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NEW IDENTITIES FOR THE HORADAM QUATERNIONS

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Abstract. In this paper, we gave some new formulas for binomial sums nd some identities of Horadam quaternions by using Binet formula. Also matrix representations for Horadam quaternion matrix were deduced.

1 Introduction

In mathematics, quaternions are a number system which extends the complex numbers. Normed division algebra, which is very significant topic occur the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbf{H}$, and octonions $\mathbf{O}$. One of the most important developments in modern algebra was the discovery of quaternions in 1843. It is possible to see effects of Hamilton's discovery in 1843 from quantum physics to computer science etc.

A quaternion is represented in the form ;

$$q = (a_0, a_1, a_2, a_3) = a_0 + a_1 i + a_2 j + a_3 k$$

where $a_0, a_1, a_2,$ and a_3 are real numbers and 1, i, j, and k are fundamental quaternion units, which holds following multiplication rules :

$$i^2 = j^2 = k^2 = ijk = -1,\ ij = -ji =,\ ik = -ki, \text{ and } jk = -kj.$$

Definition 1.1. [1] The Horadam sequence is defined by

$$W_n = W_n(a, b; p, q) = \ pW_{n-1} + qW_{n-2},\ n \geq 2$$

here $a, b, p,$ and q are integers and $W_0 = a$, $W_1 = b$, and W_n is $n-$th Horadam number.

Moreover,

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2} \text{ and } \beta = \frac{p - \sqrt{p^2 + 4q}}{2}$$

are roots of the characteristic equation $x^2 - px - q = 0$. So the Binet formula for this sequence is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{1.1}$$

where $A = b - a\beta$, $B = b - a\alpha$.

It must be noticed that if we take $a = 0$, $b = 1$, $p = 1$, and $q = 1$ in (1.1) , then we get $W_n(0, 1; 1, 1)_0^\infty$, that is, we obtain Fibonacci quaternions. Similarly, if we take $a = 0$, $b = 1$, $p = 2$, and $q = 1$, then we get $W_n(0, 1; 2, 1)_0^\infty$, that is, we obtain Pell quaternions.

Definition 1.2. [4] Horadam quaternions are defined by

$$Q_{w,n+2} = W_{n+2} + W_{n+3}i + W_{n+4}j + W_{n+5}k$$

for $n \geq 0$, where W_n is the $n-$th Horadam number and initial conditions are

$$Q_{w,0} = (a, b, pb + qa, p^2b + pqa + qb)$$

and

$$Q_{w,1} = (b, pb + qa, p^2b + pqa + qb, p^3b + p^2qa + 2pqb + q^2a).$$

Furthermore, recurrence relation for Horadam quaternion sequences is

$$Q_{w,n+2} = pQ_{w,n+1} + qQ_{w,n}.$$

Also,

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2} \text{ and } \beta = \frac{p-\sqrt{p^2+4q}}{2} \tag{1.2}$$

are roots of the characteristic equation. For $n \geq 0$, the Binet formula for the Horadam quaternions is given by

$$Q_{w,n} = \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha - \beta}$$

here $A = b - a\beta$, $B = b - a\alpha$, $\underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$, and $\underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$. Besides that generating function for the Horadam quaternions is as follow

$$g(t) = \frac{Q_{w,0} + (Q_{w,1} - pQ_{w,0})t}{1 - pt - qt^2}$$

where $Q_{w,0}$ and $Q_{w,1}$ are the initial terms of the Horadam quaternions.

Many researchers investigated the subject of quaternions. Firstly Iyer [9] and Iakin [7, 8] introduced relations of quaternions of Fibonacci and Lucas sequences so they obtained generalized quaternions. Moreover, in [16] Swamy obtained the relations of generalized Fibonacci quaternions. Note that, the second order linear recurence quaternion sequences for example in [6], Horadam defined quaternion recurrence relations. If we return to today's work, Halıcı [4, 5], İpek [11], Szynal-Liana [14, 15], Catarino [2], Polatlı [13] and many researchers studied different types of quaternions [10].

We motivated by their results in [3, 4, 12]. In [3], they studied Horadam symbol elements but we found results about sums of quaternions. In [4], they considered Horadam quaternions which is the generalizations of the second order quaternions. In [12], they studied formulas for binomial sums of quaternions. In our work, we introduced Horadam quaternions with construct the combinatoric relations of the Horadam quaternions.

During this paper, for convenience of representation, we adopt $\Delta = p^2 + 4q$ and we use following equalities

$$\begin{aligned} \alpha + \beta &= p \\ \alpha - \beta &= \sqrt{\Delta} \\ \alpha\beta &= -q \\ \alpha^2 &= p\alpha + q \\ \beta^2 &= p\beta + q \\ \alpha^2 + q &= \alpha\sqrt{\Delta} \\ \beta^2 + q &= -\beta\sqrt{\Delta}. \end{aligned} \tag{1.3}$$

2 Main Theorems of Horadam Quaternions

In [4], Halici and Karataş introduced Horadam quaternions and derived its Binet formula, generating function, and other properties for this quaternions.

In this section, by using releated Binet formula we found formulas for binomial sums of Horadam quaternions were obtained. And also, we presented exponential generating function, d'Ocagne identity and some identities of Horadam quaternions.

Theorem 2.1. *(d'Ocagne Identity) Let $Q_{w,n}$ denote the $n-th$ Horadam quaternions. For k, $n \geq 0$, and $k \geq n$, the d'Ocagne identity for Horadam quaternions are*

$$Q_{w,k}Q_{w,n+1} - Q_{w,k+1}Q_{w,n} = \frac{(-q)^n AB}{\alpha - \beta}(\underline{\alpha}\underline{\beta}\alpha^{k-n} - \underline{\beta}\underline{\alpha}\beta^{k-n})$$

Proof. Using the related Binet formula,

$$\begin{aligned}
& Q_{w,k}Q_{w,n+1} - Q_{w,k+1}Q_{w,n} \\
= & \left(\frac{A\underline{\alpha}\alpha^k - B\underline{\beta}\beta^k}{\alpha-\beta}\right)\left(\frac{A\underline{\alpha}\alpha^{n+1} - B\underline{\beta}\beta^{n+1}}{\alpha-\beta}\right) \\
& - \left(\frac{A\underline{\alpha}\alpha^{k+1} - B\underline{\beta}\beta^{k+1}}{\alpha-\beta}\right)\left(\frac{A\underline{\alpha}\alpha^{n} - B\underline{\beta}\beta^{n}}{\alpha-\beta}\right) \\
= & \frac{1}{(\alpha-\beta)^2}\left\{\begin{array}{c} -BA\underline{\beta}\underline{\alpha}\beta^k\alpha^{n+1} - AB\underline{\alpha}\underline{\beta}\alpha^k\beta^{n+1} \\ +BA\underline{\beta}\underline{\alpha}\beta^{k+1}\alpha^n + AB\underline{\alpha}\underline{\beta}\alpha^{k+1}\beta^n \end{array}\right\} \\
= & \frac{(-1)^n(\alpha\beta)^n}{(\alpha-\beta)}(AB\underline{\alpha}\underline{\beta}\alpha^{k-n} - AB\underline{\beta}\underline{\alpha}\beta^{k-n})
\end{aligned}$$

this completes the proof. □

Theorem 2.2. *Let $Q_{w,n}$ denote the $n-th$ Horadam quaternions. For $k, n \in \mathbb{N}$, we obtain*

$$Q_{w,k}Q_{w,n+1} + qQ_{w,k-1}Q_{w,n} = \frac{A^2\underline{\alpha}^2\alpha^{k+n} - B^2\underline{\beta}^2\beta^{k+n}}{\alpha-\beta}.$$

Proof. Using the relevant Binet formula and (1.3), we get

$$\begin{aligned}
& Q_{w,k}Q_{w,n+1} + qQ_{w,k-1}Q_{w,n} \\
= & \left(\frac{A\underline{\alpha}\alpha^k - B\underline{\beta}\beta^k}{\alpha-\beta}\right)\left(\frac{A\underline{\alpha}\alpha^{n+1} - B\underline{\beta}\beta^{n+1}}{\alpha-\beta}\right) \\
& +q\left(\frac{A\underline{\alpha}\alpha^{k-1} - B\underline{\beta}\beta^{k-1}}{\alpha-\beta}\right)\left(\frac{A\underline{\alpha}\alpha^{n} - B\underline{\beta}\beta^{n}}{\alpha-\beta}\right) \\
= & \frac{1}{(\alpha-\beta)^2}\left\{\begin{array}{c} A^2\underline{\alpha}^2\alpha^{n+k+1} - AB\underline{\alpha}\underline{\beta}\alpha^k\beta^{n+1} + B^2\underline{\beta}^2\beta^{n+k+1} \\ -AB\underline{\beta}\underline{\alpha}\beta^k\alpha^{n+1} + qA^2\underline{\alpha}^2\alpha^{n+k-1} - qAB\underline{\alpha}\underline{\beta}\alpha^{k-1}\beta^n \\ +qB^2\underline{\beta}^2\beta^{n+k-1} - qAB\underline{\beta}\underline{\alpha}\beta^{k-1}\alpha^n \end{array}\right\} \\
= & \frac{1}{(\alpha-\beta)^2}\left\{\begin{array}{c} A^2\underline{\alpha}^2\alpha^{n+k}(\alpha+\frac{q}{\alpha}) - AB\underline{\alpha}\underline{\beta}\alpha^{k-1}\beta^n(\alpha\beta+q) \\ -AB\underline{\beta}\underline{\alpha}\beta^{k-1}\alpha^n(\alpha\beta+q) - B^2\underline{\beta}^2\beta^{n+k}(-\beta-\frac{q}{\beta}) \end{array}\right\} \\
= & \frac{A^2\underline{\alpha}^2\alpha^{k+n} - B^2\underline{\beta}^2\beta^{k+n}}{\alpha-\beta}.
\end{aligned}$$

□

The following theorem provides the formulas of exponential generating function for Horadam quaternions.

Theorem 2.3. *Let $Q_{w,n}$ denote the $n-th$ Horadam quaternions. The exponential generating function for Horadam quaternions is*

$$\sum_{k=0}^{\infty} Q_{w,k}\frac{t^k}{k!} = \frac{A\underline{\alpha}e^{\alpha t} - B\underline{\beta}e^{\beta t}}{\alpha-\beta}.$$

Proof. Recall that the Binet formula for Horadam quaternions,

$$\begin{aligned}
\sum_{k=0}^{\infty} Q_{w,k}\frac{t^k}{k!} &= \sum_{k=0}^{\infty}\left(\frac{A\underline{\alpha}\alpha^k - B\underline{\beta}\beta^k}{\alpha-\beta}\right)\frac{t^k}{k!} \\
&= \frac{A\underline{\alpha}}{\alpha-\beta}\sum_{k=0}^{\infty}\frac{(\alpha t)^k}{k!} - \frac{B\underline{\beta}}{\alpha-\beta}\sum_{k=0}^{\infty}\frac{(\beta t)^k}{k!} \\
&= \frac{A\underline{\alpha}e^{\alpha t} - B\underline{\beta}e^{\beta t}}{\alpha-\beta},
\end{aligned}$$

this is valid. □

Theorem 2.4. *Let $Q_{w,n}$ denote the $n-th$ Horadam quaternions. For all $k\in\mathbb{N}$ and $m,s\in\mathbb{Z}$, we have*

$$\sum_{k=0}^{\infty} Q_{w,mk+s}x^k = \frac{Q_{w,s}-(-q)^m Q_{w,s-m}x}{1-(\alpha^m+\beta^m)x+(-q)^m x^2}.$$

Proof. Using the Binet formula,

$$\begin{aligned} & \sum_{k=0}^{\infty} Q_{w,mk+s}x^k \\ = & \sum_{k=0}^{\infty} \frac{A\underline{\alpha}\alpha^{mk+s}-B\underline{\beta}\beta^{mk+s}}{\alpha-\beta}x^k \\ = & \frac{A\underline{\alpha}\alpha^s}{\alpha-\beta}\sum_{k=0}^{\infty}\alpha^{mk}x^k-\frac{B\underline{\beta}\beta^s}{\alpha-\beta}\sum_{k=0}^{\infty}\beta^{mk}x^k \end{aligned}$$

with the help of sum formula, we get

$$\begin{aligned} & \frac{A\underline{\alpha}\alpha^s}{\alpha-\beta}\left(\frac{1}{1-\alpha^m x}\right)-\frac{B\underline{\beta}\beta^s}{\alpha-\beta}\left(\frac{1}{1-\beta^m x}\right) \\ = & \frac{A\underline{\alpha}\alpha^s(1-\beta^m x)-B\underline{\beta}\beta^s(1-\alpha^m x)}{(\alpha-\beta)(1-(\alpha^m+\beta^m)x+(\alpha\beta)^m x^2)} \end{aligned}$$

if necessary arrangements are make, then we get

$$\sum_{k=0}^{\infty} Q_{w,mk+s}x^k = \frac{\left(\frac{A\underline{\alpha}\alpha^s-B\underline{\beta}\beta^s}{\alpha-\beta}\right)-(\alpha\beta)^m\left(\frac{A\underline{\alpha}\alpha^{s-m}-B\underline{\beta}\beta^{s-m}}{\alpha-\beta}\right)x}{1-(\alpha^m+\beta^m)x+(\alpha\beta)^m x^2}.$$

□

The following theorem deals with the sum formulas of $mk+s$ terms.

Theorem 2.5. *Let $Q_{w,n}$ denote the $n-th$ Horadam quaternions. For all $n\in\mathbb{N}$ and $m,s\in\mathbb{Z}$, we have*

$$\sum_{k=0}^{n} Q_{w,mk+s} = \frac{(-q)^m Q_{w,mn+s}-Q_{w,mn+m+s}-(-q)^m Q_{w,s-m}+Q_{w,s}}{(-q)^m-(\alpha^m+\beta^m)+1}.$$

Proof. Using the Binet formula and (1.3), we have

$$\begin{aligned} & \sum_{k=0}^{n} Q_{w,mk+s} \\ = & \sum_{k=0}^{n} \frac{A\underline{\alpha}\alpha^{mk+s}-B\underline{\beta}\beta^{mk+s}}{\alpha-\beta} \\ = & \frac{A\underline{\alpha}\alpha^s}{\alpha-\beta}\sum_{k=0}^{n}\alpha^{mk}-\frac{B\underline{\beta}\beta^s}{\alpha-\beta}\sum_{k=0}^{n}\beta^{mk} \\ = & \frac{A\underline{\alpha}(\alpha^{mn+m+s}-\alpha^s)(\beta^m-1)-B\underline{\beta}(\beta^{mn+m+s}-1)(\alpha^m-1)}{(\alpha-\beta)(\alpha^m\beta^m-\alpha^m-\beta^m+1)} \end{aligned}$$

if necessary arrangements are make, then

$$= \frac{1}{(\alpha\beta)^m-(\alpha^m+\beta^m)+1}\left\{\begin{array}{c} (\alpha\beta)^m\frac{A\underline{\alpha}\alpha^{mn+s}-B\underline{\beta}\beta^{mn+s}}{\alpha-\beta}-\frac{A\underline{\alpha}\alpha^{mn+m+s}-B\underline{\beta}\beta^{mn+m+s}}{\alpha-\beta} \\ -(\alpha\beta)^m\frac{A\underline{\alpha}\alpha^{s-m}-B\underline{\beta}\beta^{s-m}}{\alpha-\beta}+\frac{A\underline{\alpha}\alpha^s-B\underline{\beta}\beta^s}{\alpha-\beta} \end{array}\right\}$$

we get the result. □

Theorem 2.6. *Let $Q_{w,n}$ denote the $n-th$ Horadam quaternions. For all $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} Q_{w,k} = Q_{w,2n}.$$

Proof. Using the related Binet formula,

$$\begin{aligned} & \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} Q_{w,k} \\ = & \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \frac{A\underline{\alpha}\alpha^k - B\underline{\beta}\beta^k}{\alpha-\beta} \\ = & \frac{A\underline{\alpha}}{\alpha-\beta} \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \alpha^k - \frac{B\underline{\beta}}{\alpha-\beta} \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \beta^k \end{aligned}$$

and also using binomial formula, we get

$$\frac{A\underline{\alpha}}{\alpha-\beta}(q+p\alpha)^n - \frac{B\underline{\beta}}{\alpha-\beta}(q+p\beta)^n$$

by (1.3), we obtain as follow

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} Q_{w,k} = \frac{A\underline{\alpha}\alpha^{2n} - B\underline{\beta}\beta^{2n}}{\alpha-\beta}.$$

□

Theorem 2.7. *Let $Q_{w,n}$ denote the $n-th$ Horadam quaternions. For $k \in \mathbb{N}$ and $s \in \mathbb{Z}$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} q^{n-k} p^k Q_{w,k+s} = Q_{w,2n+s}$$

Proof. Using the related Binet formula ,

$$\begin{aligned} & \sum_{n=0}^{\infty} Q_{w,2n+s} \frac{x^n}{n!} \\ = & \sum_{n=0}^{\infty} \frac{A\underline{\alpha}\alpha^{2n+s} - B\underline{\beta}\beta^{2n+s}}{\alpha-\beta} \frac{x^n}{n!} \\ = & \frac{A\underline{\alpha}\alpha^s e^{\alpha^2 x} - B\underline{\beta}\beta^s e^{\beta^2 x}}{\alpha-\beta} \\ = & e^{qx} \frac{A\underline{\alpha}\alpha^s e^{p\alpha x} - B\underline{\beta}\beta^s e^{p\beta x}}{\alpha-\beta}. \end{aligned}$$

If we consider binomial formula,

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} q^n \frac{x^n}{n!}\right) \left(\frac{A\underline{\alpha}\alpha^s}{\alpha-\beta} \sum_{n=0}^{\infty} p^n \alpha^n \frac{x^n}{n!} - \frac{B\underline{\beta}\beta^s}{\alpha-\beta} \sum_{n=0}^{\infty} p^n \beta^n \frac{x^n}{n!}\right) \\ = & \left(\sum_{n=0}^{\infty} q^n \frac{x^n}{n!}\right) \left(\sum_{n=0}^{\infty} p^n Q_{w,n+s} \frac{x^n}{n!}\right) \\ = & \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} q^{n-k} p^k Q_{w,k+s}\right] \frac{x^n}{n!}, \end{aligned}$$

then we get

$$\sum_{k=0}^{n}\binom{n}{k}q^{n-k}p^{k}Q_{w,k+s=}Q_{w,2n+s}.$$

□

Theorem 2.8. *Let $Q_{w,n}$ denote the $n-th$ Horadam quaternions. For $n\in\mathbb{N}$ and $s\in\mathbb{Z}$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}(-q)^{n-k}Q_{w,2k+s}=p^{n}Q_{w,n+s}$$

Proof. Using the Binet formula and (1.3), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}(-q)^{n-k}Q_{w,2k+s}\right]\frac{x^{n}}{n!}\\ =&\left(\sum_{n=0}^{\infty}(-q)^{n}\frac{x^{n}}{n!}\right)\left(\sum_{n=0}^{\infty}Q_{w,2n+s}\frac{x^{n}}{n!}\right)\\ =&\left(\sum_{n=0}^{\infty}(-q)^{n}\frac{x^{n}}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{A\underline{\alpha}\alpha^{2n+s}-B\underline{\beta}\beta^{2n+s}}{\alpha-\beta}\frac{x^{n}}{n!}\right).\end{aligned}$$

Note that if we take binomial formula

$$\begin{aligned}&e^{-qx}\frac{A\underline{\alpha}\alpha^{s}e^{\alpha^{2}x}-B\underline{\beta}\beta^{s}e^{\beta^{2}x}}{\alpha-\beta}\\ =&\frac{A\underline{\alpha}\alpha^{s}}{\alpha-\beta}e^{p\alpha x}-\frac{B\underline{\beta}\beta^{s}}{\alpha-\beta}e^{p\beta x}\\ =&\frac{A\underline{\alpha}\alpha^{s}}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(p\alpha x)^{n}}{n!}-\frac{B\underline{\beta}\beta^{s}}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(p\beta x)^{n}}{n!}\\ =&\sum_{n=0}^{\infty}p^{n}\frac{A\underline{\alpha}\alpha^{n+s}-B\underline{\beta}\beta^{n+s}}{\alpha-\beta}\frac{x^{n}}{n!},\end{aligned}$$

as a result of calculations, we obtained

$$\sum_{k=0}^{n}\binom{n}{k}(-q)^{n-k}Q_{w,2k+s=}p^{n}Q_{w,n+s}.$$

□

Also we presented the different binomial identites.

Theorem 2.9. *Let $Q_{w,n}$ denote the $n-th$ Horadam quaternions. For $n\geq 0$, we have*

$$\begin{aligned}(i)\sum_{k=0}^{n}\binom{n}{k}Q_{w,2k+s}q^{n-k}&=\begin{cases}Q_{w,n+s}\Delta^{\frac{n}{2}} & ,\textit{for even } n\\ (A\underline{\alpha}\alpha^{n+s}+B\underline{\beta}\beta^{n+s})\Delta^{\frac{n-1}{2}} & ,\textit{for odd } n\end{cases}\\ (ii)\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}Q_{w,2k+s}q^{n-k}&=\begin{cases}p^{n}Q_{w,n+s} & ,\textit{for even } n\\ -p^{n}Q_{w,n+s} & ,\textit{for odd } n\end{cases}\\ (iii)\sum_{k=0}^{n}\binom{n}{k}Q_{w,k}Q_{w,k+s}q^{n-k}&=\begin{cases}(A^{2}\underline{\alpha}^{2}\alpha^{n+s}+B^{2}\underline{\beta}^{2}\beta^{n+s})\Delta^{\frac{n-2}{2}} & ,\textit{for even } n\\ (A^{2}\underline{\alpha}^{2}\alpha^{n+s}-B\underline{\beta}^{2}\beta^{n+s})\Delta^{\frac{n-2}{2}} & ,\textit{for odd } n\end{cases}\\ (iv)\sum_{k=0}^{n}\binom{n}{k}Q_{w,k}^{2}q^{n-k}&=\begin{cases}(A^{2}\underline{\alpha}^{2}\alpha^{k}-B^{2}\underline{\beta}^{2}\beta^{k})\Delta^{\frac{n-2}{2}} & ,\textit{for even } n\\ (A^{2}\underline{\alpha}^{2}\alpha^{k}-B\underline{\beta}^{2}\beta^{k})\Delta^{\frac{n-2}{2}} & ,\textit{for odd } n.\end{cases}\end{aligned}$$

Proof. (i) Using the Binet formula and (1.3), then

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}Q_{w,2k+s}q^{n-k}\\
&= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{A\underline{\alpha}\alpha^{2k+s}-B\underline{\beta}\beta^{2k+s}}{\alpha-\beta}\right)q^{n-k}\\
&= \frac{A\underline{\alpha}\alpha^{s}}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}(\alpha^2)^k q^{n-k}-\frac{B\underline{\beta}\beta^{s}}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}(\beta^2)^k q^{n-k}\\
&= \frac{A\underline{\alpha}\alpha^{s}(\alpha^2+q)^n-B\underline{\beta}\beta^{s}(\beta^2+q)^n}{\alpha-\beta}\\
&= \frac{A\underline{\alpha}\alpha^{s}(\alpha\sqrt{\Delta})^n-B\underline{\beta}\beta^{s}(-\beta\sqrt{\Delta})^n}{\alpha-\beta}\\
&= \left(\frac{A\underline{\alpha}\alpha^{n+s}+(-1)^{n+1}B\underline{\beta}\beta^{n+s}}{\alpha-\beta}\right)\Delta^{\frac{n}{2}}.
\end{aligned}$$

The rest (ii), (iii), and (iv) can be made similar to (i). □

3 Matrix Representations of Horadam Quaternions

Matrix method is very useful to obtain results or algebraic representations in the study of recurrence relations. For this purpose, matrices for quaternion sequences have been studied by many authors before. For instance, in [5], author defined the a matrix whose entries called Fibonacci quaternions such that

$$Q=\begin{pmatrix} Q_2 & Q_1\\ Q_1 & Q_0\end{pmatrix}.$$

She obtained Cassini identity using the Fibonacci quaternion matrix. Also in [14], authors made up Pell quaternion and Pell-Lucas quaternion matrix as

$$R(n)=\begin{pmatrix} R_n & R_{n-1}\\ R_{n-1} & R_{n-2}\end{pmatrix} \text{ and } S(n)=\begin{pmatrix} S_n & S_{n-1}\\ S_{n-1} & S_{n-2}\end{pmatrix}, \text{ for } n\geq 2$$

respectively. Also in [15], Jacobsthal quaternion matrix was made up.

Inspiring by these findings, we define a new quaternion matrix, which is generalization of the previously defined quaternion matrices. It can be called Horadam quaternion matrix and it is defined by

$$H(n)=\begin{pmatrix} Q_{w,n+1} & qQ_{w,n}\\ Q_{w,n} & qQ_{w,n-1}\end{pmatrix}, \text{ for } n\geq 1$$

where entries elements are Horadam quaternions.

Theorem 3.1. *Let $Q_{w,n}$ denote the $n-th$ Horadam quaternions. For $n\geq 1$, $n\in\mathbb{Z}$, we have*

$$\begin{pmatrix} Q_{w,n+1} & qQ_{w,n}\\ Q_{w,n} & qQ_{w,n-1}\end{pmatrix}=\begin{pmatrix} Q_{w,2} & qQ_{w,1}\\ Q_{w,1} & qQ_{w,0}\end{pmatrix}\begin{pmatrix} p & q\\ 1 & 0\end{pmatrix}^{n-1}.$$

Proof. Using induction method, we can see proof easily. Let $n=1$, then first step is clear. Now let us assume that the equation is

$$\begin{pmatrix} Q_{w,k} & qQ_{w,k-1}\\ Q_{w,k-1} & qQ_{w,k-2}\end{pmatrix}=\begin{pmatrix} Q_{w,2} & qQ_{w,1}\\ Q_{w,1} & qQ_{w,0}\end{pmatrix}\begin{pmatrix} p & q\\ 1 & 0\end{pmatrix}^{k-2}$$

for $n = k-1$. For $n = k$ it becomes

$$\begin{aligned}
&\begin{pmatrix} Q_{w,2} \; qQ_{w,1} \\ Q_{w,1} \; qQ_{w,0} \end{pmatrix} \begin{pmatrix} p \; q \\ 1 \; 0 \end{pmatrix}^{k-1} \\
= &\begin{pmatrix} Q_{w,2} \; qQ_{w,1} \\ Q_{w,1} \; qQ_{w,0} \end{pmatrix} \begin{pmatrix} p \; q \\ 1 \; 0 \end{pmatrix}^{k-2} \begin{pmatrix} p \; q \\ 1 \; 0 \end{pmatrix} \\
= &\begin{pmatrix} Q_{w,k} \; qQ_{w,k-1} \\ Q_{w,k-1} \; qQ_{w,k-2} \end{pmatrix} \begin{pmatrix} p \; q \\ 1 \; 0 \end{pmatrix}
\end{aligned}$$

which is desired. □

4 Conclusion

In this paper, we discussed Horadam quaternion sequences that generalize all the quaternion sequences, like Fibonacci and Lucas. We have considered the iterative relation of this quaternion, the idea present in [4]. In this paper, we have anticipated to find new identities for Horadam quaternion sequences. For this aim, we gave the exponential generating function, d'Ocagne identity, some identities, and some binomial sum formulas. Also matrix representation formula is given related to these quaternions.

Hence, in the future, we will plan to investigate circulant matrices and skew circulant matrices whose entries are Horadam quaternions and its key features.

References

[1] A. F. Horadam, Complex Fibonacci Numbers and Fibonacci Quaternions, *The Am. Math. Month.* **70(3)**, 289–291 (1963).

[2] P. Catarino, A Note on $h(x)-$Fibonacci Quaternion Polynomials, *Chaos, Solitons and Fractals* **77**, 1–5 (2015).

[3] S. Gopal Rayaguru, D. Savin, G. Krishna Panda, On the Horadam Symbol Elements, *Notes on Number Theory and Discrete Mathematics*, **25, 25** 91–112 (2019).

[4] S. Halici, A. Karatas, On a Generalization for Fibonacci Quaternions, *Chaos, Solitons and Fractals*, **98** 178–182 (2017).

[5] S. Halici, On Fibonacci Quaternions, *Adv. Appl. Clifford Algebras*, **22, 2** 321-327 (2012).

[6] A.F. Horadam, Quaternion Recurrence Relations, *Ulam Q.*, **2** 23-33 (1993).

[7] A.L. Iakin, Generalized Quaternions with Quaternion Components, *Fibonacci Q.*, **15**, 350-352 (1977).

[8] A.L. Iakin, Generalized Quaternions of Higher Order, *Fibonacci Q.*, **15**, 343-346 (1977).

[9] M.R. Iyer, A Note on Fibonacci Quaternions, *Fibonacci Q.*, **3** 225-229 (1969).

[10] J. L. Ramirez, Some Combinatorial Properties of the k-Fibonacci and the k-Lucas Quaternions, *An. St. Univ. Ovidius Constanta*, **23(2)**, (2015).

[11] A. İpek, On $(p, q)-$Fibonacci Quaternions and Their Binet Formulas, Generating Functions and Certain Binomial Sums, *An. SÂ̧t. Univ. Ovidius Constanta*, **27**, 1343-1351 (2017).

[12] E. Polatlı, S. Kesim, On Quaternions with Generalized Fibonacci And Lucas Number Components, *Advances in Difference Equations*, **2015:169**, (2015).

[13] E. Polatlı, A Generalization of Fibonacci and Lucas Quaternions, *Adv. Appl. Clifford Algebras*, **26**, 719–730 (2016).

[14] A. Szynal-Liana, I. Włoch, The Pell Quaternions and the Pell Octonions. *Adv. Appl. Clifford Algebras*, **26**, 435–440 (2016).

[15] A. Szynal-Liana, I. Włoch. A Note on Jacobsthal Quaternions. *Adv. Appl. Clifford Algebras*, **26**, 441–447 (2016).

[16] M.N.S. Swamy. On Generalized Fibonacci Quaternions. *Fibonacci Q.*, **5** 547-550 (1973).

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