**Palestine Journal of Mathematics**

**Vol. 10(2)(2021), 710–714** © Palestine Polytechnic University-PPU 2021

# TRAVELING WAVE SOLUTION OF A KELLER-SEGEL MODEL WITH BACTERIAL POPULATION GROWTH

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Communicated by Cemil Tunc

MSC 2010 Classifications: Primary 35C07, 35K55, 92C17; Secondary 62P10, 46N60.

Keywords and phrases: Traveling wave solutions, Keller-Segel model, Population growth.

**Abstract** In this paper we study the existence of traveling wave solutions for a Keller-Segel model with bacterial population growth. We show the existence of exactly two heteroclinic orbits. The result is verified by using appropriate numerical simulations.

## 1 Introduction

The mathematical study of chemotaxis began with the pioneering work of Keller and Segel [4] in which

$$\begin{cases} u_t = Du_{xx} - \chi \left(u \left(\phi\left(v\right)\right)_x\right)_x + h(u,\,v) \\ v_t = \varepsilon v_{xx} + g(u,\,v) \end{cases} \tag{1.1}$$

was proposed for $x \in (-\infty,\,\infty)$, $t > 0$, $\phi(v) = \log(v)$, $g(u,\,v) = -uv^m$, $h(u,\,v) \equiv 0$. When $0 \leq m < 1$, it was shown that model (1.1) has a traveling wave solution which agrees fully with the experiments of Adler [1, 2] on bacteria *Escherichia Coli*. Subsequently, many works on various aspects of traveling wave solutions of system (1.1) with $\varepsilon \geq 0$ and $h(u,\,v) \equiv 0$ have been carried out [6, 7, 8, 9, 10, 11] and the reference therein. When $m > 1$, this model does not have a traveling wave solution [11, 14]. For $m = 1$, the existence of traveling wave solution of system (1.1) was obtained in [12] and in [13] for $\varepsilon > 0$.

It is known that the bacterial population growth was not considered in Keller Segel model (1.1); that is, $h(u,\,v) \equiv 0$. Since $v$ corresponds to nutrient source (like arginine, glucose, or oxygen), it is natural to consider the bacterial population growth. Hence, it would be interesting to investigate whether the bacterial population growth plays a significant role in the existence of traveling wave solution. Ai and Wang [3] have considered the model (1.1) for $\phi(v) = \log(v)$, $g(u,\,v) = -uv^m$, $h(u,\,v) = ruv^m$, $\varepsilon = 0$, $D = 1$, $0 < r \leq 1$ and found that when the bacterial population growth is considered, the profile of traveling bands, the minimum wave speed and the range of the chemical consumption rate for the existence of traveling wave solutions will alter. Li and Park [5] have considered the model (1.1) for $\varepsilon = 0$, $D > 0$, $\phi(v) = \log(v)$, $g(u,\,v) = -u + \gamma v$, $h(u,\,v) = \mu u(1-u)$ and found that the existence and nonexistence of traveling wave solutions, and proved the existence of heteroclinic orbit.

In this paper, we consider the following Keller-Segel model with bacterial population growth:

$$\begin{cases} u_t = u_{xx} - \chi \left(u \left(\log\left(v\right)\right)_x\right)_x + ruv(1-v) \\ v_t = -uv \end{cases} \tag{1.2}$$

where $u(x,\,t)$ and $v(x,\,t)$ at $x \in (-\infty, \infty)$, $t > 0$, represent the bacterial density and chemical concentration respectively. $\chi$ is the chemotactic sensitivity coefficient describing the strength of chemotaxis; $r$ is the growth rate of the bacteria.

This paper is organized as follows: In Section 2, we show the existence of exactly two heteroclinic traveling wave solutions of system (1.2). In Section 3, we verify the result using appropriate numerical simulations.

## 2 Main result

In this section, we investigate the existence of two heteroclinic traveling wave solutions of system (1.2).

Assume that $(u, v)(x, t) = (U, V)(z)$, $z = x - st$, is a traveling wave solution of system (1.2), where $s > 0$ denotes the wave speed. Then, the traveling wave solution $(U, V)$ of system (1.2) satisfies the following system of ordinary differential equations:

$$\begin{cases} U'' + sU' + rsV'(1-V) - \chi\left(U\frac{V'}{V}\right)' = 0 \\ sV' - UV = 0 \end{cases} \tag{2.1}$$

where $(\,.\,)' := d/dz$. As it can be seen from the second equation of (2.1), $V$ is an increasing wave front. Hence, without loss of generality, we can assume that $V(+\infty) = 1$ and $V'(+\infty) = U(+\infty) = U'(+\infty) = 0$. With these conditions, (2.1) leads to the equation

$$U' + sU - \chi\left(U\frac{V'}{V}\right) + rsV - rs\frac{V^2}{2} - \frac{rs}{2} = 0.$$

Thus, traveling wave solution of system (1.2) satisfies the system

$$\begin{cases} U' = s\left(\frac{r}{2} - rV - U\right) + \frac{\chi}{s}U^2 + rs\frac{V^2}{2} \\ V' = \frac{UV}{s} \end{cases} \tag{2.2}$$

and the conditions

$$0 < U, \quad 0 < V < 1, \quad (U, V)(+\infty) = E_0 := (0, 1). \tag{2.3}$$

Let us now find equilibrium points of system (2.2). The equilibrium points satisfy the following two equations:

**(i)** $s\left(\frac{r}{2} - rV - U\right) + \frac{\chi}{s}U^2 + rs\frac{V^2}{2} = 0,$

**(ii)** $\frac{UV}{s} = 0.$

If $U = 0$, then we obtain the first equilibrium point of system (2.2) as $E_0 = (0, 1)$. Otherwise, $V = 0$, and we have the equation $2\chi U^2 - 2s^2U + rs^2 = 0$. Hence, we obtain the following three cases:

**(a)** When $s > \sqrt{2r\chi}$, we have three equilibriums, that is one is $E_0 = (0, 1)$ and two of them are $E_1 := (u_1^*, 0)$ and $E_2 := (u_2^*, 0)$ where $u_1^* = \frac{s^2 - s\sqrt{\Delta}}{2\chi}$ and $u_2^* = \frac{s^2 + s\sqrt{\Delta}}{2\chi}$ for $\Delta := s^2 - 2r\chi$ are the roots of the equation $2\chi U^2 - 2s^2U + rs^2 = 0$.

**(b)** When $\sqrt{2r\chi} > s > 0$, we have only one equilibrium $E_0 = (0, 1)$.

**(c)** When $s = \sqrt{2r\chi}$, we have two equilibriums $E_0 = (0, 1)$ and $E_3 = \left(\frac{s^2}{2\chi}, 0\right)$.

Throughout of this paper, we consider only the case $s > \sqrt{2r\chi}$.
Let us now find the stability of the equilibriums.

**Lemma 2.1.** *Assume that* $s > \sqrt{2r\chi}$.

**(i)** $E_0$ *is a stable equilibrium of system* (2.2).

**(ii)** $E_1$ *is a saddle point of system* (2.2) *with the unstable manifold* $W^u(E_1)$ *which is tangent to the eigenvector of* $J(E_1)$, $V_1 := \left[1, \frac{(2\chi-1)u_1^* - s^2}{rs^2}\right]^T$ *corresponding to the eigenvalue* $\beta_1 = u_1^*/s$.

**(iii)** $E_2$ *is an unstable node of system* (2.2).

*Proof.* Firstly, since the Jacobian matrix $J(E_0) = \begin{bmatrix} -s & 0 \\ 1/s & 0 \end{bmatrix}$ has eigenvalues $\gamma_1 = 0$ and $\gamma_2 = -s$, part (i) follows.

Secondly, the Jacobian matrix $J(E_{1,2}) = \begin{bmatrix} -s + 2\chi u^*_{1,2}/s & -rs \\ 0 & u^*_{1,2}/s \end{bmatrix}$ has the eigenvalues $\alpha_{1,2} = -s + 2\chi u^*_{1,2}/s$ and $\beta_{1,2} = u^*_{1,2}/s$. As $-s^2 + 2\chi u^*_1 < 0$, the eigenvalue $\alpha_1 = -s + 2\chi u^*_1/s < 0$ and the other eigenvalue $\beta_1 = u^*_1/s > 0$; hence $E_1$ is a saddle point of system (2.2). Moreover, $V_1$ is an eigenvector of $J(E_1)$ corresponding to the eigenvalue $\beta_1$. Thus, using unstable manifold theorem we see that there is an unstable manifold $W^u(E_1)$ tangent to the eigenvector $V_1$. The proof of part (ii) is completed. Thirdly, it can be seen that $\alpha_2$ and $\beta_2$ are all positive; hence part (iii) follows. □

**Lemma 2.2.** *Assume that* $s > \max\{\sqrt{u^*_1}, \sqrt{2r\chi}\}$. *Let* $O := (0, 0)$ *be the origin of the plane and* $\alpha = \max\left\{u^*_2, s^2\right\}$.

**(i)** *Let* $R_1$ *be the region bounded by the line segments* $\overline{O\,E_0}$, $\overline{O\,E_1}$, $\overline{E_0\,E_1}$. *Then,* $R_1$ *is a positively invariant set of system* (2.2).

**(ii)** *Let* $P_\alpha := (\alpha, 0)$, *and* $R_2$ *the region bounded by the line segments* $\overline{E_0\,E_1}$, $\overline{E_0\,P_\alpha}$, $\overline{E_1\,P_\alpha}$. *Then,* $R_2$ *is a negatively invariant set of system* (2.2).

*Proof.* **(i)** Let $(U, V)$ be an arbitrary point on $int\left(\overline{O\,E_0}\right)$. Then

$$U' = rs\frac{(V-1)^2}{2} > 0.$$

Let $(U, V)$ be an arbitrary point on $int\left(\overline{E_0\,E_1}\right)$. Then, for $V = 1 - \frac{U}{u^*_1}$ we have

$$\frac{dV}{dU} = \frac{UV}{\frac{rs^2}{2}(V-1)^2 + \chi U^2 - s^2 U} = \frac{(u^*_1 - U)}{\left(rs^2\frac{U}{2(u^*_1)^2} + \chi U - s^2\right)u^*_1} = -\frac{1}{s^2} > -\frac{1}{u^*_1}.$$

This implies that the vector field of system (2.2) on $int\left(\overline{E_0\,E_1}\right)$ points into the interior of $R_1$. Therefore, $R_1$ is a positively invariant set of system (2.2). This completes the proof of part (i).

**(ii)** Let $(U, V)$ be an arbitrary point on $int\left(\overline{E_0\,P_\alpha}\right)$. Then, for $V = 1 - \frac{U}{\alpha}$ we have

$$K_\alpha := (1, \alpha) . \left(\frac{dU}{dt}, \frac{dV}{dt}\right) = \frac{U}{s}\left(\left(\frac{rs^2}{2\alpha^2} + \chi - 1\right)U - \left(s^2 - \alpha\right)\right).$$

If $\alpha = u^*_2 \geq s^2$, then we have

$$K_{u^*_2} = \frac{U}{su^*_2}\left(s^2 - u^*_2\right)\left(U - u^*_2\right) \geq 0.$$

If $\alpha = s^2 > u^*_2$, then we have

$$K_{s^2} = \frac{U^2}{s}\left(\frac{r}{2s^2} + \chi - 1\right) > 0$$

whenever $\frac{r}{2s^2} + \chi > 1$. We note that $s^2 \leq u^*_2$ is satisfied if $2\chi \leq 1$ or $\frac{r}{2s^2} + \chi \leq 1$.

Hence, the vector field of system (2.2) on $int\left(\overline{E_0\,P_\alpha}\right)$ points out of the interior of $R_2$. Therefore, $R_2$ is a negatively invariant set of system (2.2). This completes the proof of part (ii).

□

Now we shall give the existence of exactly two heteroclinic solutions of system (2.2).

**Theorem 2.3.** *For every* $s > \max\left\{\sqrt{u_1^*},\ \sqrt{2r\chi}\right\}$*, there exist exactly two heteroclinic orbits* $(U, V)$ *of system* (2.2) *satisfying* $V' > 0$ *on* $(-\infty, \infty)$ *such that the first orbit lying in the region* $R_1$ *satisfies* $(U, V)(-\infty) = E_1$*,* $(U, V)(+\infty) = E_0$*, and the second orbit lying in the region* $R_2$ *satisfies* $(U, V)(-\infty) = E_2$*,* $(U, V)(+\infty) = E_0$*, where the regions* $R_1$ *and* $R_2$ *are defined in Lemma* 2.2.

*Proof.* We note that as $V' > 0$, system (2.2) does not have any periodic orbit for $U, V > 0$, and the interior of invariant regions $R_1$ and $R_2$ does not contain any equilibrium point. Hence, using Lemma 2.1, Lemma 2.2 and Poincare Bendixson Theorem, the proof is completed. □

## 3 Numerical simulations

Let us consider several situations to verify the result.

For $s = 2$, $\chi = 0.5$, $r = 3$, we have the numerical simulation showing two heteroclinic orbits moving from $E_1 = (2, 0)$ and $E_2 = (6, 0)$ to $E_0 = (1, 0)$ in the regions $R_1$ and $R_2$ respectively as in Figure 1. For $s = 2$, $\chi = 1.8$, $r = 0.9$, we have the numerical simulation showing two heteroclinic orbits moving from $E_1 = (0.6268, 0)$ and $E_2 = (1.5954, 0)$ to $E_0 = (1, 0)$ in the regions $R_1$ and $R_2$ respectively as in Figure 2.

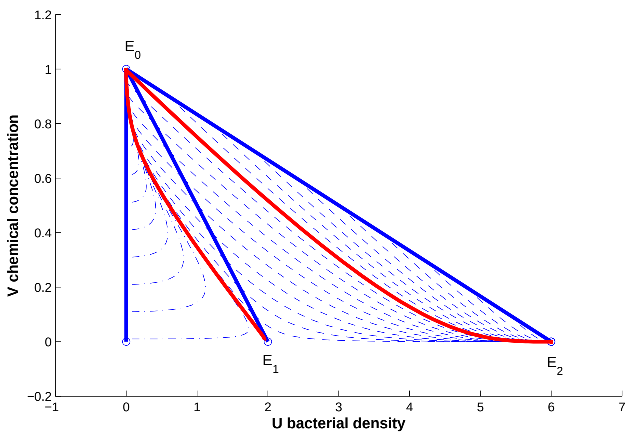


**Figure 1.** The heteroclinic orbits of system (2.2) for $s = 2$, $\chi = 0.5$, $r = 3$ moving from $E_1 = (2, 0)$ and $E_2 = (6, 0)$ to $E_0 = (0, 1)$.

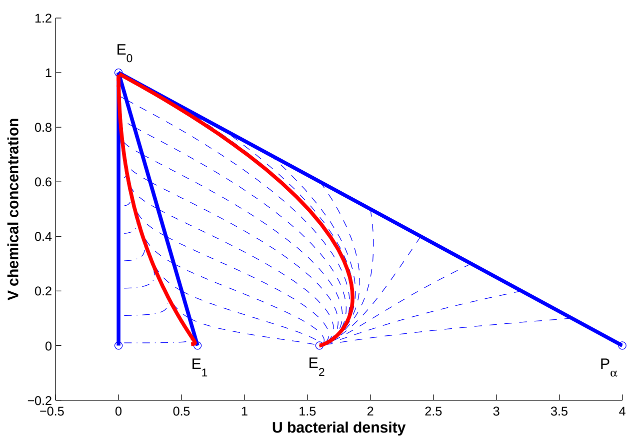


**Figure 2.** The heteroclinic orbits of system (2.2) for $s = 2$, $\chi = 1.8$, $r = 0.9$ moving from $E_1 = (0.6268, 0)$ and $E_2 = (1.5954, 0)$ to $E_0 = (0, 1)$.

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Received: September 11, 2019.
Accepted: December 27, 2019.