Palestine Journal of Mathematics

Vol. 10(2)(2021) , 614–624

ON WEIGHTED FRACTIONAL INEQUALITIES USING HADAMARD FRACTIONAL INTEGRAL OPERATOR

Asha B. Nale[1], Satish K. Panchal[2] and Vaijanath L. Chinchane[3]

Communicated by Thabet Abdeljawad

MSC 2010 Classifications: 26A99, 26D10.

Keywords and phrases: Hadamard fractional integral operator, inequality and Weighted inequality.

The authors are thankful to the anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Abstract In this paper, firstly we introduced the Hadamard fractional integral operator whose kernel is of $\log(\frac{x}{t})$ instead of the form of $(x-t)$, which is involves both in the Riemann-Liouville and Caputo integral. In view of these, we obtain some new weighted fractional integral inequalities for positive and continuous function by employing Hadamard fractional integral operator.

1 Introduction

Fractional calculus has significant importance due to its application in various fields of science and engineering such as life sciences, chemical science and physical sciences. Fractional integral inequalities plays a very important role in different fields of mathematics, especially for continuous dependence solution and uniqueness of solution in fractional differential equation. In the recent decades, many mathematicians have studied on the different type of fractional integral inequalities and its applications by using the Riemann-Liouville, Erdelyi-Kober, Saigo, Hadamard, Atangana-Baleanu, generalized fractional integral, k-fractional integral operator and generalized k-fractional integral operator, see [1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 23, 24, 26, 28]. In [4, 5], Chinchane V. L. and Pachpatte D. B. established fractional integral inequalities for Chebychev and Extended Chebychev functional using the generalized Hadamard integral operator. In [18], Mohammed P.O and et al. fractional Hermite–Hadamard–Fejer inequalities for a convex function by using weighted fractional operators. Mohammed P.O and et al proposed few integral inequalities of Hermite–Hadamard's type for a σ-convex function with respect to an increasing function involving the ϕ-Riemann–Liouville fractional integral operator and have performed a connection between the Atangana–Baleanu and Riemann–Liouville fractional integrals of a function with respect to an increasing function with nonsingular kernel, see [19, 20, 21]. Recently, Mohammed P.O and et al. have works on midpoint and trapezoid type for twice differentiable convex functions in a form classical integral and Riemann-Liouville fractional integrals, see [22]. In [28], authors have proposed some new integral inequalities of Gruss type by using one or two parameters by employing Hadamard Fractional integral operators. W. Sudsuta et al. [27], Several new integral inequalities are obtained including the Gruss type Hadamard fractional integral inequality by considering Young and weighted AM-GM inequalities. In [15], Houas M. obtained certain weighted integral inequalities by considering the fractional hypergeometric operators. Motivated from above work aimed to establish some new weighted fractional integral inequalities by using Hadamard fractional integral operators. The paper has been organized as follows. In Section 2, we define basic definitions and proposition related to Hadamard fractional derivatives and integrals. In Section 3, we give weighted fractional integral inequalities by employing Hadamard fractional integral operator. In section 4, concluding remarks are given.

2 Preliminaries

The necessary details of fractional Hadamard calculus are given in the book A. A. Kilbas et al. [16], and in book of S. G. Samko et al. [25], here we present some definitions of Hadamard derivative and integral as given in [3].

Definition 2.1. The Hadamard fractional integral of order $\alpha \in R^+$ of function $f(x)$, for all $x > 1$ is defined as

$$ {}_H\mathcal{D}_{1,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x (\log \frac{x}{t})^{\alpha-1} f(t) \frac{dt}{t}, \tag{2.1}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2.2. The Hadmard fractional derivative of order $\alpha \in [n-1, n)$, $n \in Z^+$, of function $f(x)$ is given as follows

$$ {}_H\mathcal{D}_{1,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} (x\frac{d}{dx})^n \int_1^x (\log \frac{x}{t})^{n-\alpha-1} f(t) \frac{dt}{t}. \tag{2.2}$$

From the above definitions, we can see obviously the difference between Hadamard fractional and Riemann-Liouville fractional derivative and integrals, which include two aspects. The kernel in the Hadamard integral has the form of $\log(\frac{x}{t})$ instead of the form of $(x - t)$, which is involves both in the Riemann-Liouville and Caputo integral. The Hadamard derivative has the operator$(x\frac{d}{dx})^n$, whose construction is well suited to the case of the half-axis and is invariant relation to dilation [25], while the Riemann-Liouville derivative has the operator $(\frac{d}{dx})^n$.

We give some image formulas under the operator (2.1) and (2.2), which would be used in the derivation of our main result.

Proposition 2.1. [3] If $0 < \alpha < 1$, the following relation hold:

$$ {}_H\mathcal{D}_{1,x}^{-\alpha} (\log x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\log x)^{\beta+\alpha-1}, \tag{2.3}$$

$$ {}_H\mathcal{D}_{1,x}^{\alpha} (\log x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\log x)^{\beta-\alpha-1}, \tag{2.4}$$

respectively.
For the convenience of establishing the result, we give the semigroup property,

$$ ({}_H\mathcal{D}_{1,x}^{-\alpha})({}_H\mathcal{D}_{1,x}^{-\beta}) f(x) = {}_H\mathcal{D}_{1,x}^{-(\alpha+\beta)} f(x). \tag{2.5}$$

3 Weighted Fractional Integral Inequalities

Here, we obtain new some weighted fractional integral inequalities using Hadamard fractional integral operator.

Theorem 3.1. *Let z be positive and continuous functions on $[1, \infty)$, such that*

$$ (\sigma^\varrho z^\varrho(\tau) - \tau^\varrho z^\varrho(\sigma))(z^{\varpi-\lambda}(\tau) - z^{\varpi-\lambda}(\sigma)) \geq 0, \tag{3.1}$$

and $w : [1, \infty) \to \mathbb{R}^+$ be positive continuous function. Then for all $x > 1, \alpha, \varrho > 0, \varpi \geq \lambda > 0$, we have

$$ \begin{aligned} &{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) z^{\varrho+\lambda}(x)] \, {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) x^\varrho z^{\varpi}(x)] \\ &\leq {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) z^{\varrho+\varpi}(x)] \, {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) x^\varrho z^{\lambda}(x)]. \end{aligned} \tag{3.2}$$

Proof:- Since z be positive and continuous functions on $[1,\infty)$, then for all $\varrho > 0, \varpi \geq 0, \lambda > 0, \tau,\sigma \in (1,x), x > 1$. From (3.2), we have

$$\begin{aligned}&\sigma^{\varrho} z^{\varpi-\lambda}(\sigma) z^{\varrho}(\tau) + \tau^{\varrho} z^{\varpi-\lambda}(\tau) z^{\varrho}(\sigma)\\ &\leq \sigma^{\varrho} z^{\varpi+\varrho-\lambda}(\tau) + \tau^{\varrho} z^{\varpi+\varrho-\lambda}(\sigma).\end{aligned} \tag{3.3}$$

Again, multiplying both sides of (3.3) by $\frac{(\log\frac{x}{\tau})^{\alpha-1}}{\tau\Gamma(\alpha)} w(\tau) z^{\lambda}(\tau), \tau \in (1,x), x > 1$, then integrating resulting identity with respect to τ from 1 to x, we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_1^x (\log\frac{x}{\tau})^{\alpha-1}\sigma^{\varrho} z^{\varpi-\lambda}(\sigma) w(\tau) z^{\varrho+\lambda}(\tau)\frac{d\tau}{\tau}\\ &+\frac{1}{\Gamma(\alpha)}\int_1^x (\log\frac{x}{\tau})^{\alpha-1}\tau^{\varrho} z^{\varpi}(\tau) z^{\varrho}(\sigma) w(\tau)\frac{d\tau}{\tau}\\ &\leq \frac{1}{\Gamma(\alpha)}\int_1^x (\log\frac{x}{\tau})^{\alpha-1}\sigma^{\varrho} z^{\varpi+\varrho}(\tau) w(\tau)\frac{d\tau}{\tau}\\ &+\frac{1}{\Gamma(\alpha)}\int_1^x (\log\frac{x}{\tau})^{\alpha-1}\tau^{\varrho} z^{\varpi+\varrho-\lambda}(\sigma) w(\tau) z^{\lambda}(\tau)\frac{d\tau}{\tau},\end{aligned} \tag{3.4}$$

consequently,

$$\begin{aligned}&\sigma^{\varrho} z^{\varpi-\lambda}(\sigma)\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) z^{\varrho+\lambda}(x)] + z^{\varrho}(\sigma)\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) x^{\varrho} z^{\varpi}(x)]\\ &\leq \sigma^{\varrho}\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) z^{\varrho+\varpi}(x)] + z^{\varrho+\varpi-\lambda}(\sigma)\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) x^{\varrho} z^{\lambda}(x)].\end{aligned} \tag{3.5}$$

Multiplying both side of equation (3.5) by $\frac{(\log\frac{x}{\sigma})^{\alpha-1}}{\sigma\Gamma(\alpha)} w(\sigma) z^{\lambda}(\sigma), \sigma \in (1,x), x > 1$ which is positive, and integrating the obtain result with respective to σ from 1 to x, we have

$$\begin{aligned}&{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) z^{\varrho+\lambda}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) x^{\varrho} z^{\varpi}(x)]\\ &+\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) x^{\varrho} z^{\varpi}(x)]{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) z^{\varrho+\lambda}(x)]\\ &\leq\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) z^{\varrho+\varpi}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) x^{\varrho} z^{\lambda}(x)]\\ &+\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) x^{\varrho} z^{\lambda}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) z^{\varrho+\varpi}(x)],\end{aligned} \tag{3.6}$$

which completes the proof. Now, we give our main result.

Theorem 3.2. *Let z be a positive and continuous function on $[1,\infty)$ and satisfies (3.1). Let $w : [1,\infty) \to \mathbb{R}^+$ be positive continuous function. Then for all $x > 1, \alpha, \varrho > 0$, $\varpi \geq \lambda > 0$, we have*

$$\begin{aligned}&{}_H\mathcal{D}_{1,x}^{-\beta}[w(x) x^{\varrho} z^{\varpi}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) z^{\varrho+\lambda}(x)]\\ &+\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) x^{\varrho} z^{\varpi}(x)]{}_H\mathcal{D}_{1,x}^{-\beta}[w(x) z^{\varrho+\lambda}(x)]\\ &\leq\ {}_H\mathcal{D}_{1,x}^{-\beta}[w(x) x^{\varrho} z^{\lambda}(x)]{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) z^{\varrho+\varpi}(x)]\\ &+\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x) x^{\varrho} z^{\lambda}(x)]\ {}_H\mathcal{D}_{1,x}^{-\beta}[w(x) z^{\varrho+\varpi}(x)].\end{aligned} \tag{3.7}$$

Proof:- Multiplying both sides of (3.3) by $\frac{(\log\frac{x}{\sigma})^{\beta-1}}{\sigma\Gamma(\beta)} w(\sigma) z^{\lambda}(\sigma)$, ($\sigma \in (1,x)$, $x > 1$), this function remains positive under the conditions stated with the theorem. Integrating the obtain

result with respective to σ from 1 to x, we get

$$\begin{aligned}
&\frac{z^{\varrho}(\tau)}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\tau})^{\beta-1}w(\sigma)\sigma^{\varrho}z^{\varpi}(\sigma)\frac{d\sigma}{\sigma}\\
&+\frac{\tau^{\varrho}z^{\varpi-\lambda}(\tau)}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\tau})^{\beta-1}w(\sigma)z^{\varrho+\lambda}(\sigma)\frac{d\sigma}{\sigma}\\
&\leq\frac{z^{\varpi+\varrho-\lambda}(\tau)}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\tau})^{\beta-1}w(\sigma)\sigma^{\varrho}z^{\lambda}(\sigma)\frac{d\sigma}{\sigma}\\
&+\frac{\tau^{\varrho}}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\tau})^{\beta-1}w(\sigma)z^{\varpi+\varrho}(\sigma)\frac{d\sigma}{\sigma},
\end{aligned}\tag{3.8}$$

consequently

$$\begin{aligned}
&z^{\varrho}(\tau)\ {}_H\mathcal{D}_{1,x}^{-\beta}[w(x)x^{\varrho}z^{\varpi}(x)]+\tau^{\varrho}z^{\varpi-\lambda}(\tau)\ {}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z^{\varrho+\lambda}(x)]\\
&\leq z^{\varrho+\varpi-\lambda}(\tau)\ \ {}_H\mathcal{D}_{1,x}^{-\beta}[w(x)x^{\varrho}z^{\lambda}(x)]+\tau^{\varrho}\ {}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z^{\varrho+\varpi}(x)].
\end{aligned}\tag{3.9}$$

Multiplying both side of equation (3.9) by $\frac{(\log\frac{x}{\tau})^{\alpha-1}}{\tau\Gamma(\alpha)}w(\tau)z^{\lambda}(\tau),\tau\in(1,x),x>1$ which is positive, and integrating the obtain result with respective to τ from 1 to x, we get

$$\begin{aligned}
&{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)x^{\varrho}z^{\varpi}(x)]\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}w(\tau)z^{\varrho+\lambda}\frac{d\tau}{\tau}\\
&+_H\mathcal{D}_{1,x}^{-\beta}[w(x)z^{\varrho+\lambda}(x)]\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}w(\tau)\tau^{\varrho}z^{\varpi}(\tau)\frac{d\tau}{\tau}\\
&\leq_H\mathcal{D}_{1,x}^{-\beta}[w(x)x^{\varrho}z^{\lambda}(x)]\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}w(\tau)z^{\varpi+\varrho}(\tau)\frac{d\tau}{\tau}\\
&+_H\mathcal{D}_{1,x}^{-\beta}[w(x)z^{\varrho+\varpi}(x)]\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}w(\tau)\tau^{\varrho}z^{\lambda}(\tau)\frac{d\tau}{\tau},
\end{aligned}\tag{3.10}$$

This complete the proof of Theorem 3.2.

Theorem 3.3. *Let z and y be two positive and continuous functions on $[1,\infty)$, such that*

$$(y^{\varrho}(\sigma)z^{\varrho}(\tau)-y^{\varrho}(\tau)z^{\varrho}(\sigma))(z^{\varpi-\lambda}(\tau)-z^{\varpi-\lambda}(\sigma))\geq 0,\tag{3.11}$$

and let $w:[1,\infty)\to\mathbb{R}^+$ be positive continuous function.Then for all $x>1,\varrho>0$, $\varpi\geq\lambda>0$ we have

$$\begin{aligned}
&{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varrho+\lambda}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\varpi}(x)]\\
&\leq\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varrho+\varpi}(x)]_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\lambda}(x)].
\end{aligned}\tag{3.12}$$

Proof:- Let $(\tau,\sigma)\in(1,x)$, $x>1$, for any $\varpi>\lambda>0$, $\varrho>0$. From (3.11), we have

$$(y^{\varrho}(\sigma)z^{\varpi-\lambda}(\sigma)z^{\varrho}(\tau))+y^{\varrho}(\tau)z^{\varrho}(\sigma)z^{\varpi-\lambda}(\tau)\leq(y^{\varrho}(\sigma)z^{\varpi+\varrho-\lambda}(\tau)+y^{\varrho}(\tau)z^{\varpi+\varrho-\lambda}(\sigma).\tag{3.13}$$

Multiplying both sides of (3.13) by $\frac{(\log\frac{x}{\tau})^{\alpha-1}}{\tau\Gamma(\alpha)}w(\tau)z^{\lambda}(\tau)$, $\tau\in(1,x)$, $x>1$, then integrating resulting identity with respect to τ from 1 to x, we obtain

$$\begin{aligned}
&\frac{y^{\varrho}(\sigma)z^{\varpi-\lambda}(\sigma)}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)z^{\varrho+\lambda}(\tau)]\frac{d\tau}{\tau}\\
&+\frac{z^{\varrho}(\sigma)}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)y^{\varrho}(\tau)z^{\varpi}(\tau)]\frac{d\tau}{\tau}\\
&\leq\ \frac{\mathrm{h}^{\varrho}(\sigma)}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)z^{\varpi+\varrho}(\tau)]\frac{d\tau}{\tau}\\
&+\frac{z^{\varrho+\varpi-\lambda}(\sigma)}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)y^{\varrho}(\tau)z^{\lambda}(\tau)]\frac{d\tau}{\tau}.
\end{aligned}\tag{3.14}$$

Thus, we obtain

$$\begin{aligned} & y^{\varrho}(\sigma) z^{\varpi-\lambda}(\sigma)_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varrho+\lambda}(x)] + z^{\varrho}(\sigma)_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\varpi}(x)] \\ & \leq y^{\varrho}(\sigma)_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varpi+\varrho}(x)] + z^{\varrho+\varpi-\lambda}(\sigma)_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\lambda}(x)]. \end{aligned} \tag{3.15}$$

Multiplying both sides of (3.15) by $\frac{(\log\frac{x}{\sigma})^{\alpha-1}}{\sigma\Gamma(\alpha)} w(\sigma)z^{\lambda}(\sigma)$, then integrating the resulting inequality with respect to σ over $(1,x)$, we obtain

$$\begin{aligned} & _H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varrho+\lambda}(x)]\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\sigma})^{\alpha-1}w(\sigma)y^{\varrho}(\sigma)z^{\varpi}(\sigma)\frac{d\sigma}{\sigma} \\ & +_H \mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\varpi}(x)]\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\sigma})^{\alpha-1}z^{\varrho+\lambda}(\sigma)w(\sigma)\frac{d\sigma}{\sigma} \\ & \leq_H \mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varpi+\varrho}(x)]\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\sigma})^{\alpha-1}z^{\lambda}(\sigma)w(\sigma)y^{\varrho}(\sigma)\frac{d\sigma}{\sigma} \\ & +_H \mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\lambda}(x)]\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\sigma})^{\alpha-1}w(\sigma)z^{\varpi+\varrho}(\sigma)\frac{d\sigma}{\sigma}, \end{aligned} \tag{3.16}$$

which implies that

$$\begin{aligned} & _H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varrho+\lambda}(x)]_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\varpi}(x) \\ & +_H \mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\varpi}(x)]_H\mathcal{D}_{1,x}^{-\alpha}[z^{\varrho+\lambda}(x)w(x)] \\ & \leq\ _H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varpi+\varrho}(x)]_H\mathcal{D}_{1,x}^{-\alpha}[z^{\lambda}(x)w(x)y^{\varrho}(x)] \\ & +_H \mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\lambda}(x)]_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varpi+\varrho}(x)], \end{aligned} \tag{3.17}$$

which completes the proof.

Theorem 3.4. *Let z and y be two positive and continuous functions on $[1,\infty)$ and satisfying (3.11). Let $w:[1,\infty)\to\mathbb{R}^+$ be positive continuous function. Then for all $x>1, \varrho>0$, we have*

$$\begin{aligned} & _H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}(x)z^{\varpi}(x)]_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varrho+\lambda}(x)] \\ & +_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varrho+\varpi}(x)]_H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}(x)z^{\varpi}(x)] \\ & \leq\ _H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}z^{\lambda}(x)]_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varpi+\varrho}(x)] \\ & +_H\mathcal{D}_{1,x}^{-\beta}[w(x)z^{\varrho+\varpi-\lambda}(x)]_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\lambda}(x)]. \end{aligned} \tag{3.18}$$

Proof:- Multiplying the inequality (3.15) by $\frac{(\log\frac{x}{\sigma})^{\beta-1}}{\sigma\Gamma(\beta)} w(\sigma)z^{\lambda}(\sigma)$, $\sigma\in(1,x), x>1$, this function remains positive under the conditions stated with the theorem. Integrating the obtain result with respective to σ from 1 to x, we get

$$\begin{aligned} & _H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varrho+\lambda}(x)]\frac{1}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\sigma})^{\beta-1}w(\sigma)y^{\varrho}(\sigma)z^{\varpi}(\sigma)\frac{d\sigma}{\sigma} \\ & +_H \mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\varpi}(x)]\frac{1}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\sigma})^{\beta-1}z^{\varrho+\lambda}(\sigma)w(\sigma)\frac{d\sigma}{\sigma} \\ & \leq\ _H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varpi+\varrho}(x)]\frac{1}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\sigma})^{\beta-1}z^{\lambda}(\sigma)w(\sigma)y^{\varrho}(\sigma)\frac{d\sigma}{\sigma} \\ & +_H \mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\lambda}(x)]\frac{1}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\sigma})^{\beta-1}w(\sigma)z^{\varpi+\varrho}(\sigma)\frac{d\sigma}{\sigma}, \end{aligned} \tag{3.19}$$

which implies that,

$$\begin{aligned} & {}_H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}(x)z^{\varpi}(x)]{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varrho+\lambda}(x)] \\ & +{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z^{\varrho+\lambda}(x)]{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\varpi}(x)] \\ & \leq {}_H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}z^{\lambda}(x)]{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z^{\varpi+\varrho}(x)] \\ & +{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z^{\varrho+\varpi}(x)]{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z^{\lambda}(x)], \end{aligned} \tag{3.20}$$

This completes the proof of Theorem 3.4.
Next, we shall present a new generalization of weighted fractional integral inequalities using a family of n positive functions defined on $[1,\infty)$.

Theorem 3.5. *Let $z_i, i=1,...,n$ be n positive and continuous functions on $[1,\infty)$ such that*

$$(\sigma^{\varrho}z_r^{\varrho}(\tau)-\tau^{\varrho}z_r^{\varrho}(\sigma))(z_r^{\varpi-\lambda_r}(\tau)-z^{\varpi-\lambda_r}(\sigma))\geq 0. \tag{3.21}$$

Let $w:[1,\infty)\to\mathbb{R}^+$ be positive continuous function. Then for all $x>1$, $\varrho>0$, $\varpi\geq\lambda_r>0, r\in\{1,...,n\}$, the following fractional inequality

$$\begin{aligned} & {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)] \\ & \leq {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)], \end{aligned} \tag{3.22}$$

valid.

Proof:- Suppose $z_i, i=1,...,n$ be n positive and continuous functions on $[1,\infty)$, then for any fixed $r\in\{1,...,n\}$ and for any $\varrho>0,\varpi\geq\lambda_r>0$, $\tau,\sigma\in(1,x)$, $x>1$. From (3.21), we have

$$\begin{aligned} & \sigma^{\varrho}z_r^{\varpi-\lambda_r}(\sigma)z_r^{\varrho}(\tau)+\tau^{\varrho}z_r^{\varrho}(\sigma)z_r^{\varpi-\lambda_r}(\tau) \\ & \leq \sigma^{\varrho}z_r^{\varpi+\varrho-\lambda_r}(\tau)+\tau^{\varrho}z_r^{\varpi+\varrho-\lambda_r}(\sigma), \end{aligned} \tag{3.23}$$

multiplying both sides of (3.23) by $\frac{(\log\frac{x}{\tau})^{\alpha-1}}{\tau\Gamma(\alpha)}w(\tau)\Pi_{i=1}^{n}z_i^{\lambda_i}(\tau)$, then integrating the resulting inequality with respect to τ over $(1,x)$, we obtain

$$\begin{aligned} & \sigma^{\varrho}z_r^{\varpi-\lambda_r}(\sigma)\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)z_r^{\varrho}(\tau)\Pi_{i=1}^{n}z_i^{\lambda_i}(\tau)]\frac{d\tau}{\tau} \\ & +z^{\varrho}(\sigma)\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)\tau^{\varrho}\Pi_{i\neq r}^{n}z_i^{\lambda_i}(\tau)]\frac{d\tau}{\tau} \\ & \leq \sigma^{\varrho}\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)z_r^{\varpi+\varrho}(\tau)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(\tau)]\frac{d\tau}{\tau} \\ & +z_r^{\varpi+\varrho-\lambda_r}(\sigma)\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)\tau^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(\tau)]\frac{d\tau}{\tau}, \end{aligned} \tag{3.24}$$

consequently

$$\begin{aligned} & \sigma^{\varrho}z_r^{\varpi-\lambda_r}(\sigma)\,{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)] \\ & +z^{\varrho}(\sigma)\,{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)] \\ & \leq \sigma^{\varrho}\,{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)] \\ & +z_r^{\varpi+\varrho-\lambda_r}(\sigma)\,{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]. \end{aligned} \tag{3.25}$$

Again, multiplying the inequality (3.25) by $\frac{(\log\frac{x}{\sigma})^{\alpha-1}}{\sigma\Gamma(\alpha)}w(\sigma)\Pi_{i=1}^{n}z_i^{\lambda_i}(\sigma),\sigma\in(1,x),x>1$, this function remains positive under the conditions stated with the theorem. Integrating the obtain

result with respective to σ from 1 to x, we get

$$\begin{aligned} & {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\times \\ & \frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\sigma})^{\alpha-1}w(\sigma)\sigma^{\varrho}z_r^{\varpi}(\sigma)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(\sigma)\frac{d\sigma}{\sigma} \\ & +\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\times \\ & \frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\sigma})^{\alpha-1}w(\sigma)z_r^{\varrho}(\sigma)\Pi_{i=1}^{n}z_i^{\lambda_i}(\sigma)\frac{d\sigma}{\sigma} \\ & \leq\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\times \\ & \frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\sigma})^{\alpha-1}w(\sigma)\sigma^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(\sigma)\frac{d\sigma}{\sigma} \\ & +\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\times \\ & \frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\sigma})^{\alpha-1}w(\sigma)z_r^{\varpi+\varrho}(\sigma)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(\sigma)\frac{d\sigma}{\sigma}, \end{aligned} \tag{3.26}$$

therefore,

$$\begin{aligned} & {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[x^{\varrho}z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)] \\ & +\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)] \\ & \leq\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(x)] \\ & +\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]. \end{aligned} \tag{3.27}$$

This completes the proof of Theorem 3.5.

Theorem 3.6. *Let $z_i, i = 1, ..., n$ be n positive and continuous functions on $[1,\infty)$and satisfying (3.21). Let $w : [1,\infty) \to \mathbb{R}^+$ be positive continuous function. Then for all $x > 1$, $\varrho > 0$, $\varpi \geq \lambda_r > 0, r \in \{1, ..., n\}$, then we have inequality*

$$\begin{aligned} & {}_H\mathcal{D}_{1,x}^{-\beta}[w(x)x^{\varrho}z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)] \\ & +\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\ {}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)] \\ & \leq\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\ {}_H\mathcal{D}_{1,x}^{-\beta}[w(x)x^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(x)] \\ & +\ {}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\ {}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(x)], \end{aligned} \tag{3.28}$$

is valid.

Proof:- We multiplying the inequality (3.25) by $\frac{(\log\frac{x}{\sigma})^{\beta-1}}{\sigma\Gamma(\beta)}w(\sigma)\Pi_{i=1}^{n}z_i^{\lambda_i}(\sigma), \sigma \in (1,x), x > 1$, this function remains positive under the conditions stated with the theorem. Integrating the obtain

result with respective to σ from 1 to x, we get

$$\begin{aligned}
&{}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)] \\
&\frac{1}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\sigma})^{\beta-1}w(\sigma)\sigma^{\varrho}z_r^{\varpi}(\sigma)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(\sigma)\frac{d\sigma}{\sigma} \\
&+ {}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)] \\
&\frac{1}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\sigma})^{\beta-1}w(\sigma)z_r^{\varrho}(\sigma)\Pi_{i=1}^{n}z_i^{\lambda_i}(\sigma)\frac{d\sigma}{\sigma} \\
&\leq {}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)] \\
&\frac{1}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\sigma})^{\beta-1}w(\sigma)\sigma^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(\sigma)\frac{d\sigma}{\sigma} \\
&+ {}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)x^{\varrho}\Pi_{i=1}^{n}z_i^{\lambda_i}(x)] \\
&\frac{1}{\Gamma(\beta)}\int_1^x(\log\frac{x}{\sigma})^{\beta-1}w(\sigma)z_r^{\varpi+\varrho}(\sigma)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(\sigma)\frac{d\sigma}{\sigma},
\end{aligned} \tag{3.29}$$

which gives the inequality (3.28).

Theorem 3.7. *Let $z_i, i = 1, ..., n$ and y be two positive and continuous functions on $[1,\infty)$. such that*

$$(y^{\varrho}(\sigma)z_r^{\varrho}(\tau) - y^{\varrho}(\tau)z_r^{\varrho}(\sigma))(z_r^{\varpi-\lambda_r}(\tau) - z_r^{\varpi-\lambda_r}(\sigma)) \geq 0. \tag{3.30}$$

and let $w : [1,\infty) \to \mathbb{R}^+$ be positive continuous function. Then for all $x > 1$, $\varrho > 0$, $\varpi \geq \lambda_r > 0, r \in \{1, ..., n\}$, the following fractional inequality

$$\begin{aligned}
&{}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\ {}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)\mathrm{h}^{\varrho}(x)z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)] \\
&\leq {}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)\mathrm{h}^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]{}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)].
\end{aligned} \tag{3.31}$$

Proof:- Let $\tau, \sigma \in (1, x)$, $x > 1$, for any $\varrho > 0, \varpi \geq \lambda_i > 0, r \in \{1, 2, ...n\}$. From (3.30), we have

$$\begin{aligned}
&y^{\varrho}(\sigma)z_r^{\varpi-\lambda_r}(\sigma)z_r^{\varrho}(\tau) + z_r^{\varrho}(\sigma)y_r^{\varrho}(\tau)z_r^{\varpi-\lambda_r}(\tau) \\
&\leq y^{\varrho}(\sigma)z_r^{\varpi+\varrho-\lambda_r}(\tau) + z_r^{\varpi+\varrho-\lambda_r}(\sigma)y^{\varrho}(\tau),
\end{aligned} \tag{3.32}$$

multiplying both sides of (3.32) by $\frac{(\log\frac{x}{\tau})^{\alpha-1}}{\tau\Gamma(\alpha)}w(\tau)\Pi_{i=1}^{n}z_i^{\lambda_i}(\tau)$, then integrating the resulting inequality with respect to τ over $(1, x)$, we obtain

$$\begin{aligned}
&y^{\varrho}(\sigma)z_r^{\varpi-\lambda_r}(\sigma)\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)z_r^{\varrho}(\tau)\Pi_{i=1}^{n}z_i^{\lambda_i}(\tau)]\frac{d\tau}{\tau} \\
&+ z_r^{\varrho}(\sigma)\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)y^{\varrho}(\tau)z_r^{\varpi}(\tau)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(\tau)]\frac{d\tau}{\tau} \\
&\leq y^{\varrho}(\sigma)\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)y^{\varrho}(\tau)\Pi_{i=1}^{n}z_i^{\lambda_i}(\tau)]\frac{d\tau}{\tau} \\
&+ z_r^{\varpi+\varrho-\lambda_r}(\sigma)\frac{1}{\Gamma(\alpha)}\int_1^x(\log\frac{x}{\tau})^{\alpha-1}[w(\tau)z_r^{\varpi+\varrho}(\tau)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(\tau)\frac{d\tau}{\tau}],
\end{aligned} \tag{3.33}$$

consequently

$$\begin{aligned}
&y^{\varrho}(\sigma)z_r^{\varpi-\lambda_r}(\sigma){}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]+ \\
&z_r^{\varrho}(\sigma){}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)] \\
&\leq y^{\varrho}(\sigma){}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)] \\
&z_r^{\varpi+\varrho-\lambda_r}(\sigma){}_{H}\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)].
\end{aligned} \tag{3.34}$$

Multiplying both sides of (3.34) by $\frac{(\log\frac{x}{\sigma})^{\beta-1}}{\sigma\Gamma(\beta)}w(\sigma)\Pi_{i=1}^{n}z_i^{\lambda_i}(\sigma)$, then integrating the resulting inequality with respect to σ over $(1,x)$, we have

$$\begin{aligned}&2\,{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\;{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\\&\leq 2\,{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\,{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)].\end{aligned}\tag{3.35}$$

This completes the proof of Theorem 3.7.

Theorem 3.8. *Let $z_i, i=1,...,n$ and y be two positive and continuous functions on $[1,\infty)$. such that*

$$(y^{\varrho}(\sigma)z_r^{\varrho}(\tau)-y^{\varrho}(\tau)z_r^{\varrho}(\sigma))(z_r^{\varpi-\lambda_r}(\tau)-z_r^{\varpi-\lambda_r}(\sigma))\geq 0,\tag{3.36}$$

and let $w:[1,\infty)\to\mathbb{R}^+$ be positive continuous function. Then for all $x>1$, $\varrho>0$, $\varpi\geq\lambda_r>0, r\in\{1,...,n\}$, then we have inequality

$$\begin{aligned}&{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}(x)z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\;{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]+\\&{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\;{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\\&\leq\;{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\\&+{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)].\end{aligned}\tag{3.37}$$

Proof:- Multiplying both sides of (3.32) by $\frac{(\log\frac{x}{\sigma})^{\beta-1}}{\sigma\Gamma(\beta)}w(\sigma)\Pi_{i=1}^{n}z_i^{\lambda_i}(\sigma)$, then integrating the resulting inequality with respect to σ over $(1,x)$, we have

$$\begin{aligned}&z_r^{\varrho}(\tau){}_H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}(x)z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\;+\\&y^{\varrho}(x)z_r^{\varpi-\lambda_r}(\tau){}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\\&\leq\;z_r^{\varpi+\varrho-\lambda_r}(\tau){}_H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]+\\&y^{\varrho}(\tau){}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)].\end{aligned}\tag{3.38}$$

Multiplying both sides of (3.38) by $\frac{(\log\frac{x}{\tau})^{\alpha-1}}{\tau\Gamma(\tau)}w(\tau)\Pi_{i=1}^{n}z_i^{\lambda_i}(\tau)$, then integrating the resulting inequality with respect to τ over $(1,x)$, we have

$$\begin{aligned}&{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}(x)z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\;{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]+\\&{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)z_r^{\varpi}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]\;{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z_r^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]\\&\leq\;{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)y^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)]+\\&{}_H\mathcal{D}_{1,x}^{-\beta}[w(x)z_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}z_i^{\lambda_i}(x)]{}_H\mathcal{D}_{1,x}^{-\alpha}[w(x)y^{\varrho}(x)\Pi_{i=1}^{n}z_i^{\lambda_i}(x)],\end{aligned}\tag{3.39}$$

which completes the proof.

4 Concluding Remarks

In this paper, we studied the Hadamard fractional integral operators and then we obtained some weighted fractional integral inequalities using the Hadamard fractional integral operators. The weighted fractional integral inequalities established in this paper give some contribution in the fields of fractional calculus and Hadamard fractional integral operators. Moreover, they are expected to lead to some application for finding uniqueness of solutions in fractional differential equations.

References

[1] Anastassiou, G. A., *Fractional Differentiation Inequalities,* Springer Publishing Company, New York, NY, 2009.

[2] Baleanu, D., Purohit,S. D. and Agarwal, P,. *On fractional integral inequalities involving hypergeometric operators,* Chinese Journal of Mathematics, Article ID **609476**, 2014,(2014), 1-5.

[3] Baleanu, D., Machado J. A. T. and Luo, C. J., *Fractional Dynamic and Control,* Springer, **2012**, pp.159-171.

[4] Chinchane, V. L. and Pachpatte, D. B., *A note on some integral inequalities via Hadamard integral,* J. Fractional Calculus Appl., **4**(11)(2013), 1-5.

[5] Chinchane, V. L. and Pachpatte, D. B., *On some integral inequalities using Hadamard fractional integral,* Malaya J. Math., **1**(1)(2012), 62-66.

[6] Chinchane, V. L. and Pachpatte, D. B., *Some new integral inequalities using Hadamard fractional integral operator*. Adv. Inequal. Appl. **2014**, 2014:12.

[7] Chinchane, V. L., *On Chebysshev type inequalities using generalized k-fractional integral operator,* Progr. Fract. Differ. Appl., **3**(3), (2017), 219-226.

[8] Chinchane, V. L. and Pachpatte, D. B., *On some Grüss-type fractional inequalities using Saigo fractional integral operator,* Journal of Mathematics, Article ID **527910**, 2014(2014), 1-9.

[9] Chinchane, V. L., *New approach to Minkowski fractional inequalities using generalized K-fractional integral operator,* Journal of the Math. Soc., **(1-2)**(85), (2018), 32-41.

[10] Choi, J. and Purohit, S. D., *A Grüss type integral inequalities associated with Guss hupergeometric integral operator,* Commun. Korean Math. Sco., **2**(30), (2015), 81-92.

[11] Dahmani, Z., *The Riemann-Liouville operator to genarate some new inequalities,* Int. J. Nonlinear Sci. volume **12** (2011), No.4, pp.452-455.

[12] Dahmani, Z., *Some results associate with fractional integrals involving the extended Chebyshev,* Acta Univ. Apulensis Math. Inform., **27**(2011), 217-224.

[13] Farid, G. and Habibullah, G. M., *An extension of Hadamard fractional integral,* International Journal of Mathematical Analysis **10**(9), (2015), 471-482.

[14] Fernandez A. and Mohammed P. O., *Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels,* Math. Meth. Appl. Sci. **2020** 1–18. https://doi.org/10.1002/mma.6188

[15] Houas, M., *Certain weighted integralinequalities involving the fractional hypergeometric operators,* Scientia Series A: Mathematical Science, **27**(2016), 87-97.

[16] Kilbas, A. A., Srivasatava, H. M. and Trujillo, J. J., *Theory and Application of Fractional Differential Equation,* North-Holland Mathematiccal Studies Vol. Elsevier Science Publishers, Amsterdam, London and New York, 2006.

[17] Kiryakova, V., *On two Saigo's fractional integral operator in the class of univalent functions,* Fract. Calc. Appl. Anal., **9** (2), (2006).

[18] Mohammed, P.O. Abdeljawad, T. Kashuri, A *Fractional Hermite–Hadamard–Fejer Inequalities for a Convex Function with Respect to an Increasing Function Involving a Positive Weighted Symmetric Function,* Symmetry 2020, **12**(9), 1503, 1-17. https://doi.org/10.3390/sym12091503

[19] Mohammed, P.O. Abdeljawad, T. Zeng, S. Kashuri, A. *Fractional Hermite-Hadamard Integral Inequalities for a New Class of Convex Functions,* Symmetry 2020, **12**(9), 1485, 1-12.

[20] Mohammed, P.O.; Brevik, I. A *New Version of the Hermite–Hadamard Inequality for Riemann–Liouville Fractional Integrals.,* Symmetry 2020, **12**(4), 610, 1-11. https://doi.org/10.3390/sym12040610

[21] Mohammed, P.O., Abdeljawad, T. *Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel,* Adv. Differ. Equ. 2020, **363** (2020), 1-19. https://doi.org/10.1186/s13662-020-02825-4

[22] Mohammed, P.O., Sarikaya M. Z., *IOn generalized fractional integral inequalities for twice differentiable convex functions,* Journal of Computational and Applied Mathematics **372** (2020) 112740, 1-15. https://doi.org/10.1016/j.cam.2020.112740

[23] Prabhakaran A. R., and Rao, K. S., *Saigo operator of fractional integration of Hypergeometric functions,* International Journal of Pure and Applied Mathematics, **81**(5), (2012), 755-763.

[24] Purohit, S. D. and Raina, R. K., *Chebyshev type inequalities for the Saigo fractional integral and their q-analogues,* J. Math. Inequal., **7**(2)(2013), 239-249.

[25] Somko, S. G., Kilbas, A. A., and Marichev, O. I., *Fractional Integral and Derivative Theory and Application,* Gordon and Breach, Yverdon, Switzerland, 1993.

[26] Sroysang, B., *A Study on Hadamard Fractional Integral,* Int. Journal of Math. Analysis, **39**(7),(2013) 1903 - 1906.

[27] Sudsutad, W., Ntouyas, S. K. and Tariboon, J. , *Fractional integral inequalities via Hadamard's fractional integral,* Abstract and Applied Analysis Volume 2014, Article ID **563096**, 1-11.

[28] Tafa, S. and Brahimb, K., *Some new results using Hadamard fractional integral,* Int. J. Nonlinear Anal. Appl.**1**(7) (2016), 103-109.

Author information

Asha B. Nale[1], Satish K. Panchal[2] and Vaijanath L. Chinchane[3]
, [1,2]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431 004. INDIA
[3]Department of Mathematics, Deogiri Institute of Engineering and Management Studies Aurangabad-431005, INDIA.
E-mail: ashabnale@gmail.com/drskpanchal@gmail.com/chinchane85@gmail.com

Received: July 18, 2020.
Accepted: May 29, 2021.