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EXISTENCE AND UNIQUENESS OF MILD AND STRONG SOLUTIONS FOR FRACTIONAL EVOLUTION EQUATION

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Abstract. In this paper, we investigate the existence and uniqueness of mild and strong solutions of fractional semilinear evolution equations in the Hilfer sense, by means of the Banach fixed point theorem and the Gronwall inequality. In order to elucidate the results obtained, we performed an application with respect to the Caputo partial fractional derivative.

1 Introduction

As is well known, full order differential and integral calculus, or simply differential and integral calculus, or just calculus, is a branch of what is known as Analysis, one of the pillars of mathematics. Calculus, as we know it today, is a succession of contributions from how it was independently proposed by Newton and Leibniz in the late seventeenth century. Several mathematicians contributed to evolution and improvement, which we mention here: Euler, Lagrange, Cauchy, Weierstrass, Riemann, among others [4].

At the same time, the non-integer calculus was conceived, popularly known as the fractional calculus. This name comes from a famous correspondence, dated September 30, 1695, exchanged between L'Hopital and Leibniz, where the first, through a simple questioning, wanted to know the meaning of a middle order derivative. Leibniz, in tone audacious, not to say prophetic, presented the result and stated with certainty that this paradox would one day generate several important consequences. Today, after more than three hundred years, we are sure that fractional calculus has become a source of discussion, controversy, and much research [5].

The calculus is completely consolidated and contemplates a series of applications of which we highlight the study of differential, integral and integrodifferential equations. Such equations can be found in various branches of science, spanning a huge say, from physics to engineering to biology, without forgetting economics, among others. On the other hand, fractional calculus, besides being present in several applications, despite not having a geometric interpretation, stands out in the approach of problems involving the concepts of nonlocality and memory effect which cannot be explained by the calculus. in particular by the concept of derivative which in the calculus is a local operator while in fractional calculus it is a nonlocal operator [17].

In recent years, many researchers have looked in particular at the field of fractional calculus, especially for fractional differential equations. It is already more consolidated and proven that in fact, investigating the properties of solutions of fractional differential equations, seems to be better than the integer case. In addition, there is some sense the modeling, in the fractional setting, several species , which in turn has also been of great value, because it is possible to obtain more consistent results with respect to the reality [16]. Investigating the existence, uniqueness and stability of mild, strong and classical solutions of fractional differential equations, has gained prominence and strength in the scientific community. Due to the previous facts, the fractional calculus produced important and high quality papers, see for instance [6, 13].

In 2012, Shu and Wang [20] investigated the existence and uniqueness of mild solution for non-local fractional differential equations with non-local conditions using the Banach fixed point theorem. In 2016, Shu and Shi [21] performed the work on the expressions obtained so far that were related to mild solutions to impulsive fractional evolution equations. In the following year, Gou and Li [10] investigated the existence of mild solution in global and local context, for impulsive semilinear integral equations in the fractional sense with non-compact semigroup in Banach spaces, in which the authors emphasized the importance and effectiveness of these fractional integro-differential equations has in preexisting problems. In this sense, many other works have been published and investigated. We suggest some works [12, 22, 25, 29] and more recent [3, 14, 28].

Motivated by the works [2, 10, 21], in this paper, we consider the fractional semilinear evolution equation

$$\left\{\begin{array}{rcl} {}^{H}\mathbb{D}^{\alpha,\beta}_{t_{0+}}\xi\left(t\right)+\mathcal{A}\xi\left(t\right) & = & \phi\left(t,\xi\left(\sigma\left(t\right)\right)\right) \\ I^{1-\gamma}_{t_{0+}}\xi\left(t_{0}\right)+\varphi\left(t_{1},t_{2},...,t_{p},\xi\left(\cdot\right)\right) & = & \xi_{0} \end{array}\right. \tag{1.1}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $\mathbb{F}\left(t\right)_{t\geq 0}$ on a Banach space Λ, ${}^{H}\mathbb{D}^{\alpha,\beta}_{t_{0+}}\left(\cdot\right)$ is the Hilfer fractional derivative of order $\alpha\left(0<\alpha<1\right)$ and type $\beta\left(0\leq\beta\leq 1\right)$, $I^{1-\gamma}_{t_{0+}}\left(\cdot\right)$ is the Riemann-Liouville fractional integral of order $1-\gamma\left(\gamma=\alpha+\beta\left(1-\beta\right)\right)$, $0\leq t_0<t_1<\cdots<t_p\leq t_0+a$, $a>0$, $\xi_0\in X$ and $\phi:\left[t_0,t_0+a\right]\times\Lambda\to\Lambda$, $\varphi:\left[t_0,t_0+a\right]^p\times\Lambda\to\Lambda$ and $\sigma:\left[t_0,t_0+a\right]\to\left[t_0,t_0+a\right]$ are given functions.

The motivation of this work, besides investigating the properties of the mild and strong solutions, is to provide to the many researchers that investigate the results on the existence and uniqueness of several types of fractional differential equations, new results that allow to further strengthen the field as well as provide a range of tools and news.

We highlight below the main points that motivated us to investigate the existence and uniqueness of mild and strong solutions of fractional semilinear evolution equation:

(i) We present a new class of solutions for the semilinear fractional evolution equation, Eq.(1.1);

(ii) We investigate the existence and uniqueness of mild and strong solutions for Eq.(1.1), using the Banach fixed point theorem and the Gronwall inequality;

(iii) From the choice of the $\beta\to 1$ limit on the problem investigated as well as the mild and strong solution in which Theorem 2 and Theorem 3 are addressed, we obtain the results investigated here for the Caputo fractional derivative. On the other hand, performing the same procedure as highlighted, for $\beta\to 0$, we obtain the results for the Riemann-Liouville fractional derivative;

(iv) The special case is from choosing $\alpha=1$ with $\beta\to 1$ and/or $\beta\to 0$, we get the results here investigated for the integer case;

(v) We performed an application using the Caputo fractional derivative to elucidate the results investigated here. Numerous other applications can be made.

The paper is organized as follows: In section 2, we present the definition of integral and fractional derivative with respect to another function and the concept of q-times integrated p-resolvent operator function of an (p,q)-resolvent operator function (ROF), as well as other key concepts for article development. In section 3, we investigate the main results of the paper, namely the existence and uniqueness of mild and strong solutions for a fractional evolution equation introduced via the Hilfer fractional derivative and investigated using the Banach fixed point theorem and the Gronwall inequality. In section 4, we present an application in order to elucidate the results obtained. Concluding remarks close the article.

2 Preliminaries

Let the following interval $I'=[c,d]$. The weighted space of continuous functions is given by

$$C_{1-\gamma}(I',\Omega)=\left\{\psi\in C(I',\Omega),\, t^{1-\gamma}\xi(t)\in C(I',\Omega)\right\}$$

where $0 \leq \gamma \leq 1$, with norm

$$||\xi||_{C_{1-\gamma}} = \sup_{t\in I} ||t^{1-\gamma}\xi(t)||.$$

Let (c,d) $(-\infty \leq c < d \leq \infty)$ be a finite interval (or infinite) of the real line $\mathbb{R}$ and let $\alpha > 0$. Also let $\psi(x)$ be an increasing and positive monotone function on $(c,d]$, having a continuous derivative $\psi'(x)$ (we denote first derivative as $\dfrac{d}{dx}\psi(x) = \psi'(x)$) on (c,d). The left-sided fractional integral of a function f with respect to a function ψ on $[c,d]$ is defined by [23]

$$I_{c+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x \psi'(s) (\psi(x) - \psi(s))^{\alpha-1} f(s)\, ds. \tag{2.1}$$

On the other hand, let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, let $I' = [c,d]$ be an interval such that $-\infty \leq c < d \leq \infty$ and let $f, \psi \in C^n[c,d]$ be two functions such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I'$. The left-sided $\psi-$Hilfer fractional derivative ${}^H\mathbb{D}_{c+}^{\alpha,\beta;\psi}(\cdot)$ of a function f of order α and type $0 \leq \beta \leq 1$, is defined by [23]

$${}^H\mathbb{D}_{c+}^{\alpha,\beta;\psi} f(x) = I_{c+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n I_{c+}^{(1-\beta)(n-\alpha);\psi} f(x), \tag{2.2}$$

where $I_{c+}^{\alpha}(\cdot)$ is ψ-Riemann-Liouville fractional integral.

Let $\phi : I \to X$. Consider the fractional initial value problem

$$\begin{cases} {}^H\mathbb{D}_{t_{0+}}^{\alpha,\beta}\xi(t) & = & \mathcal{A}\xi(t) + \phi(t),\ t \in (t_0, t_0 + a] \\ I_{t_{0+}}^{1-\gamma}\xi(t_0) & = & \xi_0. \end{cases} \tag{2.3}$$

Definition 2.1. [7] Let $\alpha > 0$ and $\beta \geq 0$. A function $\mathbb{F}_{\alpha,\beta} : \mathbb{R}_+ \to L(\Omega)$ is called a β-times integrated α-resolvent operator function of an (α,β)-resolvent operator function (ROF) if the following conditions are satisfied:

- (A) $\mathbb{F}_{\alpha,\beta}(\cdot)$ is strongly continuous on $\mathbb{R}_+$ and $\mathbb{F}_{\alpha,\beta}(0) = g_{\beta+1}(0)I$;
- (B) $\mathbb{F}_{\alpha,\beta}(s)\mathbb{F}_{\alpha,\beta}(t) = \mathbb{F}_{\alpha,\beta}(t)\mathbb{F}_{\alpha,\beta}(s)$ for all $t, s \geq 0$;
- (C) the function equation $\mathbb{F}_{\alpha,\beta}(s)I_t^{\alpha}\mathbb{F}_{\alpha,\beta}(t) - I_s^{\alpha}\mathbb{F}_{\alpha,\beta}(s)\mathbb{F}_{\alpha,\beta}(t)$
$= g_{\beta+1}(s)I_t^{\alpha}\mathbb{F}_{\alpha,\beta}(t) - g_{\beta+1}(t)I_s^{\alpha}\mathbb{F}_{\alpha,\beta}(s)$ for all $t, s \geq 0$.

The generator $\mathcal{A}$ of $\mathbb{F}_{\alpha,\beta}$ is defined by

$$D(\mathcal{A}) := \left\{ x \in \Omega : \lim_{t\to 0^+} \frac{\mathbb{F}_{\alpha,\beta}(t)\, x - g_{\beta+1}(t)\, x}{g_{\alpha+\beta+1}(t)} \text{ exists} \right\} \tag{2.4}$$

and

$$\mathcal{A}\, x := \lim_{t\to 0^+} \frac{\mathbb{F}_{\alpha,\beta}(t)\, x - g_{\beta+1}(t)\, x}{g_{\alpha+\beta+1}(t)}, \quad x \in D(\mathcal{A}), \tag{2.5}$$

where $g_{\alpha+\beta+1}(t) := \dfrac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta)}$ $(\alpha+\beta > 0)$.

The following is the definition of the Mainardi function, fundamental in mild solution of the Eq.(1.1). Then, the Mainardi function, denoted by $M_\alpha(Q)$, is defined by [11]

$$M_\alpha(Q) = \sum_{n=1}^{\infty} \frac{(-Q)^{n-1}}{(n-1)!\Gamma(1-\alpha n)}, \quad 0 < \alpha < 1, \quad Q \in \mathbb{C}$$

satisfying the relation

$$\int_0^\infty \theta^{\overline{\delta}} M_\alpha(\theta)\, d\theta = \frac{\Gamma(1+\overline{\delta})}{\Gamma(1+\alpha\overline{\delta})}, \quad \text{for } \theta, \overline{\delta} \geq 0\cdot$$

Theorem 2.2. *Consider the following conditions:*

(i) Λ *is a reflexive Banach space and* $\mathcal{A}$ *is the infinitesimal generator of* $\mathbb{F}_{\alpha,\beta}(t)_{t\geq 0}$ *on* Λ*;*

(ii) $\phi : \Lambda \to \Lambda$ *is Lipschitz continuous on* I *and* $\xi_0 \in D(\mathcal{A})$*;*

The Eq.(2.3) *has a unique strong solution* ξ *on* I *given by the formula* [11]

$$\xi(t) = \mathbb{F}_{\alpha,\beta}(t - t_0)\,\xi_0 + \int_{t_0}^{t} \mathcal{K}_\alpha(t - s)\,\phi(s)\,ds,\ t \in I.$$

The mild solution for the nonlocal Cauchy problem Eq.(1.1) on I in the sense of Hilfer fractional derivative, is given by means of the integral equation

$$\xi(t) = \mathbb{F}_{\alpha,\beta}(t - t_0)\,\xi_0 - \mathbb{F}_{\alpha,\beta}(t - t_0)\,\varphi(t_1, t_2, ..., t_p, \xi(\cdot)) + \int_{t_0}^{t} \mathcal{K}_\alpha(t - s)\,\phi(s, \xi(\sigma(s)))\,ds,\ t \in I.$$

A function ξ is said to be a strong solution of problem Eq.(1.1) on I if ξ is differentiable a.e. on I ${}^{H}\mathbb{D}^{\alpha,\beta}_{t_{0+}} \in \left(L^1((t_0, t_0 + a], X)\right)$ and satisfies Eq.(1.1).

3 Main results

In this section, we will investigate the existence and uniqueness of mild and strong solutions for the fractional evolution equation introduced by means of the Hilfer fractional derivative. In order to obtain the main results of the paper, we will use the Banach fixed point theorem and the Gronwall inequality. Finally, we performed an application involving the Caputo partial fractional derivative, in order to elucidate the investigated results.

Before investigating the main results of this paper, consider some conditions:

(i) $0 \leq t_0 < t_1 < \cdots < t_p \leq t_0 + a$ and $B_R := \{\mu : \|\mu\| \leq R\} \subset \Lambda$;

(ii) $\sigma : I \to I$ is absolutely continuous and $\exists b > 0$ a constant such that $\sigma'(t) \geq b$ for $t \in I$;

(iii) $\varphi : I^p \times \Lambda \to \Lambda$ and $\exists \lambda > 0$ a constant such that

$$\|\varphi(t_1, t_2, ..., t_p, \xi_1(\cdot)) - \varphi(t_1, t_2, ..., t_p, \xi_2(\cdot))\| \leq \lambda \|\xi_1 - \xi_2\|_{C_{1-\gamma}}$$

$\xi_1, \xi_2 \in C_{1-\gamma}(I, B_R)$ and $\varphi(t_1, t_2, ..., t_p) \in D(\mathcal{A})$;

(iv) $-\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $\mathbb{F}(t)_{t\geq 0}$ on Λ;

(v) $\zeta_1 = \max_{t\in[0,a]} \|\mathbb{F}_{\alpha,\beta}(t)\|$, $\zeta_2 = \max_{s\in I} \|\phi(s, 0)\|$ and $\zeta_3 = \max_{\xi\in C_{1-\gamma}(I,B_R)} \|\varphi(t_1, t_2, ..., t_p, \xi(\cdot))\|$

(vi) $\zeta_1(\|\xi_0\| + \zeta_3 + (ar\delta/b) + a\zeta_2) \leq r$ and $\zeta_1\lambda + \left(\zeta_1\delta a^2/b\right) < 1$.

Theorem 3.1. *Assume* (1)-(6) *and consider the following conditions:*

(i) Λ *is a Banach space with norm* $\|(\cdot)\|_{C_{1-\gamma}}$ *and* $\xi_0 \in \Lambda$*;*

(ii) $\phi : I \times \Lambda \to \Lambda$ *is continuous in* t *on* I *and* $\exists\delta > 0$ *constant such that*

$$\|\phi(s, \mu_1) - \phi(s, \mu_2)\| \leq \delta \|\mu_1 - \mu_2\|_{C_{1-\gamma}}, \text{for } s \in I \text{and } \mu_1, \mu_2 \in B_R.$$

Then problem Eq.(1.1) *has a unique mild solution on* I.

Proof. To realize the proof, we consider $\Omega := C_{1-\gamma}(I, B_R)$ and define the following operator $\mathcal{F}$ on Ω, given by

$$(\mathcal{F}\mu)(t) = \mathbb{F}_{\alpha,\beta}(t - t_0)\,\xi_0 - \mathbb{F}_{\alpha,\beta}(t - t_0)\,\varphi(t_1, t_2, ..., t_p, \mu(\cdot)) + \int_{t_0}^{t} \mathcal{K}_\alpha(t - s)\,\phi(s, \mu(\sigma(s)))\,ds.$$

Then, by definition of the norm in Ω, we get

$$\begin{aligned}
\|(\mathcal{F}\mu)(t)\| &\leq \|\mathbb{F}_{\alpha,\beta}(t-t_0)\|\,\|\xi_0\| + \|\mathbb{F}_{\alpha,\beta}(t-t_0)\|\,\|\varphi(t_1,t_2,...,t_p,\mu(\cdot))\| \\
&\quad + \int_{t_0}^{t} \|\mathcal{K}_\alpha(t-s)\|\,\|\phi(s,\mu(\sigma(s)))\|\,ds \\
&\leq \zeta_1\|\xi_0\| + \zeta_1\zeta_3 + \zeta_1 \int_{t_0}^{t} \|\phi(s,\mu(\sigma(s)))\|\,ds \\
&\leq \zeta_1\|\xi_0\| + \zeta_1\zeta_3 + \zeta_1\delta \int_{t_0}^{t} \|\mu(\sigma(s))\|_{C_{1-\gamma}} \left(\frac{\sigma'(s)}{b}\right) ds + \zeta_1\zeta_2 \int_{t_0}^{t} ds \\
&\leq \zeta_1\|\xi_0\| + \zeta_1\zeta_3 + \frac{\zeta_1\delta r}{b}(\sigma(t)-\sigma(t_0)) + \zeta_1\zeta_2(t-t_0) \\
&= \zeta_1\left[\|\xi_0\| + \zeta_3 + \frac{\delta r}{b}a + \zeta_2 a\right] \leq r.
\end{aligned}$$

Therefore, $\mathcal{F}(\Omega) \subset \Omega$. Let investigate the norm, for every $\mu_1,\mu_2 \in \Omega$ and $t \in I$, we obtain

$$\begin{aligned}
&\|(\mathcal{F}\mu_1)(t) - (\mathcal{F}\mu_2)(t)\| \\
\leq\;& \|\mathbb{F}_{\alpha,\beta}(t-t_0)\|\,\|\varphi(t_1,t_2,...,t_p,\mu_1(\cdot)) - \varphi(t_1,t_2,...,t_p,\mu_2(\cdot))\| \\
&+ \int_{t_0}^{t} K_\alpha(t-s)\,\|\varphi(s,\mu_1(\sigma(s))) - \varphi(s,\mu_2(\sigma(s)))\|\,ds \\
\leq\;& \zeta_1\zeta_3\|\mu_1-\mu_2\|_{C_{1-\gamma}} + \frac{\zeta_1\delta}{b}\int_{t_0}^{t} \|\mu_1(\sigma(s)) - \mu_2(\sigma(s))\|_{C_{1-\gamma}} \left(\frac{\sigma'(s)}{b}\right) ds \\
\leq\;& \zeta_1\zeta_3\|\mu_1-\mu_2\|_{C_{1-\gamma}} + \frac{\zeta_1\delta}{b}\int_{\sigma(t_0)}^{\sigma(t)} \|\mu_1(s) - \mu_2(s)\|_{C_{1-\gamma}}\,ds \\
\leq\;& \left(a\zeta_1\zeta_{13} + \frac{\zeta_1\delta a^2}{b}\right)\|\mu_1-\mu_2\|_{C_{1-\gamma}}.
\end{aligned}$$

Now, taking $q := \left(a\zeta_1\zeta_3 + \dfrac{\zeta_1\delta a^2}{b}\right)$, we have

$$\|\mathcal{F}\mu_1 - \mathcal{F}\mu_2\|_{C_{1-\gamma}} \leq q\,\|\mu_1-\mu_2\|_{C_{1-\gamma}}, \text{with } 0<q<1.$$

Thus, we guarantee that $\mathcal{F}$ is a contraction in the metric space Ω. Then, by means of the Banach fixed point theorem for $\mathcal{F}$ in the space Ω, we conclude that, in fact, this point is the mild solution of the problem Eq.(1.1) on I. □

The second main result is to investigate the existence and uniqueness of strong solution for Eq.(1.1). So we have the following result.

Theorem 3.2. *Assume* (1)-(6) *and consider the following conditions:*

(i) Λ *is a reflexive Banach space with norm* $\|(\cdot)\|_{C_{1-\gamma}}$ *and* $\xi_0 \in \Lambda$*;*

(ii) $\phi : I \times \Lambda \to \Lambda$ *is continuous in* t *on* I *and* $\exists\delta > 0$ *a constant such that*

$$\|\phi(s_1,\mu_1) - \phi(s_2,\mu_2)\| \leq \delta\left(\|s_1-s_2\|_{C_{1-\gamma}} + \|\mu_1-\mu_2\|_{C_{1-\gamma}}\right)$$

for $s_1,s_2 \in I$ *and* $\mu_1,\mu_2 \in B_R$*;*

(iii) ξ *is the mild solution of problem* Eq.(1.1) *on* I *and there exists a constant* $\widetilde{R} > 0$ *such that*

$$\|\xi(\sigma(s)) - \xi(\sigma(t))\| \leq \widetilde{R}\,\|\xi(s) - \xi(t)\|_{C_{1-\gamma}}, \text{for } s,t \in I. \tag{3.1}$$

Then ξ *is a strong solution of problem* Eq.(1.1) *on* I.

Proof. By Theorem 1, the problem Eq.(1.1), admits a unique mild solution in $C_{1-\gamma}(I,\Lambda)$, once the conditions are satisfied. In order to obtain the existence and uniqueness of the strong solution, we will use the fact that the solution ξ, is mild for Eq.(1.1) on I. Then, for any $t\in I$, we get

$$\begin{aligned}
\|\xi(t+h)-\xi(t)\| \quad & \leq [\|\mathbb{F}_{\alpha,\beta}(t+h-t_0)\|+\|\mathbb{F}_{\alpha,\beta}(t-t_0)\|]\,\|\xi_0\| \\
& +[\|\mathbb{F}_{\alpha,\beta}(t+h-t_0)\|+\|\mathbb{F}_{\alpha,\beta}(t-t_0)\|]\,\|\varphi(t_1,t_2,...,t_p,\xi(\cdot))\| \\
& +\int_{t_0}^{t_0+h}\|\mathcal{K}_\alpha(t+h-s)\|\,\|\phi(s,\xi(\sigma(s)))-\phi(s,0)\|\,ds \\
& +\int_{t_0}^{t_0+h}\|\mathcal{K}_\alpha(t+h-s)\|\,\|\phi(s,0)\|\,ds \\
& +\int_{t_0}^{t}\|\mathcal{K}_\alpha(t-s)\|\,\|\phi(s+h,\xi(\sigma(s+h)))-\phi(s,\xi(\sigma(s)))\|\,ds \\
& \leq 2\zeta_1 h\,\|\xi_0\|+2\zeta_1\zeta_3 h+\frac{\delta\zeta_1 hr}{b}+\zeta_1\zeta_2 b+\zeta_1\delta\int_{t_0}^{t}\|h\|_{C_{1-\gamma}}\,ds \\
& +\zeta_1\delta\int_{t_0}^{t}\|\xi(\sigma(s+h))-\xi(\sigma(s))\|_{C_{1-\gamma}}\,ds \\
& \leq \theta+\zeta_1\delta R\widetilde{C}\int_{t_0}^{t}(t-s)^{\alpha-1}\,\|\xi(s+h)-\xi(s)\|_{C_{1-\gamma}}\,ds,
\end{aligned}$$

where $\theta := 2a\zeta_1 h\,\|\xi_0\|_{C_{1-\gamma}}+2a\zeta_1\zeta_3 h+a\dfrac{\zeta_1\delta hr}{b}+a\zeta_1\zeta_2 b+\zeta_1\delta ha^2$.

By means of the Gronwall inequality (see[26]), we obtain

$$\|\xi(t+h)-\xi(t)\|\leq\theta\mathbb{E}_\alpha\left[\zeta_1\delta R\widetilde{C}a^\alpha\Gamma(\alpha)\right],\text{for t}\in I.$$

Thus, ξ is Lipschitz continuous on I. Note that, because u is Lipschitz in I and condition (iii), we have that $t\to\phi(t,\xi(t))$ is Lipschitz continuous on I. In this sense, by means of Theorem 1 and Theorem 2, we have that the fractional Cauchy problem with its initial condition Eq.(1.1), admits a unique solution in the interval I, which satisfies the integral equation

$$\left\{\begin{array}{rcl} {}^H\mathbb{D}_{t_0+}^{\alpha,\beta}\mu(t)+\mathcal{A}\mu(t) & = & \phi(t,\xi(\sigma(t)))\,,\ t\in[t_0,t_0+a] \\ I_{t_0+}^{1-\gamma}\mu(t_0) & = & \xi_0-\varphi(t_1,t_2,...,t_p,\xi(\cdot)) \end{array}\right. \tag{3.2}$$

has a unique solution on I satisfying the equation

$$\mu(t)=\mathbb{F}_{\alpha,\beta}(t-t_0)\,\xi_0-\mathbb{F}_{\alpha,\beta}(t-t_0)\,\varphi(t_1,t_2,...,t_p,\xi(\cdot))+\int_{t_0}^{t}\mathcal{K}_\alpha(t-s)\,\phi(s,\xi(\sigma(s)))\,ds=\xi(t)\,.$$

Thus, we conclude that, ξ is a strong solution of fractional Cauchy problem Eq.(1.1) in the interval I. □

4 Application

Now by means of Theorem 2, we shall done an application involving Caputo partial fractional derivative.

Consider the fractional semilinear evolution equation

$$\left\{\begin{array}{rcl} \dfrac{\partial^\alpha}{\partial t^\alpha}\xi(t,x) & = & \dfrac{\partial^2}{\partial x^2}\xi(t,x)+\phi(t,\xi(\sigma(t))) \\ \xi(t,0) & = & \xi(t,\pi)=0 \\ \xi(t,0)+\varphi(t_1,t_2,...,t_p,\xi(\cdot)) & = & \xi_0 \end{array}\right. \tag{4.1}$$

where $t\in I=[0,1]$, $x\in(0,\pi)$, $0<\alpha<1$, $\xi_0\in L^2([0,\pi])$. Let $\Lambda=L^2([0,\pi])$ and consider the operator

$$\mathcal{A}:D(\mathcal{A})\subset\Lambda\to\Lambda$$

defined by

$$D(\mathcal{A}) = \left\{ \xi \in \Lambda; \frac{\partial \xi}{\partial x}, \frac{\partial^2 \xi}{\partial x^2} \in \Lambda, \xi(0) = \xi(\pi) = 0 \right\}.$$

Clearly, $\mathcal{A}$ is densely defined in Λ and is the infinitesimal generator of a resolvent family $\{\mathbb{F}_\alpha\}_{t\geq 0}$ on Λ and let $\xi, \xi_1 \in C_{1-\gamma}([0,1], \Lambda)$. Define the operators $\sigma : [0,1] \to [0,1]$, $\varphi : [0,1]^p \times \Lambda \to \Lambda$ and $\phi : [0,1] \times \Lambda \to \Lambda$ by

$$\sigma(t) = e^t, \ \ \phi(t, \xi) = \frac{e^{-t}\,|\xi(t,x)|}{(12 + e^t)(1 + |\xi(t,x)|)} \ \ \varphi(t_1, t_2, ..., t_p, \xi(\cdot)) = \frac{e^t}{\sqrt{72} + |\xi(t,x)|}.$$

Now, let's check the conditions of Theorem 2.

Note that the conditions 1, 4 and $\sigma : I \to I$ is continuous absolutely are straightforward. Note that $\exists b = 1 > 0$, such that $\sigma'(t) = e^t \geq 1$, $\forall t \in [0,1]$. With this, condition 2 is satisfied.

On the other hand, we have

$$\|\varphi(t_1, t_2, ..., t_p, \xi_1(\cdot)) - \varphi(t_1, t_2, ..., t_p, \xi_2(\cdot))\| \leq \frac{e}{72} \|\xi - \xi_1\|_{C_{1-\gamma}} = \lambda \|\xi - \xi_1\|_{C_{1-\gamma}}$$

with $\lambda = \dfrac{e}{72}$. Thus we get condition 3.

We also have

$$\|\phi(t, \xi(\cdot)) - \phi(t, \xi_1(\cdot))\| \leq \frac{e^{-t}}{12 + e^t} \|\xi - \xi_1\|_{C_{1-\gamma}} \leq \delta \|\xi - \xi_1\|_{C_{1-\gamma}}$$

with $\delta = \dfrac{1}{12} > 0$.

Now, note that $\|\mathbb{F}_\alpha(t)\| \leq M$ [8]. For example, in the case $M = 1$, we get $\|\mathbb{F}_\alpha(t)\| \leq 1$ and consequently, $\zeta_1 = 1$. We also have

$$\zeta_2 = \max_{t\in[0,a]} \left\| \frac{e^{-s}}{12 + e^s} \right\| \leq \frac{1}{12}.$$

By the definition of φ, we have

$$\zeta_3 = \max_{\xi \in C_{1-\gamma}} \left\| \frac{e^t}{\sqrt{72} + |\xi(t,x)|} \right\| \leq \frac{e}{\sqrt{72}}.$$

Thus, condition 5 is verified. Finally, the condition 6 remains to be verified.

We have $\zeta_1 \lambda + \zeta_1 \dfrac{\delta a^2}{b} = \dfrac{e+6}{72} < 1$. On the other hand, it is desirable that

$$\zeta_1 \left(\|\xi_0\| + \zeta_3 + \frac{ar\delta}{b} + a\zeta_2 \right) \leq \left(\|\xi_0\| + \frac{e}{\sqrt{72}} + \frac{r}{12} + \frac{1}{12} \right) \leq r. \tag{4.2}$$

Note that the inequality (4.2) is satisfied if we consider that

$$\frac{12}{11} \left(\|\xi_0\| + \frac{e}{\sqrt{72}} + \frac{1}{12} \right) \leq r.$$

Then, by means of Theorem 2, the problem Eq.(1.1), has a unique mild solution in I.

5 Concluding remarks

The existence and uniqueness of mild and weak solutions of evolution fractional differential equations, among others, has been considered by several researchers. In this manuscript, by applying the Banach fixed point theorem and the Gronwall inequality, it was studied the existence and uniqueness of mild and strong solutions for Eq.(1.1) in an arbitrary Banach space. A nontrivial application of the abstract results obtained was also considered. Thus, we believe that the results presented in this paper contributes to the study of differential equations involving fractional operators. We also believe that the results presented in this work can be extended to more general fractional operators, for example by considering the ψ-Hilfer fractional derivative which was recently introduced and motivated several studies involving fractional calculus. However, to investigate the existence, uniqueness, stability of mild solutions of fractional differential equations towards the ψ-Hilfer fractional derivative, the inverse Laplace transform with respect to another function is required. This is an open problem and many researchers in the area have been trying to solve it, as it will allow us to investigate a wide range of other problems arising from it.

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