: Set-valued operators, Convex minimization problem, Proximal point algorithm, Fixed points.

Abstract Let K be a nonempty, closed convex subset of real Hilbert H and let $F : K \to (-\infty,\ +\infty]$ be a proper, lower semi-continuous and convex function. Let $f : K \to K$ be an b-contraction mapping and $T : K \to CB(K)$ be a multivalued quasi-nonexpansive mapping such that $Fix(T \circ J_\lambda^F) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows:

$$\begin{cases} x_0 \in K, \\ y_n = \theta_n x_n + (1-\theta_n)u_n, \ \ u_n \in T \circ J_\lambda^F x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \end{cases} \tag{0.1}$$

where J_λ^F is the Moreau-Yosida resolvent of F in K and $\{\theta_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying certain conditions. Then it is proved that, under some mild conditions, $\{x_n\}_{n\geq 1}$ converges strongly to an element of $Fix(T) \cap \text{argmin}_{u\in K} F(u)$ without imposing any compactness-type condition on the mapping or the space. Applications are also considered. The presented results in the paper are new.

1 Introduction

Let (X,d) be a metric space, K be a nonempty subset of X and $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $Fix(T) := \{x \in D(T) : x \in Tx\}$.

The theory of set-valued mappings is one of the most powerful and important tools of modern mathematics and may be considered a core subject of nonlinear analysis. In the last few decades, the problem of nonlinear analysis with its relation to fixed point theory has emerged as a rapidly growing area of research because of its applications in game theory, optimization problem, control theory, integral and differential equations and inclusions, dynamic systems theory, signal and image processing, and so on. The crucial key of this success is due to the possibility of representing various problems arising in the above disciplines, in the form of an equivalent fixed point problem with multivalued mappings. Until now there have been many effective algorithms for solving fixed point problem with set-valued mappings, the reader can consult [19, 20, 22, 23, 26, 29, 17]. We describe briefly the connection of fixed point theory for multi-valued mappings with these applications.

Game Theory. Nash showed the existence of equilibria for non-cooperative *static* games as a direct consequence of *multivalued* Brouwer or Kakutani fixed point theorem. More precisely, under some regularity conditions, given a game, there always exists a *multi-valued mapping* whose fixed points coincide with the equilibrium points of the game. This, among other things, made Nash a recipient of Nobel Prize in Economic Sciences in 1994. However, it has been remarked that the applications of this theory to equilibrium are mostly *static*: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. This is part of the problem that is being addressed by *iterative methods for fixed point of multi-*

valued mappings. Consider a game $G = (u_n, K_n)$ with N players denoted by n, $n = 1, \cdots, N$, where $K_n \subset \mathbb{R}^{m_n}$ is the set of possible strategies of the n'th player and is assumed to be nonempty, compact and convex and $u_n : K := K_1 \times K_2 \cdots \times K_N \to \mathbb{R}$ is the payoff (or gain function) of the player n and is assumed to be continuous. The player n can take *individual actions*, represented by a vector $\sigma_n \in K_n$. All players together can take a *collective action*, which is a combined vector $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$. For each n, $\sigma \in K$ and $z_n \in K_n$, we will use the following standard notations:

$$K_{-n} := K_1 \times \cdots \times K_{n-1} \times K_{n+1} \times \cdots \times K_N,$$

$$\sigma_{-n} := (\sigma_1, \cdots, \sigma_{n-1}, \sigma_{n+1}, \cdots, \sigma_N),$$

$$(z_n, \sigma_{-n}) := (\sigma_1, \cdots, \sigma_{n-1}, z_n, \sigma_{n+1}, \cdots, \sigma_N).$$

A strategy $\bar{\sigma}_n \in K_n$ permits the n'th player to maximize his gain *under the condition* that the *remaining players* have chosen their strategies σ_{-n} if and only if

$$u_n(\bar{\sigma}_n, \sigma_{-n}) = \max_{z_n \in K_n} u_n(z_n, \sigma_{-n}).$$

Now, let $T_n : K_{-n} \to 2^{K_n}$ be the multivalued map defined by

$$T_n(\sigma_{-n}) := \operatorname*{Arg\,max}_{z_n \in K_n} u_n(z_n, \sigma_{-n}) \;\forall\, \sigma_{-n} \in K_{-n}.$$

Definition. A collective action $\bar{\sigma} = (\bar{\sigma}_1, \cdots, \bar{\sigma}_N) \in K$ is called a *Nash equilibrium point* if, for each n, $\bar{\sigma}_n$ is the best response for the n'th player to the action $\bar{\sigma}_{-n}$ made by the remaining players. That is, for each n,

$$u_n(\bar{\sigma}) = \max_{z_n \in K_n} u_n(z_n, \bar{\sigma}_{-n}) \tag{1.1}$$

or equivalently,

$$\bar{\sigma}_n \in T_n(\bar{\sigma}_{-n}). \tag{1.2}$$

This is equivalent to $\bar{\sigma}$ is a fixed point of the multivalued map $T : K \to 2^K$ defined by

$$T(\sigma) := [T_1(\sigma_{-1}), T_2(\sigma_{-2}), \cdots, T_N(\sigma_{-N})].$$

From the point of view of social recognition, game theory is perhaps the most successful area of application of *fixed point theory of multivalued mappings.*

Differential Equations . Consider the following Cauchy problem:

$$\frac{du}{dt} = f(t, u), \text{ p.p.}\, t \in I := [-a, a],\; u(0) = u_0. \tag{1.3}$$

if $f : I \times \mathbb{R} \to \mathbb{R}$ is discontinuous with bounded jumps, so we look in the sens of Filippov solutions [11], i.e. the solutions of differential inclusions:

$$\frac{du}{dt} \in F(t, u), \text{ p.p.}\, t \in I,\; u(0) = u_0, \tag{1.4}$$

where

$$F(t, x) = [\liminf_{y \to x} f(t, y),\; \limsup_{y \to x} f(t, y)]. \tag{1.5}$$

Put $H := L^2(I)$. Let $N_F : H \to 2^H$ be a Nemytskii operator defined by:

$$N_F(u) := \{v \in H \;:\; v(t) \in F(t, u(t))\; p.p., \; t \in I\},$$

and let $T : H \to 2^H$ be a multivalued mapping such that $T := L^{-1} \circ N_F$, L^{-1} be an inverse differential operator $Lu := u'$, defined by

$$L^{-1}v(t) := u_0 + \int_0^t v(s)ds.$$

Therefore, u is a solution of (1.4) if and only if u is a fixed point of T, i.e., $u \in Tu$.

Let D be a nonempty subset of a normed space E. The set D is called *proximinal* (see, *e.g.*, [23]) if for each $x \in E$, there exists $u \in D$ such that

$$d(x,u) = \inf\{\|x-y\| : y \in D\} = d(x,D),$$

where $d(x,y) = \|x-y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(D)$, $K(D)$ and $P(D)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D respectively. The Pompeiu *Hausdorff metric* on $CB(D)$ is defined by:

$$H(A,B) = \max\Big\{ \sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\}$$

for all $A, B \in CB(D)$ (see, Berinde [3]). A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called L- *Lipschitzian* if there exists $L > 0$ such that

$$H(Tx,Ty) \le L\|x-y\| \;\; \forall x,y \in D(T). \tag{1.6}$$

When $L \in (0,1)$, we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.
A multivalued map T is called quasi-nonexpansive if

$$H(Tx,Tp) \le \|x-p\|$$

holds for all $x \in D(T)$ and $p \in Fix(T)$. A mapping $T : D \to CB(D)$ is said to be nonspreading-type multi-valued mapping [6] if

$$2H(Tx,Ty)^2 \le d(x,Ty)^2 + d(y,Tx)^2, \;\; x,y \in D.$$

Remark 1.1. Easily, we obtain the following conclusions:

1. Every multivalued nonexpansive mapping is quasi-nonexpansive.
2. Every multivalued nonspreading mapping is quasi-nonexpansive.

On the other hand, let K be a nonempty, closed convex subset of real Hilbert H and let $F : K \to (-\infty, \; +\infty]$ be a proper, lower semi-continuous. Some major problems in optimization is to find $x \in K$ such that such that

$$F(x) = \min_{y\in H} F(y). \tag{1.7}$$

The set of all minimizers of F on H is denoted by $\operatorname{argmin}_{y\in H} F(y)$. A successful and powerful tool for solving this problem is the well-known proximal point algorithm (shortly, the PPA) which was initiated by Martinet [18] in 1970 and later studied by Rockafellar [25] in 1976. The PPA is defined as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \operatorname{argmin}_{y\in H}\Big[F(y) + \dfrac{1}{2\lambda_n}\|x_n - y\|^2\Big], \end{cases} \tag{1.8}$$

where $\lambda_n > 0$ for all $n \ge 1$. In [25] Rockafellar proved that the sequence $\{x_n\}$ given by (1.8) converges weakly to a minimizer of F. He then posed the following question:
Q1: does the sequence $\{x_n\}$ converges strongly? This question was resolved in the negative by Güler [13] (1991). He produced a proper lower semi-continuous and convex function F in l_2 for which the PPA *converges weakly* but not *strongly*. This leads naturally to the following question: **Q2**: Can the PPA be modified to guarantee *strong convergence*? In response to Q2, several works have been done (see, *e.g.*, Güler [13], Kamimura and Takahashi [15], Chidume and Djitte [9] and the references therein). In the recent years, the problem of finding a common element of the set of solutions of various convex minimization problems and the set of fixed points for nonlinear mapping in the framework of Hilbert spaces and Banach spaces have been intensively studied by many authors.

Very recently, Chang et al. [6], proved the following theorem.

Theorem 1.2 (Chang et al. [6]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $g : C \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Let $T : K \to K(C)$ be a multivalued nonspreading-type multivalued mapping such that $\Omega := Fix(T) \cap argmin_{u\in C}\, g(u) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated as follows:*

$$\begin{cases} x_1 \in C, \\ u_n = argmin_{y\in H}\Big[g(y) + \dfrac{1}{2\lambda_n}\|x_n - y\|^2\Big], \\ z_n = (1-\beta_n)x_n + \beta_n w_n, \;\; w_n \in Tu_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n v_n, \;\; v_n \in Tz_n, \end{cases} \tag{1.9}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\}$ are real sequences in $[0,1]$ such that $0 < a \le \alpha_n, \beta_n \le b < 1$ for all $n \ge 1$, and $\{\lambda_n\}$ is a real sequence such that $\lambda_n \ge \lambda > 0$ for all $n \ge 1$ and some λ. Then the sequence $\{x_n\}$ converges weakly to an element of Ω.

Motivated and inspired by the ongoing results in this field, we introduce a new iterative approach and prove a strong convergence theorem for convex minimization problem and fixed point problem with multivalued quasi-nonexpansive mappings in Hilbert spaces. Finally, our method of proof is of independent interest.

2 Preliminaries

The following lemmas will be useful in the sequel. Let H be a real Hilbert space. Let $\{x_n\}$ be a sequence in H, and $x \in H$. We denote the weak convergence of x_n to x $x_n \rightharpoonup x$ and the strong convergence x_n to x by $x_n \longrightarrow x$. Let K be a nonempty, closed convex subset of H. The nearest point projection from H to K denoted by P_K, assigns to each $x \in H$ the unique point of K, $P_K x$ such that

$$\|x - P_K x\| \le \|x - y\|,$$

for all $y \in K$. It is well known that for every $x \in H$,

$$\langle x - P_K x, y - P_K x\rangle \le 0, \tag{2.1}$$

for all $y \in K$.

Definition 2.1. Let H be a real Hilbert space and $T : D(T) \subset H \to 2^H$ be a multivalued mapping. The multivalued map $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $d(x_n, Tx_n)$ converges to zero, then $p \in Tp$, where I is the identity map of H.

Lemma 2.2 (Cholamjiak, [7]). *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $T : C \to K(C)$ be a multivalued nonspreading mapping. Then $I - T$ is demi-closed at zero.*

Lemma 2.3 (Chidume, [8]). *Let H be a real Hilbert space. Then, for every $x, y \in H$, and every $\lambda \in [0,1]$, the following hold:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle.$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - (1-\lambda)\lambda\|x - y\|^2, \;\; \lambda \in (0,1).$$

Lemma 2.4 (Xu, [27]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1-\alpha_n)a_n + \sigma_n$ for all $n \ge 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

$(a)\ \sum_{n=0}^{\infty} \alpha_n = \infty$, $(b)\ \limsup_{n\to\infty} \dfrac{\sigma_n}{\alpha_n} \le 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Let H be a real Hilbert space and K be a nonempty convex subset of H. Let $g : K \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. For every $\lambda > 0$, the Moreau-Yosida resolvent of g , J^g_λ is defined by :

$$J^g_\lambda x = \text{argmin}_{u\in K}\Big[g(u) + \frac{1}{2\lambda}\|x-u\|^2\Big],$$

for all $x \in H$. It was shown in [13] that the set of fixed points of the resolvent associated to g coincides with the set of minimizers of g. Also, the resolvent J^g_λ of g is nonexpansive for all $\lambda > 0$.

Lemma 2.5. *(Miyadera [21]) Let H be a real Hilbert space and K be a nonempty convex subset of H. Let $g : K \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. For every $r > 0$ and $\mu > 0$, the following holds:*

$$J^g_r x = J^g_\mu\Big(\frac{\mu}{r}x + (1-\frac{\mu}{r})J^g_r x\Big).$$

Lemma 2.6 (Sub-differential inequality, Ambrosio *et al.,* [2]). *Let H be a real Hilbert space and $g : H \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Then, for every $x, y \in H$ and $\lambda > 0$, the following sub-differential inequality holds:*

$$\frac{1}{\lambda}\|J^g_\lambda x - y\|^2 - \frac{1}{\lambda}\|x-y\|^2 + \frac{1}{\lambda}\|x - J^g_\lambda x\|^2 + g(J^g_\lambda x) \le g(y). \tag{2.2}$$

Lemma 2.7 (Mainge, [19]). *Let $\{t_n\}$ be a sequence of real numbers that does not decreases at infinity in the sense that there exists a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \le t_{n_{i+1}}$ for all $i \ge 0$. For $n \in \mathbb{N}$, sufficiently large, let $\{\tau(n)\}$ be the sequence of integers defined as follows:*

$$\tau(n) = \max\{k \le n : \; t_k \le t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \; t_n\} \le t_{\tau(n)+1}.$$

Lemma 2.8. *[1] Let K be a nonempty, closed convex subset of real Hilbert H and let $F : K \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. Let $m \ge 1$ be a fixed number, $1 \le i \le m$, let $T_i : K \to CB(K)$ be a multivalued nonexpansive mapping such that $T_i p = \{p\} \; \forall p \in \bigcap_{i=1}^m F(T_i)$. Then, $Fix(T_i \circ J^F_\lambda) = Fix(T_i) \cap \text{argmin}_{u\in K} F(u), \; 1 \le i \le m$.*

Remark 2.9. Let $F : K \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function and $T : K \to CB(K)$ be a multivalued quasi-nonexpansive mapping such that $Tp = \{p\} \; \forall p \in Fix(T)$. By a similar argument as in Lemma 2.8, we can show that $Fix(T \circ J^F_\lambda) = Fix(T) \cap \text{argmin}_{u\in K} F(u)$ and $T \circ J^F_\lambda$ is a multivalued quasi-nonexpansive mapping.

3 Main Result

We show the main result of this paper, that is, the strong convergence analysis for Algorithm 3.1.

Algorithm 3.1. Step 0. Take $\{\alpha_n\} \subset (0,1), \; \{\theta_n\} \subset (0,1)$ and $\lambda > 0$ arbitrarily choose $x_0 \in K$; and let $n := 0$.
Step 1. Given $x_n \in K$, compute $x_{n+1} \in K$ as

$$\begin{cases} x_0 \in K, \\ y_n = \theta_n x_n + (1-\theta_n)u_n, \;\; u_n \in T \circ J^F_\lambda x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \end{cases} \tag{3.1}$$

Update $n := n+1$ and go to Step 1.

Now we perform the convergence analysis for Algorithm 3.1.

Theorem 3.2. *Let K be a nonempty, closed convex subset of real Hilbert H and let $F : K \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. Let $f : K \to K$ be an b-contraction mapping and $T : K \to CB(K)$ be a multivalued quasi-nonexpansive mapping such that $Fix(T \circ J_\lambda^F) \neq \emptyset$. Assume that $I - T \circ J_\lambda^F$ is demiclosed at origin and $Tp = \{p\}, \ \forall\, p \in Fix(T)$. Suppose that $\{\alpha_n\}$ and $\{\theta_n\}$ are the sequences such that:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$

(ii) $\lim_{n\to\infty} \inf(1-\theta_n)\theta_n > 0.$

Then, the sequence $\{x_n\}$ defined by Algorithm 3.1 converges strongly to $x^* \in Fix(T) \cap argmin_{u\in K} F(u)$, which solves the variational inequality:

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in Fix(T) \cap argmin_{u\in K} F(u). \tag{3.2}$$

Proof. From $(I - f)$ is strongly monotone, then the variational inequality (3.2) has a unique solution in $Fix(T \circ J_\lambda^F)$. In what follows, we denote x^* to be the unique solution of (3.2). Now we show that $\{x_n\}$ is bounded. Let $p \in Fix(T \circ J_\lambda^F)$. Using (3.1), the fact that $Tp = \{p\}$, $T \circ J_\lambda^F$ is quasi-nonexpansive and Lemma 2.3, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\theta_n x_n + (1-\theta_n)u_n - p\|^2 \\
&= \theta_n\|x_n - p\|^2 + (1-\theta_n)\|u_n - p\|^2 - (1-\theta_n)\theta_n\|x_n - u_n\|^2 \\
&\leq \theta_n\|x_n - p\|^2 + (1-\theta_n)H(T(J_\lambda^F x_n), Tp)^2 - (1-\theta_n)\theta_n\|x_n - u_n\|^2 \\
&\leq \theta_n\|x_n - p\|^2 + (1-\theta_n)\|J_\lambda^F x_n - p\|^2 - (1-\theta_n)\theta_n\|x_n - u_n\|^2 \\
&\leq \|x_n - p\|^2 - (1-\theta_n)\theta_n\|x_n - u_n\|^2.
\end{aligned}$$

Hence,

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - (1-\theta_n)\theta_n\|x_n - u_n\|^2 \tag{3.3}$$

Since $\theta_n \in]0,1[$, we have,

$$\|y_n - p\| \leq \|x_n - p\|. \tag{3.4}$$

By using (3.1) and (3.4), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - p\| \\
&\leq \alpha_n\|f(x_n) - f(p)\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq (1-\alpha_n(1-b))\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-b}\}.
\end{aligned}$$

By induction, we conclude that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-b}\}, \quad n \geq 1.$$

Hence $\{x_n\}$ is bounded, also $\{y_n\}$ and $\{f(x_n)\}$ are all bounded.
Thus we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|\alpha_n f(x_n) + (1-\alpha_n)y_n - p\|^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)(1-\theta_n)\theta_n\|u_n - x_n\|^2.
\end{aligned}$$

Since $\{x_n\}$ is bounded, then there exists a constant $C > 0$ such that

$$(1-\alpha_n)(1-\theta_n)\theta_n\|u_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n C. \tag{3.5}$$

Now we prove that $\{x_n\}$ converges strongly to x^*. We divide the rest of the proof into two cases.
Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\lim_{n\to\infty} \left[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right] = 0. \tag{3.6}$$

Using the fact that $\lim_{n\to\infty} \inf(1-\theta_n)\beta_n > 0$, we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.7}$$

Since $u_n \in T \circ J_\lambda^F x_n$, it follows that

$$\lim_{n\to\infty} d(x_n, T \circ J_\lambda^F x_n) = 0. \tag{3.8}$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle \leq 0$. Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle = \lim_{j\to+\infty}\langle x^* - f(x^*), x^* - x_{n_j}\rangle.$$

From (3.8) and the fact that $I - T\circ J_\lambda^F$ is demiclosed, we obtain $a \in Fix(T \circ J_\lambda^F)$. Using Remark 2.9, we have $a \in Fix(T) \cap \operatorname{argmin}_{u\in K} F(u)$. Hence,

$$\begin{aligned}\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle &= \lim_{j\to+\infty}\langle x^* - f(x^*), x^* - x_{n_j}\rangle \\ &= \langle x^* - f(x^*), x^* - a)\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$. From (3.1) and Lemma 2.3, we get that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - x^*\|^2 \\ &\leq \|\alpha_n(f(x_n) - f(x^*)) + (1-\alpha_n)(y_n - x^*)\|^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\ &\leq \Big(\alpha_n\|f(x_n) - f(x^*)\| + \|(1-\alpha_n)(y_n - x^*)\|\Big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\ &\leq \Big(\alpha_n b\|x_n - x^*\| + (1-\alpha_n)\|y_n - x^*\|\Big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\ &\leq \Big((1-\alpha_n(1-b))\|x_n - x^*\|\Big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\ &\leq (1-\alpha_n(1-b))\|x_n - x^*\|^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle.\end{aligned}$$

From Lemma 2.4, its follows that $x_n \to x^*$.
Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing. Set $B_n = \|x_n - x^*\|^2$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ \ B_k \leq B_{k+1}\}$. We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. Let $i \in \mathbb{N}^*$, from (3.5), we have

$$(1-\alpha_{\tau(n)})(1-\theta_{\tau(n)})\theta_{\tau(n)}\Big\|u_{\tau(n)} - x_{\tau(n)}\Big\|^2 \leq \alpha_{\tau(n)} C.$$

Furthermore, we have

$$(1-\alpha_{\tau(n)})(1-\theta_{\tau(n)})\theta_{\tau(n)}\Big\|u_{\tau(n)} - x_{\tau(n)}\Big\|^2 = 0.$$

Since $\lim_{n\to\infty} \inf(1-\theta_{\tau(n)})\theta_{\tau(n)} > 0$, we can deduce

$$\lim_{n\to\infty} \Big\|u_{\tau(n)} - x_{\tau(n)}\Big\|^2 = 0. \tag{3.9}$$

Since $u_{\tau(n)} \in T \circ J_\lambda^F x_{\tau(n)}$, it follows that

$$\lim_{n\to\infty} d\Big(x_{\tau(n)}, T \circ J_\lambda^F x_{\tau(n)}\Big) = 0. \tag{3.10}$$

By a similar argument as in case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ are bounded in K and $\limsup\limits_{\tau(n)\to+\infty} \langle x^* - f(x^*), x^* - x_{\tau(n)}\rangle \le 0$. We have for all $n \ge n_0$,

$$0 \le \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \le \alpha_{\tau(n)}[-(1-b)\|x_{\tau(n)} - x^*\|^2 + 2\langle x^* - f(x^*), x^* - x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \le \frac{2}{1-b}\langle x^* - f(x^*), x^* - x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Thus, by Lemma 2.7, we conclude that

$$0 \le B_n \le \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim\limits_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

We now apply Theorem 3.2 when multivalued mapping is a nonspreading-type mapping. In this case demiclosedness assumption ($I - T \circ J_\lambda^F$ is demiclosed at origin) is not necessary.

Theorem 3.3. *Let K be a nonempty, closed convex subset of real Hilbert H and let $F : K \to (-\infty,\ +\infty]$ be a proper, lower semi-continuous and convex function. Let $f : K \to K$ be an b-contraction mapping and $T : K \to CB(K)$ be a multivalued nonspreading mapping such that $Fix(T \circ J_\lambda^F) \neq \emptyset$ and $Tp = \{p\}\ \forall p \in Fix(T)$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} x_0 \in K, \\ y_n = \theta_n x_n + (1-\theta_n)u_n, \quad u_n \in T \circ J_\lambda^F x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n. \end{cases} \tag{3.11}$$

Suppose that $\{\alpha_n\}$ and $\{\theta_n\}$ are the sequences such that:
(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=0}^{\infty} \alpha_n = \infty,$
(ii) $\lim\limits_{n\to\infty} \inf(1-\theta_n)\theta_n > 0.$
Then, the sequence $\{x_n\}$ defined by (3.11) converges strongly to $x^* \in Fix(T) \cap \text{argmin}_{u\in K} F(u)$.

Proof. Since every multivalued nonspreading mapping is quasi-nonexpansive, then, the proof follows Lemma 2.2 and Theorem 3.2. □

Let K be a nonempty, closed and convex subset of a real Hilbert space, $T : K \to P(K)$ be a multivalued map and $P_T : K \to CB(K)$ be defined by

$$P_T(x) := \{y \in Tx : \ \|y - x\| = d(x, Tx)\}.$$

We will need the following result.

Lemma 3.4 (Song and Cho [28]). *Let K be a nonempty subset of a real Banach space and $T : K \to P(K)$ be a multi-valued map. Then the following are equivalent:*
(i) $x^* \in Fix(T)$;
(ii) $P_T(x^*) = \{x^*\}$;
(iii) $x^* \in Fix(P_T)$. *Moreover, $Fix(T) = Fix(P_T)$.*

Now, using the similar arguments as in the proof of Theorem 3.2 and Lemma 3.4, we obtain the following result by replacing $T\circ J_\lambda^F$ by $P_{T\circ J_\lambda^F}$ and removing the rigid restriction on $Fix(T)$ $(Tp = \{p\}\ \forall p\in Fix(T))$.

Theorem 3.5. *Let K be a nonempty, closed convex subset of real Hilbert H and let $F: K\to(-\infty,\ +\infty]$ be a proper, lower semi-continuous and convex function. Let $f: K\to K$ be an b-contraction mapping and $T: K\to CB(K)$ be a multivalued mapping such that $Fix(T\circ J_\lambda^F)\neq\emptyset$. Assume that P_T is quasi-nonexpansive and $I-P_{T\circ J_\lambda^F}$ is demiclosed at origin. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0\in K$ by:*

$$\begin{cases} x_0\in K,\\ y_n=\theta_n x_n+(1-\theta_n)u_n,\ \ u_n\in P_{T\circ J_\lambda^F}x_n,\\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)y_n.\end{cases}\tag{3.12}$$

Suppose that $\{\alpha_n\}$ and $\{\theta_n\}$ are the sequences such that:

(i) $\lim_{n\to\infty}\alpha_n=0,\quad \sum_{n=0}^{\infty}\alpha_n=\infty,$

(ii) $\lim_{n\to\infty}\inf(1-\theta_n)\theta_n>0.$

Then, the sequence $\{x_n\}$ defined by (3.12) converges strongly to $x^*\in Fix(T)\cap argmin_{u\in K}F(u)$.

Since single-valued nonexpansive mappings is a particular case of multivalued nonexpansive mappings, we have the following result.

Corollary 3.6. *Let K be a nonempty, closed convex subset of real Hilbert H and let $F: K\to(-\infty,\ +\infty]$ be a proper, lower semi-continuous and convex function. Let $f: K\to K$ be an b-contraction mapping and $T: K\to K$ be a nonexpansive mapping such that $Fix(T\circ J_\lambda^F)\neq\emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0\in K$ by:*

$$\begin{cases} x_0\in K,\\ y_n=\theta_n x_n+(1-\theta_n)T\circ J_\lambda^F x_n,\\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)y_n.\end{cases}\tag{3.13}$$

Suppose that $\{\alpha_n\}$ and $\{\theta_n\}$ are the sequences such that:

(i) $\lim_{n\to\infty}\alpha_n=0,\quad \sum_{n=0}^{\infty}\alpha_n=\infty,$

(ii) $\lim_{n\to\infty}\inf(1-\theta_n)\theta_n>0.$

Then, the sequence $\{x_n\}$ defined by (3.13) converges strongly to $x^*\in Fix(T)\cap argmin_{u\in K}F(u)$.

4 APPLICATION

In this section, we apply our main results for solving split feasiblity problem and convex minimization problem. The split feasibility problem (SFP) was first introdued by Censor and Elfving [4] in 1994. The SFP is to find

$$x\in K,\ \text{ such that }\ Ax\in Q,\tag{4.1}$$

where K is a nonempty, closed convex subset of a Hilbert space H_1, Q is a nonempty closed convex subset of a Hilbert space H_2, and $A: H_1\to H_2$ is a bounded linear operator.
The problem (4.1) arises in signal processing and image reconstruction. Let Γ be the solution set of the split feasibility problem (4.1). From an optimization point of view, $x^*\in\Gamma$ if and only if x^* is a solution of the following minimization problem with zero optimal value:

$$\min_{x\in K} f(x)\ \text{ where }\ f(x):=\frac{1}{2}\|Ax-P_QAx\|^2.$$

The following lemma appears in [8].

Lemma 4.1. *Given $x^* \in H$, then x^* solves SFP (4.1) if and only if x^* is the solution of the fixed point equation $x = P_K(I - \gamma A^*(I - P_Q)A)x$, where $\gamma > 0$ is a suitable constant.*

The following Proposition was also given in [10].

Proposition 4.2. *[10] Let K be a nonempty, closed and convex subset of a Hilbert space H_1, Q be a a nonempty, closed and convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator. L et P_K, P_Q denote the orthogonal projection onto set K, Q respectively. Let $0 < \gamma < \frac{2}{\rho}$, ρ is the spectral raduis of A^*A, and A^* is the adjoint of A. Then, the operator $T := P_K(I - \gamma A^*(I - P_Q)A)$ is nonexpansive on K.*

Theorem 4.3. *Let H_1 and H_2 be two real Hilbert space. Let $A : H_1 \to H_2$ is a bounded linear operator, and $A^* : H_2 \to H_1$ be a adjoint operator of A. Let K be a nonempty, closed and convex subset of a Hilbert space H_1, Q be a a nonempty, closed and convex subset of a Hilbert space H_2. Let $F : K \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function such that $\Gamma \cap argmin_{u\in K} F(u) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} x_0 \in K, \\ y_n = \theta_n x_n + (1-\theta_n)P_K(I - \gamma A^*(I - P_Q)A) \circ J_\lambda^F x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n. \end{cases} \tag{4.2}$$

Suppose that $\{\alpha_n\}$ and $\{\theta_n\}$ are the sequences such that:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$

(ii) $\lim_{n\to\infty} \inf(1-\theta_n)\theta_n > 0.$

Assume that $0 < \gamma < \frac{2}{\rho}$, ρ is the spectral raduis of A^*A. Then, the sequence $\{x_n\}$ defined by (4.2) converges strongly to a common solution of Problem (4.1) and Problem (1.7).

Proof. From Lemma 4.1, we know $x \in \Gamma$ if and only if $x = P_K(I - \gamma A^*(I - P_Q)A)x$. From Proposition 4.2, we have the operator $T := P_K(I - \gamma A^*(I - P_Q)A)$ is nonexpansive on K. Then, the proof follows Theorem 3.2. □

Now, we give some remarks on our results as follows:

(1) The proof methods of our result are very different from the ones of Chang et al. [6] for multivalued nonspreading-type mappings . Further, Our Theorem 3.2 improves and extends the corresponding results of Chang et al. [6], and M. Akindele [1] from multivalued nonspreading-type mappings and mutivalued nonexpansive mappings respectively to quasi-nonexpansive mappings.

(2) Weak convergence results were proved for finding common elements of the set of minimizers of a convex functions and the set of fixed points of nonspreading-type multivalued mappings in the results of Chang et al. [6], while in this paper, we obtain strong convergence result without imposing any compactness-type condition on the mapping or the space.

(3) Our results improve many recent results using fixed points technique for solving convex minimization problems.

We know give example of mapping T, functions f and F satisfying the assumptions of Theorem 3.2. Let $H = \mathbb{R}$ and $K = [1, 7]$. For each $x \in K$ we define $F : K \to (-\infty, \infty]$ by $F(x) := \frac{1}{2}\|x - 1\|^2$, $f(x) = \frac{1}{3}x$ and define a mapping $T : K \to CB(K)$ by

$$Tx = \begin{cases} \{1\}, \ \ x \in [1, 4], \\ \\ \left[1, \dfrac{2x^2+1}{x^2+1}\right], \ \ x \in (4, 7]. \end{cases} \tag{4.3}$$

It can easily be seen that F, f and T are satisfied the conditions in Theorem 3.2 and $Fix(T) \cap \text{argmin}_{u\in K} F(u) = \{1\}$. Using the proximity operator [5], we know that

$$\text{argmin}_{u\in K}\Big[F(u) + \frac{1}{2}\|u - x\|^2\Big] = prox_F x = \frac{x+1}{2}.$$

Remark 4.4. Prototypes of sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\theta_n\}_{n\geq 1}$ in this paper are:

$$\alpha_n = \frac{1}{n}, \quad n \geq 1;$$

$$\theta_n = \frac{1}{2n} + \frac{1}{2}, \quad n \geq 1.$$

References

[1] M. Akindele, *Proximal Point Algorithms for Finding Common Fixed Points of a Finite Family of Nonexpansive Multivalued Mappings in Real Hilbert Spaces,* Khayyam J. Math. 5 (2019) no. 2, 113-123. DOI:10.22034/kjm.2019.88426.

[2] L. Ambrosio, N. Gigli, G. Savaré, *Gradient flows in metric spaces and in the space of probability measures,* Second edition, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, (2008).

[3] Berinde, V., Pcurar, M., The role of the Pompeiu-Hausdorff metric in fixed point theory. Creat. Math. Inform. 22 (2013), no. 2, 143-150.

[4] Y. Censor, Y. Elfvng, *A multiprojection algorithm using Bergman projection in product space*, Numer Algorithm, 8 (1994), 221-239.

[5] P.L. Combettes, J.C. Pesquet, *Proximal splitting methods in signal processing,* In: H.H.Bauschke, R. Burachik, P.L. Combettes, V. Elser, D.R. Luke and H. Wolkowicz, (eds.) Fixed-Point Algorithms for Inverse Problems in Science and Engineering, pp. 185-212 Springer, New York, 2011.

[6] S.S. Chang, D.P. Wu, L. Wang, G. Wang, *Proximal point algorithms involving fixed point of nonspreading-type multivalued mappings in Hilbert spaces,* J. Nonlinear Sci. Appl. 9 (2016), no. 10, 5561-5569.

[7] W. Cholamjiak, *Shrinking projection methods for a split equilibrium problem and a nonspreading-type multivalued mapping,* J. Nonlinear Sci. Appl.,9 (2016), 5561-5569.

[8] C. E. Chidume, *Geometric Properties of Banach spaces and Nonlinear Iterations*, Springer Verlag Series: Lecture Notes in Mathematics, Vol. 1965, (2009), ISBN 978-1-84882-189-7.

[9] C.E. Chidume, N. Djitte, *Iterative algorithm for zeros of bounded m-Accretive nonlinear operators*, *to appear*, J. Nonlinear and convex analysis.

[10] Q-W. Fan, Zhangsong. Yao, *Strong convergence theorems for nonexpansive mapping and its applications for solving the split feasibility problem*, J. Nonlinear. Sci. App., 10 (2017), 1470-1477.

[11] A.F. Filippov, *Differential equations with discontinuous right hand side*, Math. Sbornik **51**, 99–128, english transl. Trans. AMS 42 (1964), 199-232.

[12] L. Gorniewicz; Topological fixed point theory of multivalued mappings, Kluwer Academic Pub., Dordrecht, Netherlands (1999).

[13] O. Güler, On the convergence of the proximal point algorithm for convex minimization, SIAM J. Control Optim., 29 (1991), 403-419.

[14] S. Kakutani, A generalization of Brouwer' s fixed point theorem, Duke Mathematical Journal (1941), no. 3, 457-459.

[15] S. Kamimura, W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. of Optimization **13**(3) (5003), 938-945.

[16] N. Lehdili, A. Moudafi, *Combining the proximal algorithm and Tikhonov regularization*, Optimization **37**(1996), 239-252.

[17] T.C. Lim; A fixed point theorem for multi-valued nonexpansive mappings in a uniformly convex Banach space, Bull. Am. Math. Soc. **80** (1974), 1123-1126.

[18] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives,* (French) Rev. Franaise Informat. Recherche Opérationnelle, 4 (1970), 154-158.

[19] P. E. Mainge, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Analysis,* 16, 899-912 (2008).

[20] A. Moudafi, *Viscosity approximation methods for fixed point problems,* J. Math. Anal. Appl. 241, 46-55 (2000).

[21] I. Miyadera, *Nonlinear semigroups,* Translations of Mathematical Monographs, American Mathematical Society, Providence, (1992).

[22] S. B. Nadler Jr.; Multivaled contraction mappings ,Pacific J. Math. **30** (1969), 475-488.

[23] B. Panyanak, Mann and Ishikawa; iteration processes for multi-valued mappings in Banach Spaces, Comput. Math. Appl. **54** (2007), 872-877.

[24] S. Reich, *Strong Convergence theorems for resolvents of accretive operators in Banach spaces*, in *J. Math. Anal. Appl.* **183** (1994), 118-120.

[25] R. T. Rockafellar, *Monotone operators and the proximal point algorithm,* SIAM J. Control Optimization, 14 (1976), 877-898.

[26] K.P.R. Sastry and G.V.R. Babu; Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point, Czechoslovak Math.J, **55** (2005), 817-826.

[27] H.K. Xu; Iterative algorithms for nonlinear operators, J. London Math. Soc. **66** (2002), no. 2, 240 - 256.

[28] Y. Song, Y.J. Cho, Some notes on Ishikawa iteration for multi-valued mappings. Bull. Korean Math. Soc. 48(3), 575-584 (2011). doi:10.4134/BKMS.2011.48.3.575

[29] N. Shahzad and H. Zegeye; On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Analysis **71** (2009) 838 844.

[30] Y. Yao, H. Zhou, Y. C. Liou, *Strong convergence of modified Krasnoselskii-Mann iterative algorithm for nonexpansive mappings,* J. Math. Anal. Appl. Comput. 29 (2009) 383-389.

Author information

T.M.M.Sow, Amadou Mahtar Mbow, Université, Dakar, Senegal.
E-mail: thierno.sow@uam.edu.sn

Received: January 20, 2020.
Accepted: April 28, 2020.