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New wavelet-Galerkin method for the numerical solution of Helmholtz equation

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Abstract This paper investigates the numerical solution of the 1-D Helmholtz equation via the new wavelet-Galerkin method (NWGM). The numerical solution of the Helmholtz equation is very expensive if attempted by traditional discretization methods (finite difference method, Galerkin method). The proposed scheme is rather simple than the existing ones, the power of this technique is illustrated by comparing numerical solutions with the exact solution. The solutions presented here are comparably good and give higher accuracy than the Haar wavelet collocation method (HWCM) with an exact solution by increasing the resolution level.

1 Introduction

Many of the science and engineering problems can be modeled mathematically as a boundary value problem. Thus, it is very important to build a method to solve a boundary value problem. For some simple boundary value problems, it may be possible to solve it analytically by separation of variables. However, in most applications, boundary value problems are much more complex and there are no available analytical methods. For this reason, some of the numerical methods such as finite differences, finite elements and multigrid are used for the solution. The wavelet method, however, offers several advantages over traditional methods. Wavelet analysis is a newly developed numerical concept which permits one to represent a function in terms of a set of basis functions, called as wavelets, which are localized in space. We can expect numerical methods based on wavelet bases to be able to attain good spatial and spectral resolutions. Recently so many efforts have been put into developing the techniques based on the properties of wavelet bases introduced in the 1980s by Stromberg and Meyer.

Nowadays, the ideas of thoughtful wavelets were provided by Daubechies, Mallat, and others [1-3]. The number of applications where the wavelets have been used has been found in the literature. Various types of wavelet functions have existed since from many years. The Daubechies family of wavelets will be reflected due to their useful properties. Since the contribution of orthogonal bases of compactly supported wavelet by Daubechies [2] and multiresolution analysis based fast wavelet transform algorithm by Beylkin et al. [4], wavelet based approximation of differential equations gained momentum in an attractive way. Wavelets have the ability to represent the solutions in different levels of resolutions, which make them generally useful for emerging hierarchical solutions in science and engineering problems.

In the approximations theory, wavelet based Galerkin method is the furthermost recurrently used technique nowadays. Daubechies wavelets based Galerkin method to solve certain differential equations requires a computational domain of modest nature. This tremendous work has been done by many researchers, for example, see reference [5-9]. Yet there is difficulty in dealing with connection coefficients for different scales. In order to demonstrate the wavelet technique i.e a new wavelet-Galerkin method (NWGM), we consider the one dimensional Helmholtz's equation. By comparison with a Haar wavelet (simple wavelet but not continuous) collocation method (HWCM) [10-15] solution to this problem, we show how a wavelet technique may be efficiently developed.

The present paper is organized as follows. Section 2 highlights the Daubechies wavelets. The method of solution is presented in section 3. Section 4 deals with the implementation of the test problem. Finally, concluding remarks of the proposed work are discussed in section 5.

2 Daubechies wavelets

Wavelets are a family of orthonormal functions which are characterized by the translation and dilation of a single function. Daubechies wavelets are compactly supported functions introduced by Daubechies [2]. This means that they have non zero values within a finite interval and have a zero value everywhere else. That's why it is useful for representing the solution of a differential equation. They are orthonormal bases for functions in $L^2(R)$. The construction of wavelet functions starts from building the scaling or dilation function, $\phi(x)$ and a set of coefficients $h_k\,,\ k \in Z$, which satisfies the two-scale relation or refinement equation,

$$\phi(x) = \sum_{k=0}^{L-1} h_k \phi(2x - k) \tag{2.1}$$

where L denotes the order of the Daubechies wavelet. The associated wavelet function is given by

$$\psi(x) = \sum_{k=0}^{L-1} g_k \phi(2x - k) \tag{2.2}$$

where $g_k = (-1)^k h_{L-1-k}$ and $\int \phi(x)dx = 1$.
The translation and dilations of the scaling function $\phi(2^J x - k)$ or the wavelet function $\psi(2^J x - k)$ form a complete and orthogonal basis.
The wavelet basis induces a multiresolution analysis (MRA) [3] on $L^2(R)$, i.e. the decomposition of the Hilbert space $L^2(R)$ into a chain of closed subspaces

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \tag{2.3}$$

such that

$$\bigcup V_J = L^2(R) \tag{2.4}$$

and

$$\bigcap V_J = \{0\} \tag{2.5}$$

By defining $W_J$as an orthogonal complement of V_J in V_{J+1},

$$V_{J+1} = V_J \oplus W_J \tag{2.6}$$

The space $L^2(R)$ is represented as a direct sum of W_j's as

$$L^2(R) = \oplus W_J \tag{2.7}$$

On each fixed scale $J(\geq\ 0)$, the wavelets $\left\{\psi_{J,\,k}(x) = 2^{J/2}\psi(2^J x - k),\, k \in Z\right\}$form an orthonormal basis of W_J and the functions $\left\{\phi_{J,\,k}(x) = 2^{J/2}\phi(2^J x - k),\, k \in Z\right\}$ form an orthonormal basis of V_J. The set of spaces V_J is called a multiresolution analysis of$L^2(R)$, these spaces will be used to approximate the solutions of differential equations using the new wavelet-Galerkin method.

3 Method of solution

The Russian engineer V. I. Galerkin had developed a projection method based on weak form in which a set of test functions are selected such that residual of the differential equation becomes orthogonal to test functions [16, 17].

If the base functions in the Galerkin method are wavelets, then it is called the wavelet-Galerkin method (WGM). Here, we have proposed the new wavelet-Galerkin method, which has advantages over the classical WGM in terms of time consumption that is; we applied the method instead of finding the connection coefficients.

Theorem 3.1. *[18] Let $V_J,\ \ J \in Z$ be a given MRA with scaling function ϕ and $P_J f$ is a projection of $f \in L^2(R)$ onto V_J so that*

$$P_J f = \sum_k c_k 2^{J/2}\phi(2^J x - k)$$

Then for sufficiently large J, $c_k \cong 2^{-J/2} f(k2^{-J})$ with $\int \overline{\phi(x)}\, dx = 1$.

Here, we develop an NWGM is followed by the basics of finite difference scheme.

Lemma 3.2. *For an unknown function $u(x)$ and large $J \in Z_+$, then*

$$u^{(n)}(x) = \frac{1}{h^n}\sum_{i=-1}^{n-1}(-1)^{n+i+1\,n}C_{i+1}\, u(x+ih).$$

Proof. For small $h = \frac{1}{2^J}$, finite difference discretizations of the unknown function$u(x)$ is as follows,

$$u'(x) = \frac{u(x) - u(x-h)}{h},$$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2},$$

and so on upto n^{th} difference, we have

$$\begin{aligned} u^n(x) &= \tfrac{u(x+(n-1)h) - {}^nC_1 u(x+(n-2)h) + {}^nC_2 u(x+(n-3)h) + \cdots + {}^nC_{n-1}u(x) - u(x-h)}{h^n} \\ &= \tfrac{1}{h^n}\textstyle\sum_{i=-1}^{n-1}(-1)^{n+i+1\,n}C_{i+1}\, u(x+ih). \end{aligned}$$

Method of solution:

Consider n^{th} order ODE

$$\sum_{p=0}^{n} A_p\, u^p(x) = F(x), \quad a < x < b, \tag{3.1}$$

where A_p is the constant/variable coefficient and $F(x)$ is a polynomial of any degree in x. Let the solution $u(x)$ of the problem be approximated by its J^{th} level wavelet series on the interval (*a, b*), i.e.

$$u(x) = \sum_k c_k 2^{J/2}\phi(2^J x - k) \tag{3.2}$$

Using the above lemma, we have

$$\begin{aligned} u^p(x) &= \tfrac{1}{h^p}\sum_{i=-1}^{p-1}\tbinom{p}{i+1}(-1)^{p+i+1}\textstyle\sum_k c_k 2^{J/2}\phi\left(2^J(x+\frac{i}{2^J}) - k\right) \\ &= \tfrac{1}{h^p}\sum_{i=-1}^{p-1}\tbinom{p}{i+1}(-1)^{p+i+1}\textstyle\sum_k c_k 2^{J/2}\phi(2^J x + i - k) \qquad , \\ &= \tfrac{1}{h^p}\sum_{i=-1}^{p-1}\tbinom{p}{i+1}(-1)^{p+i+1}\textstyle\sum_k c_{k+i}\,\phi_k \end{aligned} \tag{3.3}$$

Substituting Eqn. (3.3) in (3.1), we get

$$\sum_{p=0}^{n} A_p \frac{1}{h^p}\sum_{i=-1}^{p-1}\binom{p}{i+1}(-1)^{p+i+1}\sum_k c_{k+i}\,\phi_k = F(x) \tag{3.4}$$

By taking an inner product with ϕ_m, we get

$$\sum_{p=0}^{n} A_p \frac{1}{h^p}\sum_{i=-1}^{p-1}\binom{p}{i+1}(-1)^{p+i+1}c_{k+i} = G(x), \tag{3.5}$$

where $G(x) = \int_R F(x)\phi_m dx$.
Solving the system (3.5), for the coefficients c_k. By substituting these coefficients in Eqn. (3.2), we get the required solution of a given differential equation.

4 Numerical implementation

Here we consider the 1-D Helmholtz equation and present the numerical results by varying the wavenumber β, which shows the applicability of the method.
Test problem: Now consider the problem,

$$\frac{\partial^2 u}{\partial x^2} + \beta u = f \tag{4.1}$$

where $u = u(x),\ f = f(x)$ and β is a wave number with respect to boundary conditions $u(0) = a, u(1) = b$. The implementation of the Eqn. (4.1) as per the method explained in section 3, is as follows:
Here $A_2 = 1$, $A_0 = \beta$ and $F(x) = f = 0$, having boundary conditions $u(0) = 1, u(1) = 0$. The exact equation is $u(x) = \cos(\sqrt{\beta}x) - \cot(\sqrt{\beta})\ \sin(\sqrt{\beta}x)$.
For sufficiently large J, $c_0 = \langle u, \phi_0 \rangle = 2^{-J/2}u(0)$, $c_{2^J} = \langle u, \phi_{2^J} \rangle = 2^{-J/2}u(1)$ for $k = 0,\ k = 2^J$ and

$$c_k = 2^{-J/2}u(k/2^J) \tag{4.2}$$

Let us assume,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\sum_{i=-1}^{1}\binom{2}{i+1}(-1)^{2+i+1}c_{k+i} \tag{4.3}$$

$$u = \sum_{i=-1}^{-1}\binom{0}{i+1}(-1)^{i+1}c_{k+i} \tag{4.4}$$

Substituting Eqns. (4.3) and (4.4) in Eqn. (4.1), we get a system of algebraic equations

$$\frac{1}{h^2}\sum_{i=-1}^{1}\binom{2}{i+1}(-1)^{2+i+1}c_{k+i} + \beta\sum_{i=-1}^{-1}\binom{0}{i+1}(-1)^{i+1}c_{k+i} = 0 \tag{4.5}$$

Now, we have a system of 2^J-1 equations with 2^J-1 unknown coefficients. We obtain the coefficients c by solving Eqn. (4.5) i. e for J=4 & $\beta = -2$, c= {2.2515e-01, 2.0225e-01, 1.8111e-01, 1.6155e-01, 1.4341e-01, 1.2652e-01, 1.1076e-01, 9.5989e-02, 8.2081e-02, 6.8922e-02, 5.6405e-02, 4.4427e-02, 3.2889e-02, 2.1698e-02, 1.0764e-02}. Substitute these coefficients in Eqn. (4.2), we get the required numerical solution, the results are presented in the following figures for different values of β, which show the nature of the problem. Table 1 presents the error analysis of the problem for different β. The errors are computed by $E_{max} = \max|u_e - u_a|$, and $E_{RMS} = \max\left(\sqrt{\frac{\sum|u_e-u_a|^2}{N}}\right)$, where u_e and u_a are exact and approximate solutions respectively.

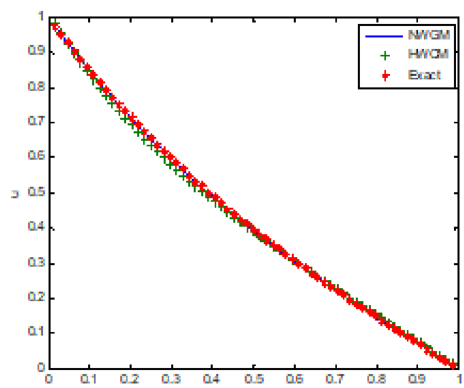

Figure 1. Comparison of numerical solutions with the exact solution for J=6 of the problem for $\beta = -2$.

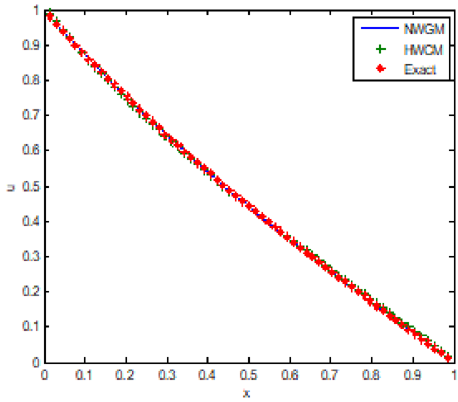

Figure 2. Comparison of numerical solutions with the exact solution for J=5 of the problem for $\beta = -1$.

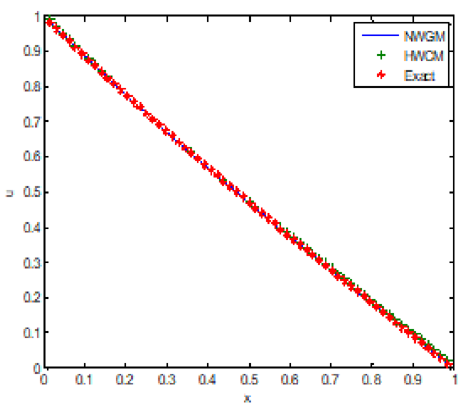

Figure 3. Comparison of numerical solutions with the exact solution for J=6 of the problem for $\beta = -0.5$.

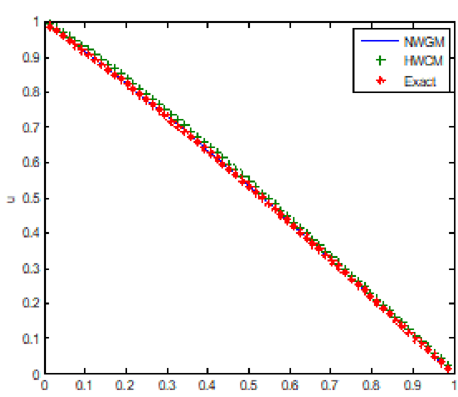

Figure 4. Comparison of numerical solutions with the exact solution for J=6 of the problem for $\beta = 0.5$.

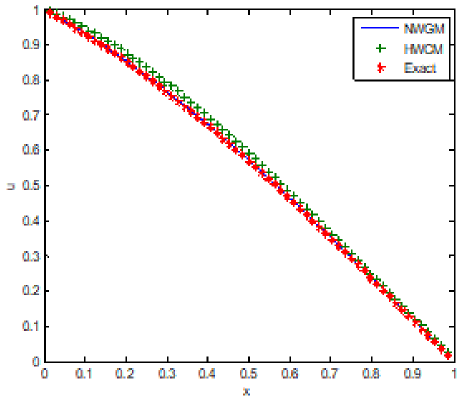

Figure 5. Comparison of numerical solutions with the exact solution for J=6 of the problem for $\beta = 1$.

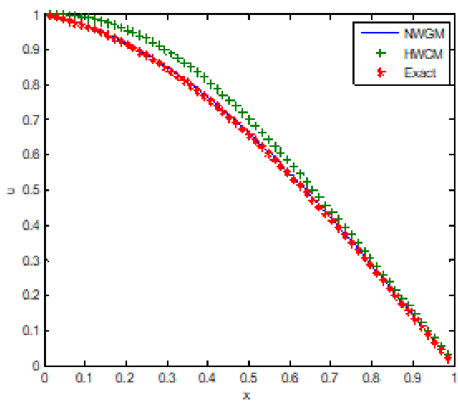

Figure 6. Comparison of numerical solutions with the exact solution for J=6 of the problem for $\beta = 2$.

Table 1. Error analysis of the Test problem for different β.

5 Conclusions

In this paper, we have proposed a new wavelet-Galerkin method for the numerical solution of the 1-D Helmholtz equation. Here we obtained the numerical results for different values of β and are presented in figures and table. From the figures we have seen that, the solution nature of the problem slightly changes from concave to convex as the values of βlies in [-2, 2]. From the table, we conclude that the proposed technique has superconvergence in the above said values of β especially in [-0.5, 0.5] than the existing ones. In order to obtain the solution of ODEs using the usual WGM, one can need to find the connection coefficients for the preferred scale. Whereas the method presented here is, to obtain comparable results with exact solutions in low CPU time than existing ones. Hence, the proposed scheme is a powerful technique for the fast and accurate solution of differential equations.

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J	N = 2^J	$\beta = -2$			
		E_{max}		E_{RMS}	
		NWGM	HWCM	NWGM	HWCM
3	8	2.6107e-02	3.0299e-02	9.8674e-03	1.0712e-02
4	16	1.2795e-02	3.0302e-02	3.3037e-03	7.5755e-03
5	32	6.3105e-03	3.0501e-02	1.1334e-03	5.3918e-03
6	64	3.1386e-03	3.0518e-02	3.9543e-04	3.8148e-03
7	128	1.5647e-03	3.0522e-02	1.3884e-04	2.6978e-03
8	256	7.8124e-04	3.0523e-02	4.8923e-05	1.9077e-03
9	512	3.9034e-04	3.0523e-02	1.7268e-05	1.3490e-03
10	1024	1.9510e-04	-	6.0999e-06	-
$\beta = -1$					
3	8	1.4271e-02	1.5924e-02	5.3940e-03	5.6300e-03
4	16	7.0195e-03	1.6230e-02	1.8124e-03	4.0574e-03
5	32	3.4917e-03	1.6213e-02	6.2712e-04	2.8660e-03
6	64	1.7402e-03	1.6247e-02	2.1925e-04	2.0309e-03
7	128	8.6887e-04	1.6254e-02	7.7099e-05	1.4367e-03
8	256	4.3410e-04	1.6255e-02	2.7184e-05	1.0159e-03
9	512	2.1697e-04	-	9.5981e-06	-
10	1024	1.0846e-04	-	3.3912e-06	-
$\beta = -0.5$					
3	8	7.4653e-03	8.1506e-03	2.8216e-03	2.8817e-03
4	16	3.7000e-03	8.3946e-03	9.5533e-04	2.0986e-03
5	32	1.8431e-03	8.4287e-03	3.3103e-04	1.4900e-03
6	64	9.2036e-04	8.4256e-03	1.1595e-04	1.0532e-03
7	128	4.5983e-04	8.4305e-03	4.0803e-05	7.4516e-04
8	256	2.2983e-04	8.4317e-03	1.4392e-05	5.2698e-04
9	512	1.1489e-04	-	5.0825e-06	-
10	1024	5.7440e-05	-	1.7959e-06	-
$\beta = 0.5$					
3	8	8.1773e-03	8.9600e-03	3.0907e-03	3.1678e-03
4	16	4.1304e-03	9.1789e-03	1.0665e-03	2.2947e-03
5	32	2.0741e-03	9.2098e-03	3.7252e-04	1.6281e-03
6	64	1.0395e-03	9. 2099e-03	1.3096e-04	1.1512e-03
7	128	5.2023e-04	9. 2099e-03	4.6163e-05	1.1512e-03
8	256	2.6024e-04	9. 2137e-03	1.6297e-05	8.1438e-04
9	512	1.3015e-04	9. 2143e-03	5.7573e-06	5.7589e-04
10	1024	6.5080e-05	-	2.0348e-06	-
$\beta = 1$					
3	8	1.7118e-02	1.9162e-02	6.4702e-03	6.7748e-03
4	16	8.7494e-03	1.9413e-02	2.2591e-03	4.8533e-03
5	32	4.4230e-03	1.9428e-02	7.9439e-04	3.4343e-03
6	64	2.2208e-03	1.9454e-02	2.7980e-04	2.4317e-03
7	128	1.1126e-03	1.9457e-02	9.8731e-05	1.7197e-03
8	256	5.5685e-04	1.9457e-02	3.4871e-05	1.2161e-03
9	512	2.7855e-04	-	1.2322e-05	-
10	1024	1.3931e-04	-	4.3555e-06	-
$\beta = 2$					
3	8	3.7471e-02	4.4029e-02	1.4163e-02	1.5566e-02
4	16	1.9912e-02	4.4335e-02	5.1412e-03	1.1084e-02
5	32	1.0162e-02	4.4519e-02	1.8251e-03	7.8700e-03
6	64	5.1296e-03	4.4519e-02	6.4627e-04	5.5648e-03
7	128	2.5761e-03	4.4541e-02	2.2859e-04	3.9369e-03
8	256	1.2907e-03	4.4546e-02	8.0826e-05	2.7841e-03
9	512	6.4600e-04	-	2.8577e-05	-
10	1024	3.2316e-04	-	1.0104e-05	-

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