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LOWER AND UPPER BOUNDS FOR THE BLOW UP TIME FOR GENERALIZED HEAT EQUATIONS WITH VARIABLE EXPONENTS

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Abstract. This paper deals with the initial-boundary value problem for generalized heat equations with variable exponent in a bounded domain. Under suitable conditions, we discuss the lower and upper bounds for the blow up time of solutions.

1 Introduction

In this paper, we deal with the lower and upper bounds for the blow up time of solutions of the following generalized heat equations with variable exponents

$$\begin{cases} u_t + \Delta^2 u_t - div\left(|\nabla u|^{m(x)-2}\nabla u\right) = |uv|^{p(x)-2}uv^2 & \text{in } \Omega\times(0,T), \\ v_t + \Delta^2 v_t - div\left(|\nabla v|^{n(x)-2}\nabla v\right) = |uv|^{p(x)-2}u^2 v & \text{in } \Omega\times(0,T), \\ u = 0, \quad v = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with a smooth boundary $\partial\Omega$. The exponents $m(\cdot)$, $n(\cdot)$, $p(\cdot)$ are given measurable functions on Ω satisfying

$$p^+ > \max\left\{m^+, n^+\right\}, \quad \min\left\{m^-, n^-\right\} \geq 2, \tag{1.2}$$

$$m^+ \geq n^-,\ n^+ \geq m^-, \tag{1.3}$$

where

$$\begin{cases} m^- = ess\inf_{x\in\Omega} m(x), & m^+ = ess\sup_{x\in\Omega} m(x), \\ n^- = ess\inf_{x\in\Omega} n(x), & n^+ = ess\sup_{x\in\Omega} n(x), \\ p^- = ess\inf_{x\in\Omega} p(x), & p^+ = ess\sup_{x\in\Omega} p(x), \end{cases}$$

and

$$\begin{cases} m^- \leq m(x) \leq m^+, \\ n^- \leq n(x) \leq n^+, \\ p^- \leq p(x) \leq p^+. \end{cases}$$

The problems with variable exponents arises in many branches in sciences such as image processing, electrorheological fluids and nonlinear elasticity theory [4, 6, 16].

Alaoui et al. [1] studied the following nonlinear heat equation with variable exponent

$$u_t - div\left(|\nabla u|^{m(x)-2}\nabla u\right) = |u|^{p(x)-2}u. \tag{1.4}$$

They proved the blow up of solutions. Also, Wu [18] proved the blow up of solutions for the equation (1.4). When $m(\cdot) \equiv 2$, many authors [3, 9, 13, 17, 19] studied the lower bounds for the blow up time and blow up of solutions for the equation (1.4).

Di et al. [5] considered

$$u_t - \Delta u_t - div\left(|\nabla u|^{m(x)-2}\nabla u\right) = |u|^{p(x)-2}u, \tag{1.5}$$

and established the lower and upper bounds for the blow up time of solutions. Recently, some authors was obtained the global existence, blow up and asymptotic stability of solutions for 1.5, see [11, 12, 20]. In [8] Gao and Gao studied the following equation

$$u_t - \Delta u_t - div\left(|\nabla u|^{m(x)-2}\nabla u\right) = 0,$$

and proved the existence and asymptotic behavior of solutions.

Bai and Zhang [2] considered the following system

$$\begin{cases} u_t - \Delta u = v^{p(x)}, \\ v_t - \Delta v = u^{q(x)}. \end{cases}$$

They proved the global existence and the blow up of solutions. In 2017, Qi et al. [15] discussed the following equation

$$\begin{cases} u_t - div\left(|\nabla u|^{m(x)-2}\nabla u\right) - \Delta u_t = |uv|^{p(x)-2}uv^2, \\ v_t - div\left(|\nabla v|^{n(x)-2}\nabla v\right) - \Delta v_t = |uv|^{p(x)-2}u^2v. \end{cases}$$

They proved the bounds for the blow up time of solutions.

Motivated by previous paper, we consider the lower and upper bounds for the blow up time of solutions. Therefore, we try to extend the previous results from constant exponents to variable exponents.

Our paper is organized as follows: In Section 2, we state some results about the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. In Sections 3-4, we investigate the upper bound for blow-up time and lower bound for blow up time of solutions, respectively, using the similar arguments as in [1, 5, 15].

2 Preliminaries

In this section, we state some results about the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ (see [6, 7, 10, 14]). Also, $\|\cdot\|$ and $\|\cdot\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

Let $p:\Omega\to[1,\infty]$ be a measurable function, where Ω is a bounded domain of R^n. We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u:\Omega\to R,\ u \text{ is measurable and } \rho_{p(.)}(\lambda u) < \infty,\ \text{ for some } \lambda>0\right\},$$

where

$$\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

Also endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf\left\{\lambda>0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

$L^{p(x)}(\Omega)$ is a Banach space.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{u\in L^{p(x)}(\Omega) : \nabla u \text{ exists and } |\nabla u|\in L^{p(x)}(\Omega)\right\}.$$

Variable exponent Sobolev space is a Banach space with respect to the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}.$$

Let the variable exponents $p(.)$ and $q(.)$ satisfy the log-Hölder continuity condition:

$$|p(x) - p(y)| \leq \frac{A}{\log \frac{1}{|x-y|}}, \text{ for all } x, y \in \Omega \text{ with } |x-y| < \delta, \tag{2.1}$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 2.1. *(Poincare inequality) Let Ω be a bounded domain of R^n and $p(.)$ satisfies log-Hölder condition, then*

$$\|u\|_{p(x)} \leq c\|\nabla u\|_{p(x)}, \textit{ for all } u \in W_0^{1,p(x)}(\Omega),$$

where $c = c(p^-, p^+, |\Omega|) > 0$.

Lemma 2.2. *Let $p(.) \in C(\overline{\Omega})$ and $q : \Omega \to [1, \infty)$ be a measurable function and satisfy*

$$\underset{x\in\overline{\Omega}}{essinf}(p^*(x) - q(x)) > 0.$$

Then the Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact. Where

$$p^*(x) = \begin{cases} \frac{np^-}{n-p^-}, & \textit{if } p^- < n, \\ \infty, & \textit{if } p^- \geq n. \end{cases}$$

3 Upper bound for blow-up time

In this section, we shall prove the upper bound for blow up time of solutions to system (1.1).

Theorem 3.1. *Suppose that (2.1), (1.2) and (1.3) hold. Let $u_0 \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$, $v_0 \in W_0^{1,n(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$ such that $\|u_0\|_{H_0^1}, \|v_0\|_{H_0^1} > 0$ and*

$$E(0) \leq 0. \tag{3.1}$$

Then, the solution (u, v) of the system (1.1) blows-up in finite time T^*. Moreover, an upper bound for blow-up time is given by

$$T^* \leq \frac{(F(0))^{1-\frac{1}{b}} b}{\beta(b-1)}, \tag{3.2}$$

where β and b are suitable positive constants given later and $F(0) = \|u_0\|_{H_0^1}^2 + \|v_0\|_{H_0^1}^2$.

Proof. We multiplying the first equation of (1.1) by u_t, the second equation of (1.1) by v_t, and integrating over the domain Ω, we obtain

$$\begin{aligned} &\int_\Omega \left(|u_t|^2 + |v_t|^2 + |\Delta u_t|^2 + |\Delta v_t|^2\right) dx + \frac{d}{dt}\int_\Omega \left(\frac{1}{m(x)}|\nabla u|^{m(x)} + \frac{1}{n(x)}|\nabla v|^{n(x)}\right) dx \\ = &\ \frac{d}{dt}\int_\Omega \frac{1}{p(x)}|uv|^{p(x)} dx, \end{aligned} \tag{3.3}$$

$$E'(t) = -\left(\|u_t\|^2 + \|\Delta u_t\|^2 + \|v_t\|^2 + \|\Delta v_t\|^2\right) \leq 0, \tag{3.4}$$

where

$$E(t) = \int_\Omega \left(\frac{1}{m(x)}|\nabla u|^{m(x)} + \frac{1}{n(x)}|\nabla v|^{n(x)} - \frac{1}{p(x)}|uv|^{p(x)}\right)dx. \tag{3.5}$$

We then define

$$F(t) = \|u\|^2 + \|\Delta u\|^2 + \|v\|^2 + \|\Delta v\|^2. \tag{3.6}$$

Multiplying the first equation of (1.1) by u,

$$\int_\Omega uu_t dx + \int_\Omega \Delta u \Delta u_t dx + \int_\Omega |\nabla u|^{m(x)} dx = \int_\Omega |uv|^{p(x)} dx, \tag{3.7}$$

and the second equation of (1.1) by v,

$$\int_\Omega vv_t dx + \int_\Omega \Delta v \Delta v_t dx + \int_\Omega |\nabla v|^{n(x)} dx = \int_\Omega |uv|^{p(x)} dx. \tag{3.8}$$

Adding (3.7) and (3.8), we have

$$\begin{aligned} &\int_\Omega uu_t dx + \int_\Omega \Delta u\Delta u_t dx + \int_\Omega vv_t dx + \int_\Omega \Delta v \Delta v_t dx \\ = &-\int_\Omega \left(|\nabla u|^{m(x)} + |\nabla v|^{n(x)}\right) dx + 2\int_\Omega |uv|^{p(x)} dx. \end{aligned} \tag{3.9}$$

By differentiating $F(t)$, we obtain

$$\begin{aligned} F'(t) &= 2\int_\Omega uu_t dx + 2\int_\Omega \Delta u \Delta u_t dx + 2\int_\Omega vv_t dx + 2\int_\Omega \Delta v\Delta v_t dx \\ &= 4\int_\Omega |uv|^{p(x)} dx - 2\int_\Omega \left(|\nabla u|^{m(x)} + |\nabla v|^{n(x)}\right) dx \\ &= 4\int_\Omega p(x)\left[\frac{|uv|^{p(x)}}{p(x)} - \left(\frac{|\nabla u|^{m(x)}}{m(x)} + \frac{|\nabla v|^{n(x)}}{n(x)}\right)\right] dx \\ &\quad +4\int_\Omega p(x)\left(\frac{1}{m(x)} - \frac{1}{p(x)}\right)|\nabla u|^{m(x)} dx \\ &\quad +4\int_\Omega p(x)\left(\frac{1}{n(x)} - \frac{1}{p(x)}\right)|\nabla v|^{n(x)} dx + 2\int_\Omega \left(|\nabla u|^{m(x)} + |\nabla v|^{n(x)}\right) dx. \end{aligned} \tag{3.10}$$

Since $E'(t) \leq 0$, we get

$$\begin{aligned} &\int_\Omega p(x)\left[\frac{|uv|^{p(x)}}{p(x)} - \left(\frac{|\nabla u|^{m(x)}}{m(x)} + \frac{|\nabla v|^{n(x)}}{n(x)}\right)\right] dx \\ \geq &\int_\Omega p(x)\left[\frac{|u_0v_0|^{p(x)}}{p(x)} - \left(\frac{|\nabla u_0|^{m(x)}}{m(x)} + \frac{|\nabla v_0|^{n(x)}}{n(x)}\right)\right] dx \\ \geq &\int_\Omega p^-\left[\frac{|u_0v_0|^{p(x)}}{p(x)} - \left(\frac{|\nabla u_0|^{m(x)}}{m(x)} + \frac{|\nabla v_0|^{n(x)}}{n(x)}\right)\right] dx \\ \geq &\ 0. \end{aligned} \tag{3.11}$$

Therefore (3.10) takes the form

$$\begin{aligned} F'(t) &\geq 4\int_\Omega p^-\left(\frac{1}{m^+} - \frac{1}{p^-}\right)|\nabla u|^{m(x)} dx + 4\int_\Omega p^-\left(\frac{1}{n^+} - \frac{1}{p^-}\right)|\nabla v|^{n(x)} dx \\ &\quad +2\int_\Omega \left(|\nabla u|^{m(x)} + |\nabla v|^{n(x)}\right) dx \\ &= C_1\int_\Omega |\nabla u|^{m(x)} dx + C_2\int_\Omega |\nabla v|^{n(x)} dx, \end{aligned}$$

where $C_1 = 2 + 4p^-\left(\frac{1}{m^+} - \frac{1}{p^-}\right)$, $C_2 = 2 + 4p^-\left(\frac{1}{n^+} - \frac{1}{p^-}\right)$. We define the sets

$$\Omega_+ = \{x \in \Omega \mid |\nabla u| \geq 1, |\nabla v| \geq 1\} \text{ and } \Omega_- = \{x \in \Omega \mid |\nabla u| < 1, |\nabla v| < 1\}.$$

By the embedding of $L^r(\Omega) \hookrightarrow L^2(\Omega)$ for all $r \geq 2$ ($\|\nabla u\|_2 \leq C\|\nabla v\|_r$), we get

$$\begin{aligned} F'(t) &\geq C_1\left(\int_{\Omega_-} |\nabla u|^{m^+} dx + \int_{\Omega_+} |\nabla u|^{m^-} dx\right) + C_2\left(\int_{\Omega_-} |\nabla v|^{n^+} dx + \int_{\Omega_+} |\nabla v|^{n^-} dx\right) \\ &\geq C_3\left[\int_{\Omega_-}\left(|\nabla u|^2 dx\right)^{\frac{m^+}{2}} + \int_{\Omega_+}\left(|\nabla u|^2 dx\right)^{\frac{m^-}{2}}\right] \\ &\quad + C_4\left[\int_{\Omega_-}\left(|\nabla v|^2 dx\right)^{\frac{n^+}{2}} + \int_{\Omega_+}\left(|\nabla v|^2 dx\right)^{\frac{n^-}{2}}\right]. \end{aligned}$$

This means that

$$\begin{cases} (F'(t))^a \geq C_5\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \geq 0, \\ (F'(t))^b \geq C_6\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \geq 0, \end{cases} \tag{3.12}$$

where $a = \max\left\{\frac{2}{m^+}, \frac{2}{n^+}\right\}$, $b = \max\left\{\frac{2}{m^-}, \frac{2}{n^-}\right\}$ .The Poincare inequality gives $\|u\|^2 \leq \frac{1}{\lambda_1}\|\nabla u\|^2$, where λ_1 is the first eigenvalue of the problem

$$\begin{cases} \Delta w = -\lambda w, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

Thus, we obtain

$$\begin{cases} \|\nabla u\|^2 = \frac{1}{1+\lambda_1}\|\nabla u\|^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla u\|^2 \\ \qquad \geq \frac{\lambda_1}{1+\lambda_1}\|u\|^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla u\|^2 \\ \qquad = \frac{\lambda_1}{1+\lambda_1}\|u\|^2_{H_0^1}, \\ \\ \|\nabla v\|^2 = \frac{1}{1+\lambda_1}\|\nabla v\|^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla v\|^2 \\ \qquad \geq \frac{\lambda_1}{1+\lambda_1}\|v\|^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla v\|^2 \\ \qquad = \frac{\lambda_1}{1+\lambda_1}\|v\|^2_{H_0^1}. \end{cases} \tag{3.13}$$

Combining (3.12) and (3.13) yields

$$(F'(t))^a \geq \frac{C_5\lambda_1}{1+\lambda_1}\left(\|u\|^2_{H_0^1} + \|v\|^2_{H_0^1}\right),$$

and

$$(F'(t))^b \geq \frac{C_6\lambda_1}{1+\lambda_1}\left(\|u\|^2_{H_0^1} + \|v\|^2_{H_0^1}\right).$$

Thus, adding the above inequalities, we obtain

$$(F'(t))^a + (F'(t))^b \geq \frac{\lambda_1(C_5 + C_6)}{1+\lambda_1}\left(\|u\|^2_{H_0^1} + \|v\|^2_{H_0^1}\right) = C_7F(t), \tag{3.14}$$

this implies

$$(F'(t))^b\left(1 + (F'(t))^{a-b}\right) \geq C_7F(t). \tag{3.15}$$

By (3.14) and the fact that $F(t) \geq F(0) > 0$ (since $F'(t) \geq 0$), we get

$$(F'(t))^a \geq \frac{C_7}{2}F(t) \geq \frac{C_7}{2}F(0),$$

and

$$(F'(t))^b \geq \frac{C_7}{2}F(t) \geq \frac{C_7}{2}F(0).$$

This implies that

$$F'(t) \geq C_8 (F(0))^{\frac{1}{a}},$$

and

$$F'(t) \geq C_9 (F(0))^{\frac{1}{b}}.$$

Hence $F'(t) \geq \alpha$, where $\alpha = \min\left\{C_8 (F(0))^{\frac{1}{a}}, C_9 (F(0))^{\frac{1}{b}}\right\}$. By te (1.3), it is easy to see $a - b \leq 0$. We have

$$F'(t) \geq \beta (F(t))^{\frac{1}{b}}, \tag{3.16}$$

where $\beta = \left(\frac{C_7}{1+\alpha^{a-b}}\right)^{\frac{1}{b}}$ is the constant. Consequently, we have

$$\frac{F'(t)}{(F(t))^{\frac{1}{b}}} \geq \beta. \tag{3.17}$$

A simple integrating the of (3.17) over $(0, t)$, then yields

$$(F(t))^{1-\frac{1}{b}} \leq (F(0))^{1-\frac{1}{b}} + \frac{(b-1)\beta t}{b}, \tag{3.18}$$

$$F(t) \geq \frac{1}{\left[(F(0))^{1-\frac{1}{b}} + \frac{(b-1)\beta t}{b}\right]^{\frac{b}{1-b}}}. \tag{3.19}$$

Therefore, (3.19) shows that $F(t)$ blows up at some finite time

$$T^* \leq \frac{b (F(0))^{1-\frac{1}{b}}}{(b-1)\beta}. \tag{3.20}$$

Remark 3.2. The larger $F(0)$ is the quicker the blow up takes place.

□

4 Lower bound for blow-up time

In this section, we shall prove the lower bound for blow up time of solutions to system (1.1).

Theorem 4.1. *Assume that (2.1) and (1.2) hold. Assume further that*

$$\begin{cases} 2 < p^+ \ \ if \ n \leq 2, \\ 2 < p^+ \leq \frac{2n}{n-2} \ \ if \ n \geq 3, \end{cases}$$

$u_0 \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$, $v_0 \in W_0^{1,n(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$ *and the solution* (u, v) *of the system (1.1) becomes unbounded at finite time* T^* *in* $H_0^1(\Omega)$*-norm, then a lower bound* T^* *for blow-up time is given by*

$$T^* \geq \int_{F(0)}^{\infty} \frac{d\eta}{M\eta^{p^+} + N\eta^{p^-}}, \tag{4.1}$$

where M *and* N *are suitable positive constants given later.*

Proof. We define the function $F(t)$ the same as (3.6). By (3.10), we have

$$\begin{aligned} F'(t) &= 2\int_\Omega u u_t dx + 2\int_\Omega \Delta u \Delta u_t dx + 2\int_\Omega v v_t dx + 2\int_\Omega \Delta v \Delta v_t dx \\ &\leq 4\int_\Omega |uv|^{p(x)} dx. \end{aligned} \tag{4.2}$$

We define the sets

$$\Omega_+ = \{x \in \Omega \,|\, |uv| \geq 1\} \ \text{ and } \ \Omega_- = \{x \in \Omega \,|\, |uv| < 1\}.$$

Thanks to the Cauchy-Schwarz and the Sobolev embedding inequalities, we obtain

$$\begin{aligned}\int_\Omega |uv|^{p(x)}\,dx &\le \int_{\Omega_+} |uv|^{p^+}\,dx + \int_{\Omega_-} |uv|^{p^-}\,dx \\ &\le \int_\Omega |uv|^{p^+}\,dx + \int_\Omega |uv|^{p^-}\,dx \\ &\le \left(\int_\Omega |u|^{2p^+}\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^{2p^+}\,dx\right)^{\frac{1}{2}} + \left(\int_\Omega |u|^{2p^-}\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^{2p^-}\,dx\right)^{\frac{1}{2}} \\ &\le \left(B_+^{p^+}\right)^2 \|\nabla u\|^{p^+} \|\nabla v\|^{p^+} + \left(B_-^{p^-}\right)^2 \|\nabla u\|^{p^-} \|\nabla v\|\, p^-, \end{aligned} \tag{4.3}$$

where B_+, B_- are the Sobolev embedding constants for $H_0^1(\Omega) \hookrightarrow L^{p^+}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^{p^-}(\Omega)$, respectively. By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}(F'(t))^2 &\ge \left(\int_\Omega |\nabla u|^2\,dx\right)^2 + \left(\int_\Omega |\nabla v|^2\,dx\right)^2 \\ &\ge 2\int_\Omega |\nabla u|^2\,dx \int_\Omega |\nabla v|^2\,dx.\end{aligned}$$

Then

$$(F'(t))^{p^+} \ge 2^{\frac{p^+}{2}} \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p^+}{2}} \left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{p^+}{2}},$$

and

$$(F'(t))^{p^-} \ge 2^{\frac{p^-}{2}} \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p^-}{2}} \left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{p^-}{2}},$$

which implies that

$$2^{-\frac{p^+}{2}} (F'(t))^{p^+} \ge \|\nabla u\|^{p^+} \|\nabla v\|^{p^+}, \tag{4.4}$$

and

$$2^{-\frac{p^-}{2}} (F'(t))^{p^-} \ge \|\nabla u\|^{p^-} \|\nabla v\|^{p^-}. \tag{4.5}$$

Combining (3.20) and (4.5) yields

$$F'(t) \le M (F(t))^{p^+} + N (F(t))^{p^-},$$

where $M = 2^{-\frac{p^+}{2}} \left(B_+^{p^+}\right)^2, N = 2^{-\frac{p^-}{2}} \left(B_+^{p^-}\right)^2$. Therefore

$$\frac{F'(t)}{M (F(t))^{p^+} + N (F(t))^{p^-}} \le 1. \tag{4.6}$$

A simple integrating the of (4.6) over $(0,t)$, then yields

$$\int_{F(0)}^{F(t)} \frac{d\eta}{M\eta^{p^+} + N\eta^{p^-}} \le t.$$

If (u,v) blows up in $H_0^1(\Omega)$ norm, then we obtain a lower bound T^* given by

$$T^* \ge \int_{F(0)}^{\infty} \frac{d\eta}{M\eta^{p^+} + N\eta^{p^-}}.$$

Clearly, the integral is bound since exponents $p^+ \ge p^- > 2$. □

Conclusion

In this paper, we considered the lower and upper bounds for the blow up time of solutions for a generalized heat equations with variable exponent in a bounded domain. This improves many results in the literature.

References

[1] M.K. Alaoui, S.A. Messaoudi and H.B. Khenous, A blow-up result for nonlinear generalized heat equation, *Comput. Math. Appl.,* **68**, 1723-1732 (2014).

[2] X. Bai and S. Zheng, A semilinear parabolic system with coupling variable exponents, *Annali di Matematica,* **190,** 525-537 (2011).

[3] K. Baghaei, M.B. Ghaemi and M. Hasaaraki, Lower bounds for the blow-up time in a semilinear parabolic problem involving a variable source, *Appl.Math. Lett.,* **27**, 49-52 (2014).

[4] Y. Chen, S. Levine and M. Rao, Variable Exponent, Linear Growth Functionals in Image Restoration, *SIAM J. Appl. Math.,* **66**, 1383-1406 (2006).

[5] H. Di, Y. Shang and X. Peng, Blow up phenomena for a pseuo-parabolic equation with variable exponents, *Appl.Math. Lett.,* **64**, 67-73 (2017).

[6] L. Diening, P. Hasto, P. Harjulehto and M.M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents,* Springer-Verlag, 2011.

[7] X.L. Fan, J.S. Shen and D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.,* **263**, 749-760 (2001).

[8] W. Gao and B. Gao, Existence and asymptotic behaviour of solutions for a viscous $p(x)$-Laplacian equation, *Appl. Anal.,* **91**, 879-894 (2012).

[9] A. Khelghati and K. Baghaei, Blow-up in a semilinear parabolic problem with variable source under positive initial energy, *Appl. Anal.,* **94**, 1888-1896 (2015).

[10] O. Kovacik and J. Rakosnik, On spaces $L^{p(x)}(\Omega)$, and $W^{k,p(x)}(\Omega)$, *Czech. Math. J.,* **41**, 592-618 (1991).

[11] X. Li, B. Gou and M. Liao, Asymptotic stability of solutions to quasilinear hyperbolic equations with variable sources, *Comput. Math. Appl.,* **79**, 1012-1022 (2020).

[12] M. Liao, Non-global existence of solutions to pseudo-parabolic equations with variable exponents and positive initial energy, *C. R.Mecanique,* **347**, 710-715 (2019).

[13] J.B. Pinasco, Blow-up for parabolic and hyperbolic problems with variable exponents, *Nonlinear Anal.,* **71**, 1094-1099 (2009).

[14] E. Pişkin, *Sobolev Spaces,* Seçkin Publishing, 2017. (in Turkish).

[15] Q. Qi, Y. Chen and Q. Wang, Blow up phenomena for a pseudo-parabolic system with variable exponents, *Electron. J. Qual. Theory Differ. Equ.,* **36**, 1-9 (2017).

[16] M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory,* Lecture Notes in Mathematics, Springer, 2000.

[17] H. Wang and Y. He, On blow-up of solutions for a semilinear parabolic equation involving variable source and positive initial energy, *Appl.Math. Lett.,* **26**, 1008-1012 (2013).

[18] X. Wu, The blow-up of solutions for m-Laplacian equations with variable sources under positive initial energy, *Comput. Math. Appl.,* **72**, 2516-2524 (2016).

[19] X. Wu, B. Guo, W. Gao, Blow-up of solutions for a semilinear parabolic equation involving variable source and positive initial energy, *Appl.Math. Lett.,* **26**, 539-543 (2013).

[20] X. Zhu, B. Guo, M. Liao, Global existence and blow-up of weak solutions for a pseudo-parabolic equation with high initial energy, *Appl.Math. Lett.,* **104**, 1-7 (2020).

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