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APPROXIMATION IN THE WEIGHTED GENERALIZED LIPSCHITZ CLASS BY DEFERRED CESÁRO-MATRIX PRODUCT SUBMETHODS

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Abstract. In this study we obtain the error estimates of approximation to conjugate of a function f (2π-periodic) in the weighted generalized Lipschitz class $W(L^p, \xi(t))$, $p \geq 1$, by using a new product mean of its conjugate Fourier series. We write $f \in W(L^p, \xi(t))$ if the condition

$$\|(f(x+t) - f(x))sin^{\beta}(x/2)\|_p = O(\xi(t))$$

holds, where $\xi(t)$ is a positive increasing function and $p \geq 1, \beta \geq 0$.

Here we introduce a new product mean called deferred Cesáro-Matrix (DCM) mean. Let $T = (u_{j,k})$ be an infinite triangular matrix satisfying the Silverman-Toeplitz conditions. Then the deferred Cesáro-Matrix mean is defined by

$$t_n^{DT}(f;x) := \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} \sum_{k=0}^{m} u_{m,k} s_k(f;x),$$

where $a = (a_n)$ and $b = (b_n)$ are sequences of nonnegative integers with conditions $a_n < b_n$, $n = 1, 2, 3, ...$ and $\lim\limits_{n\to\infty} b_n = +\infty$, and $s_k(f;x)$ denotes k^{th} partial sum of Fourier series of f.

1 Introduction

One of the main problems in approximation theory is to investigate the properties of functions in a subclass of a certain function space, whose properties we do not know well, using the properties of a function in a class better known using mathematical methods. It is first shown that a class of approaching functions exists. Then, the answer to the question "What is the approach speed for the functions in this class?" is sought. The idea of reaching from known to unknown reveals a feasible method in almost all fields of science. Especially, in applied sciences such as physics and engineering, where Fourier analysis has an important place, Fourier approach features are used for the output of digital filters and signals [2]. From this perspective, the focus of this study is on the determination of the degree of approximation for the functions in a certain class by the generalized summability method generated with the partial sums sequence of the Fourier series.

The estimation of error of conjugate of functions in different Lipschitz classes using single summability operators have been studied by many mathematicians like Kushwaha [13], Nigam and Sharma [12], Lal and Nigam [22], Lal [21], Qureshi [14], [15], [17], [18], Rhoades [5], Mittal et al. [19] and Kranz et al. [20] in the past few decades. Qureshi has determined the degree of approximation to functions which belong to the classes $Lip\alpha$ and $Lip(\alpha, p)$ by means of conjugate series in [15] and [17], respectively. In subsequent years, similar investigations have been made in researches such as [8], [9], [21] and [25]. The studies of estimation of error of conjugate of functions in different Lipschitz classes using different product operator, have been made by the researchers like Lal and Sing [23], [24], Dhakal [6], Nigam and Sharma [9],

[10], [11], [12] and Padhy et al. [7] recently. In this work, we are interested in a summability method in the theory of Fourier series. The results of [26] will be extended to a more general summability method. For this purpose, we shall give the following notations that are used in this paper. Let $L := L(0, 2\pi)$ denote the space of functions that are 2π-periodic and Lebesgue integrable on $[0, 2\pi]$ and let

$$S[f] = \frac{a_o}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f;x) \tag{1.1}$$

be the Fourier series of a function $f \in L$, i.e., for any $k = 0, 1, 2, \cdots$

$$a_k = a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt \quad , \quad b_k = b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt.$$

Let

$$s_n(f;x) = \frac{a_o}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=1}^{n} A_k(f;x)$$

denote the partial sum of the first $(n+1)$ terms of the Fourier series of $f \in L$ at a point x. The conjugate series of (1.1) is given by

$$\tilde{S}[f] = \sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx) \equiv \sum_{k=1}^{\infty} \tilde{A}_k(f;x).$$

Note that there is no free term in $\tilde{S}[f]$. Therefore, the series conjugate to the series $\tilde{S}[f]$ is the series $S[f]$ without free term.

The function $\tilde{f} \in L$ for which $S[\tilde{f}] = \tilde{S}[f]$ is called trigonometrically conjugate or simply conjugate to $f(.)$. It can be shown that the functions $f(.)$ and $\tilde{f}(.)$ are connected by the equality

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cot\frac{t}{2}dt$$

$$= -\frac{1}{2\pi}\int_{0}^{\pi} \eta(t)\cot\frac{t}{2}dt = -\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi} \eta(t)\cot\frac{t}{2}dt, \tag{1.2}$$

where $\eta(t) := \eta(x,t) = f(x+t) - f(x-t)$. If $f \in L$, then equation (1.2) exists for almost all x. On the other hand, we know that the series conjugate to the Fourier series doesn't have to be a Fourier series [[1],p.186].

Let

$$\tau_n(f;x) = \tau_n(f,T;x) := \sum_{k=0}^{n} u_{n,k} s_k(f;x), \quad \forall n \geq 0$$

where $T \equiv (u_{n,k})$ is a lower triangular infinite matrix satisfying the Silverman-Toeplitz [3] condition of regularity, such that

$$u_{n,k} = \begin{cases} \geq 0 & ,k \leq n \\ 0 & ,k > n \end{cases} \quad (k = 0, 1, 2, ...)$$

and

$$\sum_{k=0}^{n} u_{n,k} = 1, \quad (n = 0, 1, 2, ...). \tag{1.3}$$

The Fourier series of a function f is said to be $T-$ summable to s, if $\tau_n(f;x) \to s(x)$ as $n \to \infty$. The Fourier series of f is called Cesáro-T $(C^1.T)$ summable to $s(x)$ if

$$t_n^{CT} := \frac{1}{n+1}\sum_{m=0}^{n} \tau_m(f;x) = \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=0}^{m} u_{m,k} s_k(f;x) \to s(x),$$

as $n \to \infty$.

From this definition, we similarly write the deferred Cesáro-Matrix product mean as follow.

Let $T = (u_{j,k})$ be an infinite triangular matrix satisfying the Silverman-Toeplitz conditions. Then, the deferred Cesáro-Matrix mean is defined by

$$t_n^{DT}(f;x) := \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} \sum_{k=0}^{m} u_{m,k} s_k(f;x),$$

where $a = (a_n)$ and $b = (b_n)$ are sequences of nonnegative integers with conditions $a_n < b_n$, $n = 1, 2, 3, ...$ and $\lim_{n\to\infty} b_n = +\infty$, and $s_k(f;x)$ denotes k^{th} partial sum of Fourier series of f.

The degree of approximation of a function $f : \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial T_n of degree n is defined by

$$\|T_n - f\|_\infty = \sup\{|T_n(x) - f(x)|, x \in \mathbb{R}\}$$

with respect to the supremum norm [1]. The degree of approximation of a function $f \in L_p$ $(p \geq 1)$ is given by

$$E_n(f) = \min_n \|T_n - f\|_p,$$

where $\|.\|_p$ denotes the L_p- norm with respect to x and is defined by

$$\|f\|_p := \{\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx\}^{\frac{1}{p}}.$$

This method of approximation is called the trigonometric Fourier approximation.

We recall the following definitions:

1. A function f is said to belong to the $Lip\alpha$ class if $|f(x+t) - f(x)| = O(|t|^\alpha),\ 0 < \alpha \leq 1$;

2. A function f is said to belong to the $Lip(\alpha, p)$ class if $w_p(\delta, f) = O(\delta^\alpha)$, where

$$w_p(\delta, f) = \sup_{|t|\leq\delta} \{\frac{1}{2\pi} \int_0^{2\pi} |f(x+t) - f(x)|^p dx\}^{\frac{1}{p}},\ \ 0 < \alpha \leq 1;\ \ p \geq 1;$$

3. A function f is said to belong to the $Lip(\xi(t), p)$ class if $w_p(\delta, f) = O(\xi(t))$, where $\xi(t)$ is a positive increasing function and $p \geq 1$;

4. We write $f \in W(L^p, \xi(t))$ if the condition

$$\|f(\cdot + t) - f(\cdot))sin^\beta(\cdot/2)\|_p = O(\xi(t))$$

holds, where $\xi(t)$ is a positive increasing function and $p \geq 1, \beta \geq 0$.

If $\beta = 0$, then $W(L^p, \xi(t))$ reduces to $Lip(\xi(t), p)$; and, if $\xi(t) = t^\alpha$, the $Lip(\xi(t), p)$ class reduces to the $Lip(\alpha, p)$ class. If $p \to \infty$ then the $Lip(\alpha, p)$ class coincides with the $Lip\alpha$ class. Accordingly, we have the following inclusions:

$$Lip\alpha \subset Lip(\alpha, p) \subset Lip(\xi(t), p) \subset W(L^p, \xi(t))$$

for all $0 < \alpha \leq 1$ and $p \geq 1$.

2 Main results

The following two theorems are related with the degree of approximation to conjugate of functions belonging to the classes $W(L^p, \xi(t))$ and $Lip\alpha$ by deferred Cesáro-Matrix product means of conjugate of their Fourier series. We are going to use the following notations throughout this section and auxiliary results:

$$\eta(x, t) := \eta(t) = f(x + t) + f(x - t) - 2f(x)$$

and

$$\tilde{K}_n^{DT}(t) = \frac{1}{2\pi(b_n - a_n)} \sum_{m=a_n+1}^{b_n} \sum_{k=0}^{m} u_{m,m-k} \frac{\cos(m-k+\frac{1}{2})t}{\sin(\frac{t}{2})}.$$

Before stating the theorems, we develop the auxiliary results needed in the proofs of both of them.

Lemma 2.1. *If the condition of (1.3) holds for* $\{u_{j,k}\}$, *then*

$$\tilde{K}_n^{DT}(t) = O(t^{-1}), \quad 0 < t \leq \frac{\pi}{b_n - a_n}.$$

Proof. For $0 < t \leq \frac{\pi}{b_n - a_n}$, using Jordan's inequality, $(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$, we have

$$|\tilde{K}_n^{DT}(t)| = |\frac{1}{2\pi(b_n - a_n)} \sum_{m=a_n+1}^{b_n} \sum_{k=0}^{m} u_{m,m-k} \frac{\cos(m-k+\frac{1}{2})t}{\sin(\frac{t}{2})}|$$

$$\leq \frac{1}{2\pi(b_n - a_n)} \sum_{m=a_n+1}^{b_n} \sum_{k=0}^{m} u_{m,m-k} |\frac{\cos(m-k+\frac{1}{2})t}{\sin(\frac{t}{2})}|$$

$$\leq \frac{1}{2t(b_n - a_n)} \sum_{m=a_n+1}^{b_n} \sum_{k=0}^{m} u_{m,m-k}$$

$$= \frac{1}{2t(b_n - a_n)} \sum_{m=a_n+1}^{b_n} 1, \qquad (\sum_{k=0}^{m} u_{m,m-k} = 1)$$

$$= \frac{1}{2t(b_n - a_n)} (b_n - a_n) = O(t^{-1}). \quad \square$$

Lemma 2.2. *If the condition of (1.3) holds for* $\{u_{j,k}\}$, *then*

$$\tilde{K}_n^{DT}(t) = O(\frac{1}{t^2(b_n - a_n)}), \quad \frac{\pi}{b_n - a_n} \leq t < \pi.$$

Proof. For $\frac{\pi}{b_n - a_n} \leq t < \pi$,

$$|\tilde{K}_n^{DT}(t)| = |\frac{1}{2\pi(b_n - a_n)} \sum_{m=a_n+1}^{b_n} \sum_{k=0}^{m} u_{m,m-k} \frac{\cos(m-k+\frac{1}{2})t}{\sin(\frac{t}{2})}|$$

$$\leq O(\frac{1}{t(b_n - a_n)}) |Re \sum_{m=a_n+1}^{b_n} \sum_{k=0}^{m} u_{m,m-k} e^{i(m-k+\frac{1}{2})t}|$$

$$\leq O(\frac{1}{t(b_n - a_n)}) | \sum_{m=a_n+1}^{b_n} \sum_{k=0}^{m} u_{m,m-k} e^{i(m-k+\frac{1}{2})t}|. \tag{2.1}$$

Now we take

$$| \sum_{m=a_n+1}^{b_n} \sum_{k=0}^{m} u_{m,m-k} e^{i(m-k+\frac{1}{2})t}| \leq | \sum_{m=a_n+1}^{\lambda} \sum_{k=0}^{m} u_{m,m-k} e^{i(m-k+\frac{1}{2})t}|$$

$$+| \sum_{m=\lambda+1}^{b_n} \sum_{k=0}^{\lambda} u_{m,m-k} e^{i(m-k+\frac{1}{2})t}|$$

$$+|\sum_{m=\lambda+1}^{b_n}\sum_{k=\lambda+1}^{m}u_{m,m-k}e^{i(m-k+\frac{1}{2})t}|$$
$$:= L_1+L_2+L_3,$$

where λ denotes the integer part of $\frac{1}{t}$. Owing to (1.3) and Jordan's inequality, $(\sin(t/2))^{-1}\leq \pi/t$ for $0<t\leq\pi$, we get

$$L_1\leq\sum_{m=a_n+1}^{\lambda}\sum_{k=0}^{m}u_{m,m-k}|e^{i(m-k+\frac{1}{2})t}|=\sum_{m=a_n+1}^{\lambda}\sum_{k=0}^{m}u_{m,m-k}=$$
$$=\lambda-a_n=O(\frac{1}{t}).$$

Now we will estimate L_2. Changing the order of summation and using Abel's transformation in L_2, we get

$$L_2=|\sum_{m=\lambda+1}^{b_n}\sum_{k=0}^{\lambda}u_{m,m-k}e^{i(m-k+\frac{1}{2})t}|=|\sum_{k=0}^{\lambda}\sum_{m=\lambda+1}^{b_n}u_{m,m-k}e^{i(m-k+\frac{1}{2})t}|$$
$$=|\sum_{k=0}^{\lambda}[\sum_{m=\lambda+1}^{b_n-1}(\sum_{v=0}^{m}e^{i(v-k+\frac{1}{2})t})(u_{m,m-k}-u_{m+1,m-k+1})$$
$$-(\sum_{v=0}^{\lambda}e^{i(v-k+\frac{1}{2})t})u_{\lambda+1,\lambda-k+1}+(\sum_{v=0}^{b_n}e^{i(v-k+\frac{1}{2})t})u_{b_n,b_n-k}]|$$
$$\leq\sum_{k=0}^{\lambda}[|\sum_{m=\lambda+1}^{b_n-1}(\sum_{v=0}^{m}e^{i(v-k+\frac{1}{2})t})(u_{m,m-k}-u_{m+1,m-k+1})|$$
$$+|(\sum_{v=0}^{\lambda}e^{i(v-k+\frac{1}{2})t})u_{\lambda+1,\lambda-k+1}|+|(\sum_{v=0}^{b_n}e^{i(v-k+\frac{1}{2})t})u_{b_n,b_n-k}|]$$
$$\leq O(t^{-1})\sum_{k=0}^{\lambda}[\sum_{m=\lambda+1}^{b_n-1}(u_{m,m-k}-u_{m+1,m-k+1})+u_{\lambda+1,\lambda-k+1}+u_{b_n,b_n-k}]$$
$$=O(t^{-1})[\sum_{k=0}^{\lambda}(u_{\lambda+1,\lambda+1-k}-u_{b_n,b_n-k})+\sum_{k=0}^{\lambda}u_{\lambda+1,\lambda-k+1}+\sum_{k=0}^{\lambda}u_{b_n,b_n-k}]$$
$$=O(t^{-1})\sum_{k=0}^{\lambda}u_{\lambda+1,\lambda+1-k}=O(t^{-1}).$$

Using Abel's transformation in L_3, we obtain

$$L_3=|\sum_{m=\lambda+1}^{b_n}\sum_{k=\lambda+1}^{m}u_{m,m-k}e^{i(m-k+\frac{1}{2})t}|\leq\sum_{m=\lambda+1}^{b_n}|\sum_{k=\lambda+1}^{m}u_{m,m-k}e^{i(m-k+\frac{1}{2})t}|$$
$$=\sum_{m=\lambda+1}^{b_n}|\sum_{k=\lambda+1}^{m-1}(\sum_{v=0}^{k}e^{i(m-v+\frac{1}{2})t})(u_{m,m-k}-u_{m,m-k-1})-(\sum_{v=0}^{\lambda}e^{i(m-v+\frac{1}{2})t})u_{m,m-\lambda-1}$$
$$+(\sum_{v=0}^{m}e^{i(m-v+\frac{1}{2})t})u_{m,0}|$$
$$\leq O(t^{-1})[\sum_{m=\lambda+1}^{b_n}\sum_{k=\lambda+1}^{m-1}(u_{m,m-k}-u_{m,m-k-1})+\sum_{m=\lambda+1}^{b_n}u_{m,m-\lambda-1}+\sum_{m=\lambda+1}^{b_n}u_{m,0}]$$

$$= O(t^{-1})(\sum_{m=\lambda+1}^{b_n}(u_{m,m-\lambda-1}-u_{m,0})+\sum_{m=\lambda+1}^{b_n}u_{m,m-\lambda-1}+\sum_{m=\lambda+1}^{b_n}u_{m,0})$$

$$= O(t^{-1})\sum_{m=\lambda+1}^{b_n}u_{m,m-\lambda-1} = O(t^{-1}).$$

(By considering $u_{r,r-k} \geq u_{r+1,r+1-k} \geq u_{r+1,r-k}$ and $A_{\lambda+1,0} = 1$.) Combining L_1, L_2 and L_3 we get

$$L_1 + L_2 + L_3 = O(t^{-1}). \tag{2.2}$$

Consequently, from (2.1) and (2.2) we obtain

$$|\tilde{K}_n^{DT}(t)| = O(\frac{1}{t^2(b_n-a_n)}). \quad \square$$

Theorem 2.3. *Let $f \in L$ and $T \equiv (a_{n,k})$ be a lower triangular regular matrix with nonnegative entries and row sums 1. If $f \in Lip\alpha$ $(0 < \alpha \leq 1)$, then the degree of approximation of the conjugate function $\tilde{f}$ by the deferred Cesáro-Matrix product means of its conjugate Fourier series is given by*

$$\left\|t_n^{DT}(\tilde{f}) - \tilde{f}\right\|_\infty = \begin{cases} O((b_n-a_n)^{-\alpha}) & ,0<\alpha<1; \\ O(\frac{lne(b_n-a_n)}{b_n-a_n}) & ,\alpha = 1. \end{cases}$$

Proof. Owing to [1], the integral representation of $s_k(\tilde{f}, x)$ is given by

$$s_k(\tilde{f}, x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^\pi \eta(t)(\frac{cos(k+\frac{1}{2})t}{sin(\frac{t}{2})})dt. \tag{2.3}$$

If we denote deferred Cesáro-Matrix product means of $\{s_k(\tilde{f}, x)\}$ by $t_n^{DT}(\tilde{f}; x)$ and take into account (2.3), we have

$$t_n^{DT}(\tilde{f}; x) - \tilde{f}(x) = \frac{1}{b_n-a_n}\sum_{m=a_n+1}^{b_n}\sum_{k=0}^{m}u_{m,k}\left[s_k(\tilde{f}, x) - \tilde{f}(x)\right]$$

$$= \frac{1}{b_n-a_n}\sum_{m=a_n+1}^{b_n}\sum_{k=0}^{m}u_{m,k}\left(\frac{1}{2\pi}\int_0^\pi \eta(t)(\frac{cos(k+\frac{1}{2})t}{sin(\frac{t}{2})})dt\right)$$

$$= \frac{1}{2\pi(b_n-a_n)}\int_0^\pi \eta(t)\sum_{m=a_n+1}^{b_n}\sum_{k=0}^{m}u_{m,k}(\frac{cos(k+\frac{1}{2})t}{sin(\frac{t}{2})})dt = \int_0^\pi \eta(t)\tilde{K}_n^{DT}(t)dt$$

$$= \int_0^\pi \eta(t)\tilde{K}_n^{DT}(t)dt. \tag{2.4}$$

By considering (2.4), we write

$$|t_n^{DT}(\tilde{f}; x) - \tilde{f}(x)| = |\int_0^\pi \eta(t)\tilde{K}_n^{DT}(t)dt| \leq \int_0^\pi |\eta(t)||\tilde{K}_n^{DT}(t)|dt$$

$$\leq \int_0^{\pi/(b_n-a_n)} |\eta(t)||\tilde{K}_n^{DT}(t)|dt + \int_{\pi/(b_n-a_n)}^{\pi} |\eta(t)||\tilde{K}_n^{DT}(t)|dt := I_1 + I_2.$$

Since $f \in Lip\alpha$, $\eta(t)$ belongs to the $Lip\alpha$ class. Hence, from Lemma 2.1, we get

$$I_1 = \int_0^{\pi/(b_n-a_n)} |\eta(t)||\tilde{K}_n^{DT}(t)|dt = O(\int_0^{\pi/(b_n-a_n)} t^{\alpha}t^{-1}dt) = O(\int_0^{\pi/(b_n-a_n)} t^{\alpha-1}dt)$$

$$= \begin{cases} O((b_n-a_n)^{-\alpha}) & ,0<\alpha<1; \\ O(\frac{1}{b_n-a_n}) & ,\alpha=1. \end{cases} \tag{2.5}$$

By using Lemma 2.2, we obtain

$$I_2 = \int_{\pi/(b_n-a_n)}^{\pi} |\eta(t)||\tilde{K}_n^{DT}(t)|dt = O(\frac{1}{b_n-a_n}\int_{\pi/(b_n-a_n)}^{\pi} t^{\alpha}t^{-2}dt)$$

$$= O(\frac{1}{b_n-a_n}\int_{\pi/(b_n-a_n)}^{\pi} t^{\alpha-2}dt) = \begin{cases} O((b_n-a_n)^{-\alpha}) & ,0<\alpha<1; \\ O(\frac{ln(b_n-a_n)}{b_n-a_n}) & ,\alpha=1. \end{cases} \tag{2.6}$$

Taking into account (2.5) and (2.6), we have

$$\|t_n^{DT}(\tilde{f})-\tilde{f}\|_{\infty} = \sup_{x\in[0,2\pi]} |t_n^{DT}(\tilde{f};x)-\tilde{f}(x)|$$

$$= \begin{cases} O((b_n-a_n)^{-\alpha}) & ,0<\alpha<1; \\ O(\frac{lne(b_n-a_n)}{b_n-a_n}) & ,\alpha=1. \end{cases}$$

Therefore, the proof of Theorem 2.3 is completed. □

Theorem 2.4. *Let $f \in L$ and $\xi(t)$ be a positive increasing function. If $f \in W(L^p, \xi(t))$ with $0 \le \beta \le 1-\frac{1}{p}$, then the degree of approximation of the conjugate function $\tilde{f}$ by the deferred Cesáro-Matrix product means of its conjugate Fourier series is given by*

$$\left\|t_n^{DT}(\tilde{f})-\tilde{f}\right\|_p = O((b_n-a_n)^{\beta+\frac{1}{p}}\xi(\frac{\pi}{b_n-a_n}))$$

provided that the function $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\}$$

is a decreasing function and

$$\left\{\int_0^{\pi/(b_n-a_n)} \left(\frac{|\eta(t)|sin^{\beta}(t/2)}{\xi(t)}\right)^p dt\right\}^{1/p} = O(1), \tag{2.7}$$

$$\left\{\int_{\pi/(b_n-a_n)}^{\pi} \left(\frac{|\eta(t)|t^{-\delta}}{\xi(t)}\right)^p dt\right\}^{1/p} = O((b_n-a_n)^{\delta}), \tag{2.8}$$

where δ is an arbitrary number such that $q(\beta-\delta)-1>0$, $p^{-1}+q^{-1}=1$, $p \ge 1$, (2.7) and (2.8) hold uniformly in x.

Proof. We know that,

$$|t_n^{DT}(\tilde{f};x)-\tilde{f}(x)| = |\int_0^{\pi} \eta(t)\tilde{K}_n^{DT}(t)dt|$$

$$\leq |\int_0^{\pi/(b_n-a_n)} \eta(t)\tilde{K}_n^{DT}(t)dt| + |\int_{\pi/(b_n-a_n)}^{\pi} \eta(t)\tilde{K}_n^{DT}(t)dt| := J_1 + J_2.$$

By considering Hölder's inequality, condition (2.7), Lemma 2.1, Jordan's inequality and $\eta(t) \in W(L^p, \xi(t))$, we have

$$J_1 = \left| \int_0^{\pi/(b_n-a_n)} \frac{\eta(t)sin^{\beta}(t/2)}{\xi(t)} \frac{\xi(t)\tilde{K}_n^{DT}(t)}{sin^{\beta}(t/2)} dt \right|$$

$$\leq \left(\int_0^{\pi/(b_n-a_n)} \left|\frac{\eta(t)sin^{\beta}(t/2)}{\xi(t)}\right|^p dt \right)^{1/p} \left(\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\pi/(b_n-a_n)} \left|\frac{\xi(t)\tilde{K}_n^{DT}(t)}{sin^{\beta}(t/2)}\right|^q dt \right)^{1/q}$$

$$= O(1) \left(\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\pi/(b_n-a_n)} \left(\frac{\xi(t)t^{-1}}{sin^{\beta}(t/2)}\right)^q dt \right)^{1/q}$$

$$= O\left(\frac{\xi(\frac{\pi}{b_n-a_n})}{\frac{\pi}{b_n-a_n}}\right) \left(\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\pi/(b_n-a_n)} \left(\frac{1}{sin^{\beta}(t/2)}\right)^{\beta q} dt \right)^{1/q}$$

$$= O((b_n-a_n)\xi(\frac{\pi}{b_n-a_n})) \left(\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\pi/(b_n-a_n)} \left(\frac{\pi}{t}\right)^{\beta q} dt \right)^{1/q}$$

$$= O((b_n-a_n)\xi(\frac{\pi}{b_n-a_n})) \left(\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\pi/(b_n-a_n)} t^{-\beta q} dt \right)^{1/q}$$

$$= O((b_n-a_n)^{-\frac{1}{q}+\beta+1}\xi(\frac{\pi}{b_n-a_n})) = O((b_n-a_n)^{\beta+\frac{1}{p}}\xi(\frac{\pi}{b_n-a_n})). \tag{2.9}$$

Now let's estimate J_2. By using Lemma 2.2, we write

$$J_2 = |\int_{\pi/(b_n-a_n)}^{\pi} \eta(t)\tilde{K}_n^{DT}(t)dt| = O\left(\int_{\pi/(b_n-a_n)}^{\pi} |\eta(t)| \frac{1}{t^2(b_n-a_n)} dt \right)$$

$$= O(\frac{1}{b_n-a_n}) \left(\int_{\pi/(b_n-a_n)}^{\pi} \left(\frac{|\eta(t)|t^{-\delta}sin^{\beta}(t/2)}{\xi(t)}\right)^p dt \right)^{1/p} \left(\int_{\pi/(b_n-a_n)}^{\pi} \left(\frac{\xi(t)t^{\delta-2}}{sin^{\beta}(t/2)}\right)^q dt \right)^{1/q}$$

$$= O((b_n-a_n)^{\delta-1}) \left(\int_{\pi/(b_n-a_n)}^{\pi} \left(\frac{\xi(t)t^{\delta-2}}{sin^{\beta}(t/2)}\right)^q dt \right)^{1/q}$$

$$= O((b_n-a_n)^{\delta-1}) \left(\int_{\pi/(b_n-a_n)}^{\pi} \left(\xi(t)t^{\delta-2-\beta}\right)^q dt \right)^{1/q}$$

$$= O((b_n - a_n)^{\delta-1}) \left(\int_{1/\pi}^{\frac{b_n-a_n}{\pi}} \left(\xi(1/x)t^{-\delta+2+\beta}\right)^q x^{-2}dx \right)^{1/q}$$

$$= O((b_n - a_n)^{\delta}\xi(\frac{\pi}{b_n - a_n})) \left(\int_{1/\pi}^{\frac{b_n-a_n}{\pi}} x^{\beta q-\delta q+q-2}dx \right)^{1/q}$$

$$= O((b_n - a_n)^{\beta+\frac{1}{p}}\xi(\frac{\pi}{b_n - a_n})). \tag{2.10}$$

Combining (2.9) and (2.10), we have

$$\left\|t_n^{DT}(\tilde{f}) - \tilde{f}\right\|_p = O((b_n - a_n)^{\beta+\frac{1}{p}}\xi(\frac{\pi}{b_n - a_n})). \quad \square$$

3 Corollaries

Corollary 3.1. *If $\beta = 0$, then the weighted class $W(L^p, \xi(t))$ reduces to the class $Lip(\xi(t), p)$. Therefore, for $f \in Lip(\xi(t), p)$ we have*

$$\left\|t_n^{DT}(\tilde{f}) - \tilde{f}\right\|_p = O((b_n - a_n)^{\frac{1}{p}}\xi(\frac{\pi}{b_n - a_n}))$$

with respect to Theorem 2.4.

Corollary 3.2. *If $\beta = 0$ and $\xi(t) = t^\alpha, \quad (0 < \alpha \leq 1)$, then the weighted class $W(L^p, \xi(t))$ reduces to the class $Lip(\alpha, p)$. Therefore, for $f \in Lip(\alpha, p)$ we have*

$$\left\|t_n^{DT}(\tilde{f}) - \tilde{f}\right\|_p = O((b_n - a_n)^{\frac{1}{p}-\alpha})$$

with respect to Theorem 2.4.

Corollary 3.3. *If $p \to \infty$ in Corollary 3.2, then for $f \in Lip\alpha, \quad (0 < \alpha \leq 1)$ we have*

$$\left\|t_n^{DT}(\tilde{f}) - \tilde{f}\right\|_p = O((b_n - a_n)^{-\alpha})$$

with respect to Theorem 2.4.

4 Conclusion

The studies of error approximation of a function in Lipschitz classes using different product means have been centre of creative study among the researchers. The weighted $W(L^p, \xi(t))$ class is a generalization of the classes $Lip\alpha$, $Lip(\alpha, p)$ and $Lip(\xi(t), p)$. Taking into account this generalization of the function classes, we give two theorems are related with the degree of approximation to conjugate of functions belonging to the classes $W(L^p, \xi(t))$ and $Lip(\alpha, p)$ by deferred Cesáro-Matrix product means of conjugate of their Fourier series. These two theorems give us a generalization by a more general summability method of previous results such as [6]-[11], [24], [26]. Also the introduced method can be applied for further investigations from different classes. Since this approach method is a generalization of the previous methods, it is also applicable in signal analysis like the previous ones. Our expectation here is to provide a more effective method for general situations, both the approximation theory, the theory of Fourier series, the summability theory and their application areas.

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