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α-Centroidal Means and its Applications to Graph Labeling

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Abstract. In this paper, a judging condition for the m-power Schur convexity of the power type α-centroidal mean for two positive arguments with three parameters is established. As an application, the centroidal mean labelling of the graphs containing the cycles are discussed.

1 Introduction

The classical definition and good number of results on centroidal mean and its applications are discussed in [1]. For $a, b > 0$, then centroidal mean is

$$CT(a,b) = \frac{a^2+ab+b^2}{a+b}.$$

K. M. Nagaraja and P. S. K. Reddy [4] defined the α-centroidal mean and its dual, which are similar to oscillatory mean and its dual as follows: For $a, b > 0$ and $\alpha \in (0,1)$, α-centroidal mean and its dual form are respectively defined as follows:

$$CT(a,b;\alpha) = \alpha H(a,b) + (1-\alpha)C(a,b) \tag{1.1}$$

and

$$CT^{(d)}(a,b;\alpha) = H^{\alpha}(a,b)C^{1-\alpha}(a,b). \tag{1.2}$$

In [5], K. M. Nagaraja and P. S. K. Reddy were generalized the above said means. In [6], the authors defined power type α-centroidal mean and its dual as follows:

$$C(a,b;\alpha,p) = \begin{cases} \left[\alpha\left(\frac{2a^pb^p}{a^p+b^p}\right) + (1-\alpha)\left(\frac{a^{2p}+b^{2p}}{a^p+b^p}\right)\right]^{\frac{1}{p}}, & \text{if } p \neq 0; \\ \sqrt{ab}, & \text{if } p = 0. \end{cases} \tag{1.3}$$

and

$$C^{(d)}(a,b;\alpha,p) = \begin{cases} \left[\left(\frac{2a^pb^p}{a^p+b^p}\right)^{\alpha}\left(\frac{a^{2p}+b^{2p}}{a^p+b^p}\right)^{1-\alpha}\right]^{\frac{1}{p}}, & \text{if } p \neq 0; \\ \sqrt{ab}, & \text{if } p = 0. \end{cases} \tag{1.4}$$

Equation (3) take the alternate form with three parameters as:

$$K(\alpha,\beta,p;a,b) = [\alpha C(a^p,b^p) + \beta H(a^p,b^p)]^{\frac{1}{p}},\ p \neq 0.$$

In [10], Schur introduced the concept of Schur convexity, later on it spread over to every fields of science and technology. The convexity and Schur convexities have many important applications in analytic inequalities, linear regression, graphs and matrices, combinatorial optimization, information theoretic topics, gamma functions, stochastic orderings, reliability and other related fields. Further, researchers developed the results on Schur geometric, harmonic convexities and relevant to $\mathcal{M}$-power Schur convexity, the good number of publications are available in literature

(See [2, 7, 11]).

If the vertices and edges of a graph are assigned the values subject to certain conditions then the graph is known as graph labeling. A graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ is called a centroidal mean graph, if the function $f : V(G) \to A \subseteq N$ to label the vertices $x \in V(G)$ with distinct labels $f(x)$, and each edge $e = \{x_i, x_j\}$ is labeled with

$$f^*(\{x_i x_j\}) = \left\lceil \frac{2[(f(x_1))^2 + f(x_1)f(x_2) + (f(x_2))^2]}{3(f(x_1) + f(x_2))} \right\rceil$$

or

$$f^*(\{x_i x_j\}) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for every $x_i, x_j \in V(G)$ with $i \neq j$ (See [3]).

In [8, 9], the authors discussed several results on labelling of graphs pertaining to cycles, paths, dragon, etc., of Heron mean and contra harmonic mean. This work motivates us to establish the few results on $\mathcal{M}$-power Schur convexities and labelling of graphs of the power type α-centroidal mean. The following result proved by Dongsheng Wang et al. [2]:

Lemma 1.1. *Let $\Omega \subseteq R^n_+$ be a symmetric set with nonempty interior ω^0 and $\varphi : \Omega \to R_+$ be continuous on Ω and differentiable in Ω^0. Then φ is Schur m-power convex on Ω if and only if ϕ is symmetric on ω and*

$$\frac{x_1^m - x_2^m}{m}\left[x_1^{1-m}\frac{\partial\varphi(x)}{\partial x_1} - x_2^{1-m}\frac{\partial\varphi(x)}{\partial x_2}\right] \geq 0, if m \neq 0 \tag{1.5}$$

$$(ln x_1 - ln x_2)\left[x_1\frac{\partial\varphi(x)}{\partial x_1} - x_2\frac{\partial\varphi(x)}{\partial x_2}\right] \geq 0, if m = 0 \tag{1.6}$$

for all $x \in \omega^0$.

2 Main Results

In this section, the Schur m-power convexity of the power type α-centroidal mean $K(\alpha, \beta, p, a, b)$ is discussed.

Theorem 2.1. *For $\alpha, \beta > 0$ and $\alpha + \beta = 1, Z = \frac{a}{b} > 1$, then*

(i) For $m > 0$, the power type α-centroidal mean is convex if

- *(a) $\alpha > 2\beta$ and $|m| < p < \infty$*
- *(b) $\alpha < 2\beta$ and $\frac{m}{3} < p < -m$*

(ii) For $m < 0$, the power type α-centroidal mean is convex if $\alpha > 2\beta$ and $-\infty < p < m$

(iii) For $m > 0$, the power type α-centroidal mean is concave if

- *(a) $\alpha < 2\beta$ and $-m < p < \frac{m}{3}$.*
- *(b) $\alpha > 2\beta$ and $-\infty < p < -m$.*

Proof. We have

$$K(\alpha, \beta, p; a, b) = [\alpha C(a^p, b^p) + \beta H(a^p, b^p)]^{\frac{1}{p}}.$$

Then

$$\frac{\partial K}{\partial a} = s(a, b)\left[\alpha\left(\frac{2pa^{2p-1}}{a^p + b^p} - \frac{pa^{p-1}(a^{2p} + b^{2p})}{(a^p + b^p)^2}\right) + \beta\left(\frac{2pa^{p-1}b^{2p}}{(a^p + b^p)^2}\right)\right]$$

and

$$\frac{\partial K}{\partial b} = s(a, b)\left[\alpha\left(\frac{2pb^{2p-1}}{a^p + b^p} - \frac{pb^{p-1}(a^{2p} + b^{2p})}{(a^p + b^p)^2}\right) + \beta\left(\frac{2pb^{p-1}a^{2p}}{(a^p + b^p)^2}\right)\right].$$

Let $s(a,b) = \frac{1}{(\alpha+\beta)^{\frac{1}{p}}}\frac{1}{p}\left[\alpha\left(\frac{a^{2p}+b^{2p}}{a^p+b^p}\right)+\beta\left(\frac{2a^pb^p}{a^p+b^p}\right)\right]^{\frac{1}{p}-1}$ and $\Delta = \frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial K}{\partial a} - b^{1-m}\frac{\partial K}{\partial b}\right)$.
Then

$$\Delta = \frac{s(a,b)}{(a^p+b^p)^2}\frac{a^m-b^m}{m}\{\alpha(a^{3p-m}-b^{3p-m})+(\alpha-2\beta)(a^{2p}b^{p-m}-b^{2p}a^{p-m}) + 2\alpha(a^{2p-m}b^p - a^pb^{2p-m})\}.$$

For $z = \frac{a}{b} \geq 1$,

$$\Delta = \frac{s(a,b)b^p}{(a^p+b^p)^2}.\frac{z^m-1}{m}f(z),$$

where $f(z) = \alpha(z^{3p-m}-1)+(\alpha-2\beta)(z^{2p}-z^{p-m})+2\alpha(z^{2p-m}-z^p)$.

On regrouping leads to

$$f(z) = \alpha(z^{3p-m}+z^{2p}-z^{p-m}+2z^{2p-m}-2z^p-1)+\beta(2z^{p-m}-2z^{2p}).$$

We have,

$$\begin{aligned} f'(z) &= \alpha(3p-m)z^{3p-m-1}+2p\alpha z^{2p-1}-\alpha(p-m)z^{p-m-1} \\ &\quad +2\alpha(2p-m)z^{2p-m-1}-2\alpha pz^{p-1}+2\beta(p-m)z^{p-m-1}-4pz^{2p-1} \\ &= z^{p-m-1}[\alpha(3p-m)z^{2p}+2pz^{p+m}-(p-m)+(4p-2m)z^p-2pz^m \\ &\quad +2\beta(p-m)-4\beta pz^{p+m}] \\ &= z^{p-m-1}[\alpha(3p-m)z^{2p}+2p(\alpha-2\beta)z^{p+m}+\alpha(4p-2m)z^p \\ &\quad -2\alpha pz^m+\beta(2p-2m)-\alpha(p-m)] \\ &= z^{p-m-1}[\alpha(3p-m)z^{2p}+2p(\alpha-2\beta)z^{p+m}+\beta(2p-2m)-\alpha(p-m) \\ &\quad +\alpha z^m(4p-2m)z^{p-m}-2p] \\ &= z^{p-m-1} \quad q(z), \end{aligned}$$

where

$$q(z) = \alpha(3p-m)z^{2p}+2p(\alpha-2\beta)z^{p+m}+\beta(2p-2m)-\alpha(p-m)+\alpha z^m(4p-2m)z^{p-m}-2p$$

and

$$q'(z) = 2\alpha p(3p-m)z^{2p-1}+2p(\alpha-2\beta)(p+m)z^{p+m-1}+0+\alpha p(4p-2m)z^{p-1}-\alpha 2pmz^{m-1}.$$

Then with simple modification, we have,

$$q'(z) = 2p\{\alpha p(3p-m)z^{2p-1}+(\alpha-2\beta)(p+m)z^{p+m-1}+\alpha[2p-m(1+z^{m-p})]z^{p-1}\} \quad (2.1)$$

It is easy to see that $q'(z) > 0$ if and only if each term of (2.1) is positive. We have following two cases for which the value of $q'(z) > 0$.
Case (i): For $m > 0$, we have two subcases:
Subcase (a): $\alpha > 2\beta$, $p > \frac{m}{3}$, $p > |m|$, equivalently written as: $\alpha > 2\beta$ and $|m| < p < \infty$.
Subcase (b): $\alpha < 2\beta$, $p > \frac{m}{3}$, $p < -m$ and $p > m$, equivalently written as: $\alpha < 2\beta$ and no feasible region exist for p for these set of parameter $q' > 0$.
Case (ii): For $m < 0$, we have two subcases:
Subcase (a): $\alpha > 2\beta$, $p > \frac{m}{3}$, $p > |m|$, equivalently written as: $\alpha > 2\beta$ and $-m < p < \infty$.
Subcase (b): $\alpha < 2\beta$, $p > \frac{m}{3}$, $p < -m$ and $p > m$, equivalently written as: $\alpha < 2\beta$ and $\frac{m}{3} < p < -m$.
Similarly, it is easy to see that $q'(z) < 0$ if and only if each term of (2.1) is negative and also the above cases will exist. Now, we have:

$$\Delta = \frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial K}{\partial a} - b^{1-m}\frac{\partial K}{\partial b}\right)$$

and $f'(z) = z^{p-m-1}q(z)$ is concave for the set of values satisfy $\alpha < 2\beta$ and $-m < p < \frac{m}{3}$ and $\alpha > 2\beta$ and $-\infty < p < -m$. Further, $f'(z) = z^{p-m-1}q(z)$ is convex for the set of values satisfy $\alpha > 2\beta$ and $m < p < \infty$, $\alpha > 2\beta$ and $-m < p < \infty$. $\alpha < 2\beta$ and $\frac{m}{3} < p < -m$, $\alpha > 2\beta$ and $\infty < p < m$. □

3 Application to Graph Labeling

In this section the application of centroidal mean to label graphs containing cycles such as triangular ladder TL_n, polygonal chain G_{mn} and square graph P_n^2 are discussed.

Theorem 3.1. *A triangular ladder TL_n is a centroidal mean graph.*

Proof. Consider a triangular ladder TL_n, with n vertices $x_1, x_2, x_3, \ldots, x_n$ as one path and $y_1, y_2, y_3, \ldots, y_n$ as other path. The function $f : V(TL_n) \to 1, 2, 3, \ldots, (4n-2)$ is defined by $f(x_i) = 4i - 1, \quad 1 \le i \le n$ and $f(y_i) = 4i - 3, \quad 1 \le i \le n$ such that the induced function $f^* : E(G) \to N$ given by

$$f^*(\{x_1x_2\}) = \left\lfloor \frac{2[(f(x_1))^2 + f(x_1)f(x_2) + (f(x_2))^2]}{3(f(x_1) + f(x_2))} \right\rfloor$$

for every $x_1, x_2 \in V(G)$.

The edges $\{x_i, x_{i+1}\}$ are labeled by

$$f^*(\{x_ix_{i+1}\}) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_{i+1}) + (f(x_{i+1}))^2]}{3(f(x_i) + f(x_{i+1}))} \right\rfloor = 4i + 1; \quad 1 \le i \le (n-1).$$

The edges $\{y_i, y_{i+1}\}$ are labeled by

$$f^*(\{y_iy_{i+1}\}) = \left\lfloor \frac{2[(f(y_i))^2 + f(y_i)f(y_{i+1}) + (f(y_{i+1}))^2]}{3(f(y_i) + f(y_{i+1}))} \right\rfloor = 4i - 1; \quad 1 \le i \le (n-1).$$

The edges $\{x_i, y_i\}$ are labeled by

$$f^*(\{x_iy_i\}) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(y_i) + (f(y_i))^2]}{3(f(x_i) + f(y_i))} \right\rfloor = 4i - 2; \quad 1 \le i \le n$$

The edges $\{x_i, y_{i+1}\}$ are labeled by

$$f^*(\{x_iy_{i+1}\}) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(y_{i+1}) + (f(y_{i+1}))^2]}{3(f(x_i) + f(y_{i+1}))} \right\rfloor = 4i; \quad 1 \le i \le (n-1)$$

are all distinct. Hence, the triangular ladder TL_n is a centroidal mean graph. □

Illustration: Consider the triangular ladder TL_n of $n = 8$. The following Figure-1 shows the centroidal mean labeling of a graph.

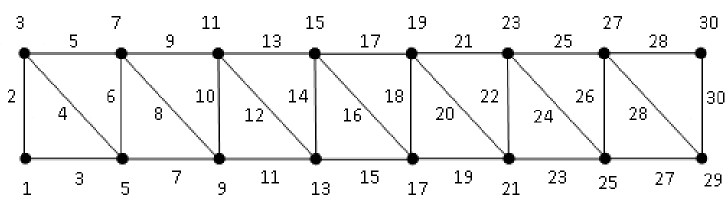

Figure 1. Triangular Ladder TL_8

Theorem 3.2. *The polygonal chain $G_{m,n}$ is a centroidal mean graph.*

Proof. Consider a polygonal chain $G_{m,n}$, in which $x_1, x_2, x_4, x_6, \ldots x_{n-4}, x_{n-2}, x_{n+1}x_{n-1}, x_{n-3}, x_{n-5}, \ldots, x_7, x_3, x_1$ be the first cycle. The second cycle is connected to the first cycle at the vertex x_{n+1}. Let

$$x_{n+1}, x_{n+2}, x_{n+4}, \ldots, x_{2n+1}, x_{2n-1}, x_{2n-3}, \ldots, x_{n+7}, x_{n+5}, x_{n+3}, x_{n+1}$$

be the second cycle. The third cycle is connected to the cycle at the vertex x_{2n+1}. Then third cycle will be

$$x_{2n+1}, x_{2n+2}, x_{2n+4}, \ldots, x_{3n+1}, x_{3n-1}, x_{n-3}, \ldots, x_{2n+5}, x_{2n+3}, x_{2n+1}.$$

In general r^{th} cycle is connected to the $(r-1)^{th}$ cycle at the vertex x_{rn+1} and the r^{th} cycle will be

$$x_{rn+1}, x_{rn+2}, x_{rn+4}, \ldots, x_{(r+1)n-4}, x_{(r+1)n-2}, \ldots, x_{rn+5}, x_{rn+3}, x_{rn+1}$$

and the graph has m cycles. Consider a function $f : V(G_{m,n}) \to \{1, 2, 3, \ldots, (q+1)\}$ by $f(x_i) = i$ for $1 \le i \le mn+1$ and $f(x_n) = mn+1$. Then the label of the edges are done by $f^*(\{x_i x_j\}) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i)+f(x_j))} \right\rfloor$; $i \neq j$, are all distinct. Hence $G_{m,n}$ is a centroidal mean graph. □

Illustration: The following Figure-2 shows the centroidal mean labeling of polygonal chain G_{mn}.

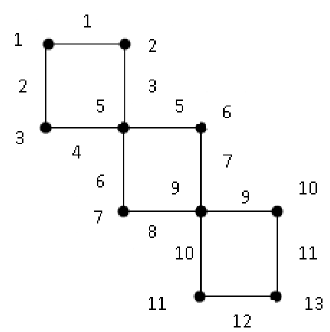

Figure 2. polygonal chain G_{mn}.

Theorem 3.3. *The square graph P_n^2 is a centroidal mean graph.*

Proof. Consider a path P_n with $x_1, x_2, x_3, \ldots x_n$ as its vertices. Then P_n^2 has n vertices, $(2n-3)$ edges and any two vertices u and v are adjacent, if $d(u, v) \le 2$. Consider a function $f : V(P_n^2) \to N$ by $f(x_i) = 2i - 1$, $\quad 1 \le i \le (n-1)$ and $f(x_n) = 2n - 2$. The edges are labeled by

$$f^*(\{x_i x_j\}) = \left\lfloor \frac{2[(f(x_i))^2 + f(x_i)f(x_j) + (f(x_j))^2]}{3(f(x_i) + f(x_j))} \right\rfloor$$

for all $e = \{x_i, x_j\} \in E(P_n^2)$ such that $f^*(e_i) \neq f^*(e_j)$ for $i \neq j$, f^* is injective. Hence, P_n^2 is a centroidal mean graph. □

Illustration: Consider a path P_8 and P_8^2 is a graph obtained by joining the vertices u and v, if $d(u, v) \le 2$. The following figure represents the square graph.

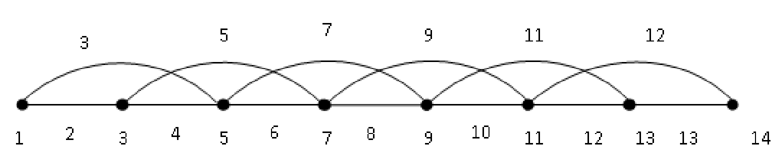

Figure 3. square graph P_8^2

4 Conclusion

The condition for Schur m-power convexity of the power type $\alpha-$centroidal mean for two variables with three parameters is investigated and also a judging condition is constructed. Further, the application of centroidal mean for labeling graphs with illustrations presented. Even, we have provided some interesting applications in differential geometry via mathematical means.

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