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A fixed point theorem on weak contraction condition (B) in complete metric spaces with w- distance

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Abstract. In this paper, establish a fixed point theorem for a weak contraction condition (B) map on a complete metric spaces endowed with w- distance. Presented fixed point theorem generalize some results existing in the literature.

1 Introduction and Preliminaries

The classical Banach's contraction principle [2] is one of the most useful results in fixed point theory. But suffer from one drawback the contractive condition forces to be continuous. In 1969, Kannan [11] proved a fixed point theorem for a map satisfying a contractive condition that didn't require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next three decades.

On the other hand, Berinde[4] introduced the concept of almost contraction and proved some fixed point theorems for almost contractions in complete metric spaces. This concept by Berinde in [3] was called weak contraction, but in [4], Berinde renamed it as almost contraction which is appropriate. In [3], Berinde shows that any Banach, Kannan, Chatterjea and Zamfirescu mappings are weak contraction. The latter has been studied in some other papers [5, 6, 7] for the case of both single valued and multi valued mappings.

Very recently, Babu *et al.*[1] considered the class of mappings that satisfy condition (B) and proved the existence of fixed point theorem for such mappings on complete metric spaces. They discussed in details about quasi-contraction, almost contraction and the class of mappings that satisfy condition (B).

Branciari [8] established a fixed point result for an integral-type inequality, which is a generalization of Banach contraction principle. Vijayaraju et al. [12] obtained a general principle, which made it possible to prove many fixed point theorems for pairs of integral type maps. In 1996, Kada *et al.*[10] introduced and studies the concept of w-distance on a metric space and also give some examples of w-distances and improved Caristi's fixed point theorem, Ekeland's ϵ-variational's principle, and the convex minimization theorem according to Takahashi.

The following definition is the concept of w-distance on metric space.

Definition 1.1. [10] Let X be a metric space endowed with a metric d. A function $p : X \times X \to [0, \infty)$ is called a w-distance on X if it satisfies the following properties:

(w_1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;

(w_2) p is lower semi-continuous in its second variable;

(w_3) for each $\epsilon > 0$, there exists a $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Example 1.2. [10] Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = c$ for every $x, y \in X$ is a w-distance on X, where c is a positive real number. But p is not a metric since $p(x, x) = c(> 0)$ for any $x \in X$.

Lemma 1.3. [*10*] *Let (X, d) be a metric space and p be a w-distance on metric space X,*

(i) *if $\{x_n\}$ is a sequence in X such that $\lim_n p(x_n, x) = \lim_n p(x_n, y) = 0$ then $x = y$,*

(ii) *if* $p(x_n, y_n) \leq \alpha_n, \ \ p(x_n, y) \leq \beta_n$ *for any* $n \in N$*, where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, \infty)$ *converging to* 0*, then* $\{y_n\}$ *converges to* y*,*

(iii) *let* p *be a* w*-distance on a metric space* (X, d) *and* $\{x_n\}$ *be a sequence in* X *such that, for each, for each* $\epsilon > 0$*, there exists an* $N_\epsilon \in N$ *such that* $m > n > N_\epsilon$ *implies* $p(x_n, x_m) < \epsilon$ *(or* $\lim_{m,n} \ p(x_n, x_m) = 0$*), then* $\{x_n\}$ *is a Cauchy sequence.*

If $p(a, b) = p(b, a) = 0$ and $p(a, a) \leq p(a, b) + p(b, a) = 0$ and by (i) of Lemma 1.3, $a = b$.

Definition 1.4. Let (X, d) be a metric space. A map $T : X \to X$

(a) is called a **weak contraction**(or (δ, L)-weak contraction) [3] if there exist a constant $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L\, d(y, Tx), \quad for\ all\ x, y \in X \tag{1.1}$$

(b) is called a **almost contraction** [4] if there exist a constant $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L\, d(x, Tx), \quad for\ all\ x, y \in X \tag{1.2}$$

(c) is called **quasi-contraction**[9] if there exists $h \in (0, 1)$ and for any $x, y \in X$ such that

$$d(Tx, Ty) \leq h\ max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{1.3}$$

(d) is said to **satisfy condition** (**B**)[1] if there exists $0 < \delta < 1, \ L \geq 0$ and for any $x, y \in X$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{1.4}$$

Remark 1.5. [1] The following observations are very interesting

(i) inequality (1.1) and (1.2) are independent,

(ii) inequality (1.1) and (1.3) are independent,

(iii) inequality (1.4) implies that (1.1) and (1.2), but its converse not true,

(iv) inequality (1.4) need not be continuous.

The aim of this paper is to establish a fixed point theorem defined on complete metric space with w-distance and using contractive condition (B).

2 A fixed point theorem

Theorem 2.1. *Let* p *be a* w*-distance on a complete metric space* (X, d) *such that* $p(x, x) = 0 \ \forall x \in X$*. Let* T *be a selfmap on* X *satisfying the condition if there exists a* $\delta \in (0, 1)$ *and some* $L \geq 0$ *such that*

$$p(Tx, Ty) \leq \delta p(x, y) + L \min\{p(x, Tx), p(x, Ty), p(y, Ty), p(y, Tx)\} \tag{2.1}$$

$\forall x, y \in X$*. Then* T *has a unique fixed point in* X*.*

Proof. Let x_0 be any arbitrary point in X and define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for each $n \geq 0$. It is easy to observe that, if $x_{n+1} = x_n$ for some n, then x_n is a fixed point of T. Therefore, we assume that $x_{n+1} \neq x_n$ for each n. Now taking $x = x_n$ and $y = x_{n-1}$ in (2.1), we obtain that

$$\begin{aligned} p(x_{n+1}, x_n) &= p(Tx_n, Tx_{n-1}) \\ &\leq \delta\, p(x_n, x_{n-1}) + L\ \min\{p(x_n, Tx_n), p(x_n, Tx_{n-1}), p(x_{n-1}, Tx_{n-1}), p(x_{n-1}, Tx_n)\} \\ &= \delta\, p(x_n, x_{n-1}) + L\ \min\{p(x_n, x_{n+1}), p(x_n, x_n), p(x_{n-1}, x_n), p(x_{n-1}, x_{n+1})\} \\ &\leq \delta\, p(x_n, x_{n-1}) \leq \cdots \leq \delta^n p(x_1, x_0), \end{aligned}$$

letting $n \to \infty$, we get

$$p(x_{n+1}, x_n) \to 0 \; as \; n \to \infty. \tag{2.2}$$

Similarly, it can be shown that $p(x_n, x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty$.

Now, we have to show that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

Suppose not. there is a $\epsilon > 0$ and two subsequences $\{m_k\}, \{n_k\}$ of $\{x_n\}$ such that

$$p(x_{n_k}, x_{m_k}) \geq \epsilon, where \;\; m_k > n_k. \tag{2.3}$$

By (2.2) there exists $k_0 \in N$ such that $n_k > k_0$ implies that

$$p(x_{n_k}, x_{n_{k+1}}) < \epsilon.$$

If $n_k > k_0$ by (2.3), $m_k \neq n_{k+1}$. We can assume that m_k is a minimal index such that

$$p(x_{n_k}, x_{m_k}) \geq \epsilon \;\; but \;\; p(x_{n_k}, x_h) < \epsilon, \quad h \in \{n_{k+1}, \cdots, m_{k-1}\}.$$

We have

$$\begin{aligned} \epsilon \leq p(x_{n_k}, x_{m_k}) &\leq p(x_{n_k}, x_{m_k-1}) + p(x_{m_k-1}, x_{m_k}) \\ &< \epsilon + p(x_{m_k-1}, x_{m_k}), \end{aligned}$$

letting $k \to \infty$ this implies that

$$\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = \epsilon. \tag{2.4}$$

Now, we have to show that $\lim_{k\to\infty} p(x_{n_k+1}, x_{m_k+1}) = \epsilon$.

$p(x_{n_k+1}, x_{m_k+1}) \leq p(x_{m_k+1}, x_{n_k}) + p(x_{n_k}, x_{m_k}) + p(x_{m_k}, x_{m_k+1})$.

Letting $k \to \infty$ and using (2.2) and (2.4), then we obtain that

$$\lim_{k\to\infty} p(x_{n_k+1}, x_{m_k+1}) \leq \epsilon. \tag{2.5}$$

Now, $p(x_{n_k}, x_{m_k}) \leq p(x_{n_k}, x_{n_k+1}) + p(x_{n_k+1}, x_{m_k+1}) + p(x_{m_k+1}, x_{m_k})$

by using (2.2) and letting $k \to \infty$, the get

$$\epsilon \leq \lim_{k\to\infty} p(x_{n_k+1}, x_{m_k+1}). \tag{2.6}$$

From (2.5) and (2.6), we have

$$\lim_{k\to\infty} p(x_{n_k+1}, x_{m_k+1}) = \epsilon. \tag{2.8}$$

Now, from (2.1), (2.2) and (2.4), we have

$$\begin{aligned} p(x_{n_k+1}, x_{m_k+1}) &= p(Tx_{n_k}, Tx_{m_k}) \\ &\leq \delta\, p(x_{n_k}, x_{m_k}) + L \min\{p(x_{n_k}, x_{n_k+1}), p(x_{n_k}, x_{m_k+1}), \\ &\qquad p(x_{m_k}, x_{m_k+1}), p(x_{m_k}, x_{n_k+1})\}. \end{aligned}$$

On taking limit as $k \to \infty$, we obtain that $\epsilon \leq \delta\epsilon < \epsilon$, a contradiction. Thus

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.8}$$

By (iii) of Lemma 1.3, $\{x_n\}$ is a cauchy sequence and since X is complete, there exist a point $t \in X$ such that $x_n \to t$ as $n \to \infty$.

Finally, we have to show that t is fixed point of T. By (2.8), for each $\epsilon > 0$ there exists an $N_\epsilon \in N$ such that $n > N_\epsilon$ implies $p(x_N, x_n) < \epsilon$. But $x_n \to t$ and $p(x, .)$ is a lower continuous. Thus

$p(x_{N_\epsilon}, t) \leq \liminf_{n\to\infty} p(x_{N_\epsilon}, x_n) \leq \epsilon.$

Therefore $p(x_{N_\epsilon}, t) \leq \epsilon$. Set $\epsilon = \frac{1}{k}, N_\epsilon = n_k$ and we have

$$\lim_{k\to\infty} p(x_{n_k}, t) = 0. \tag{2.9}$$

By the Definition 1.1, we can write

$p(x_{n_k}, Tt) \leq p(x_{n_k}, x_{n_k+1}) + p(x_{n_k+1}, Tt).$

Again by using the condition (2.1), we get

$$\begin{aligned} p(x_{n_k}, Tt) &\leq p(x_{n_k}, x_{n_k+1}) + p(Tx_{n_k}, Tt) \\ &\leq p(x_{n_k}, x_{n_k+1}) + \delta p(x_{n_k}, t) + L \min\{p(x_{n_k}, Tx_{n_k}), p(x_{n_k}, Tt), \\ &\qquad p(t, Tt), p(t, Tx_{n_k})\} \\ &= p(x_{n_k}, x_{n_k+1}) + \delta p(x_{n_k}, t) + L \min\{p(x_{n_k}, x_{n_k+1}), p(x_{n_k}, Tt), \\ &\qquad p(t, Tt), p(t, x_{n_k+1})\}. \end{aligned}$$

Letting $k \to \infty$ and by using (2.2) and (2.9), we obtain

$$\lim_{k\to\infty} p(x_{n_k}, Tt) = 0. \tag{2.10}$$

Hence by Lemma 1.3 and (2.9), (2.10) we conclude that $t = Tt$.

Uniqueness: Let t be fixed point of T. Assume that r be another fixed point of T. Now, we have to show that $t = r$. Suppose not, $p(t, r) \neq 0$.

Consider $x = t$ and $y = r$ in (2.1), then we get

$$\begin{aligned} 0 < p(t, r) = p(Tt, Tr) &\leq \delta\, p(t, r) + L\ \min\{p(t, Tt), p(t, Tr), p(r, Tr), p(r, Tt)\} \\ &= \delta\, p(t, r) < p(t, r), \end{aligned}$$

a contradiction. Thus $p(t, r) = 0$. Similarly, we get $p(r, t) = 0$.

Then by (i) of Lemma 1.3, we get $t = r$. □

Here we give a simple example illustrating Theorem 2.1.

Example 2.2. Let $X = [0, 1]$ which is a complete metric space with usual metric d of reals. Moreover, by defining $p(x, y) = y$, if $x \neq y$ and $p(x, y) = 0$, if $x = y$, p is a $w-$distance on (X, d) and T be a self map on X defined by $Tx = \frac{x}{2}, \forall x \in X$. It is easy to verify that the condition (2.1) holds with $\delta = \frac{2}{3}$ and $L = 1$. We note that 0 is a fixed point of T. And also observe that $p \neq d$.

The following example shows that the condition $p(x, x) = 0$ is necessary in Theorem 2.1.

Example 2.3. Let $X = [0, 1]$ which is a complete metric space with usual metric d of reals and define $p(x, y) = \frac{1}{2}, \forall x, y \in X$, p is $w-$ distance on (X, d). Let T be a self map on X defined by $Tx = 1$, if $x = 0$ and $Tx = \frac{x}{2}$, if $x \neq 0$. Then the condition (2.1) of Theorem 2.1 is satisfies with for any $\delta \in (0, 1)$ and $L = 1$, but $p(x, x) \neq 0$ for any $x \in X$. Clearly, T possesses a no fixed point in X.

Open problem: *What further conditions are necessary, if $p(x, x) = 0$ for all $x \in X$ is removed in Theorem 2.1.*

3 Some Applications

Denote by Ω the set of functions $\gamma : R^+ \to R^+$ satisfying the following conditions:

a) γ is a Lebesgue integrable mapping on each compact subset of R^+,

b) for every $\epsilon > 0$, we have $\int_0^\epsilon \gamma(s)\, ds > 0$.

Theorem 3.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the condition if there exists a $\delta \in (0, 1)$ and some $L \geq 0$ such that*

$$\int_0^{d(Tx,Ty)} \gamma(s)\, ds \leq \delta \int_0^{d(x,y)} \gamma(s)\, ds + L \int_0^{m(x,y)} \xi(s)\, ds$$

for all $x, y \in X$, where $\gamma, \xi \in \Omega$. Then T has a unique fixed point.(where $m(x, y)$= second part of RHS term in (2.1))

Proof. The function $t \in [0, \infty) \mapsto \int_0^t \alpha(s)\, ds$ and the function $t \in [0, \infty) \mapsto \int_0^t \beta(s)\, ds$ belongs to Ω. Now, in Theorem 2.1, set $p = d$.

Taking $\xi(s) = 0$ in Theorem 3.1. We obtain the following result.

Corollary 3.2. *[8] Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the condition if there exists a $\delta \in (0, 1)$ such that*

$$\int_0^{d(Tx,Ty)} \gamma(s)\, ds \leq \delta \int_0^{d(x,y)} \gamma(s)\, ds$$

for all $x, y \in X$, where $\gamma \in \Omega$ and $\delta \in [0, 1)$. Then T has a unique fixed point.

Taking $\gamma(s) = 1$ in corollary 3.2, then we get the following.

Corollary 3.3. *[2] Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying*

$$d(Tx, Ty) \leq \delta d(x, y)$$

for all $x, y \in X$, and there exists $\delta \in (0, 1)$. Then T has a unique fixed point.

References

[1] G.V.R.Babu, M.L.Sandhya, M.V.R.Kameswari, A note on fixed point theorem of Berinde on weak contractions, *Carpathian J. Math*. 24 (1) (2008) 08-12.

[2] S. Banach, Sur les operations dans les ensembles abstraits et leur application auxequations integerales, *Fund. Math*. 3 (1922), 133-181.

[3] V.Berinde, Approximating fixed points of weak conractions using the Picard iteration, *Nonlinear Anal. Forum* 9(1) (2004) 43-53.

[4] V.Berinde, General constructive fixed point theorems for Ciric-type almost contractions in metric spaces, *Carpathian J. math.* 24 (2) (2008) 10-19.

[5] V.Berinde, Common fixed points of noncommuting almost contractions in cone metric spaces, *Math. Commun*. 15 (1) (2010) 229-241.

[6] V.Berinde, Approximating common fixed points of noncommuting almost contractions in metric spaces, *Fixed Point Theory,* 11 (2) (2010) 179-188.

[7] V.Berinde, Approximating common fixed points of noncommuting discontinuous weakly contractive mappings in metric spaces, *Carpathian J. Math.* 25 (1) (2009), 13-22.

[8] A. Branciari, A fixed point theorem for mapping satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci*. 29 (2002) 9 531-536

[9] Lj. B. Ciric. A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.*, 45 (1974), 267-273.

[10] O. Kada, T. Suzuki, and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, *Mathematica Japonica*, vol. 44, no. 2, pp. 381-391, 1996.

[11] R.Kannan, Some results on fixed points, *Bull. Calcutta Math.Soc.,* 10 (1968) 71-76.

[12] P.Vijayaraju, B.E.Rhoades, R. Mohanraj, A fixed point theorem for a pair of maps saisfying a general contractive condition of integeral type, *Int. J. Math. Math. Sci.* 15 (2005) pp. 2350-2364.

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