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A Note on D'Ocagne's Identity on Generalized Fibonacci and Lucas numbers

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Abstract. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. The Fibonacci sequence, Lucas numbers and their generalization have many interesting properties and applications to almost every field. In this note, the D'Ocagne's identity for the generalized Fibonacci and Lucas sequences is established in terms of log convex identity of generalized Fibonacci and Lucas sequence by using mathematical induction.

1 Introduction

In recent years, many interesting properties of classic Fibonacci numbers, classic Lucas numbers and their generalizations have been shown by researchers and applied to almost every field of science and art.

A sequence is an arrangement of any objects or a set of numbers in a particular order followed by some rule, based on this the Fibonacci and Lucas number are the examples of sequence in a particular order that, by adding the previous two numbers of the sequence with different two initial values.

The Fibonacci sequence exhibits a certain numerical pattern which originated as the answer to an exercise in the first ever high school algebra text. This pattern turned out to have an interest and importance far beyond what its creator imagined. It can be used to model or describe an amazing variety of phenomena, in mathematics and science, art and nature. The mathematical ideas of the Fibonacci sequence leads to, such as the golden ratio, spirals and self- similar curves, have long been appreciated for their charm and beauty, but no one can really explain why they are echoed so clearly in the world of art and nature. The Fibonacci sequence is the series of numbers: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$. Any number in this sequence is the sum of the previous two numbers, and this pattern is mathematically written as

$$F_n = F_{n-1} + F_{n-2},$$

where n is a positive integer greater than 1, F_n is the n-th Fibonacci number with $F_0 = 0$ and $F_1 = 1$. Several interesting results and identities on Fibonacci numbers were found in [2, 3]. Some interesting results on sequences, double sequences and its applications are found in [5, 6].

The Lucas sequence has the same recursive relationship as the Fibonacci sequence, where each term is the sum of the two previous terms, but with different starting values. This produces a sequence where the ratios of successive terms approach the golden ratio, and in fact the terms themselves are rounding of integer powers of the golden ratio. The sequence also has a variety of relationships with the Fibonacci numbers, like the fact that adding any two Fibonacci numbers two terms apart in the Fibonacci sequence results in the Lucas number in between. The first few

Lucas numbers are: $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \cdots$

The Lucas numbers may thus be defined as follows:

$$L_n = \begin{cases} 2 & if\ n = 0 \\ 1 & if\ n = 1 \\ L_{n-1} + L_{n-2} & if\ n \geq 2. \end{cases}$$

where n belongs to the natural numbers. Several interesting results and identities on Lucas numbers were found in [4].

It is well known that N-bonacci numbers have huge number of application in and around all fields of study and real life. Particular values of N, we have the following:

Sl.No.	N =Sum of consecutive numbers	Name
1	2	Fibonacci number
2	3	Tribonacci number
3	4	Tetra-bonacci number
$\vdots$	$\vdots$	$\vdots$
n	N	N-bonacci number

Table 1. N-bonacci numbers

In [1], Zvonko Cerin studied on factors of sums of consecutive Fibonacci and Lucas numbers. The author discovered that the sums $\sum_{j=0}^{4i+3} F_{k+j}$ have the Fibonacci number F_{2i+2} as a common factor, the alternating sums of 20 and 22 consecutive Fibonacci numbers are all respectively divisible by F_{10} and L_{11}. Also, obtained some interesting results on sums of consecutive products, and squares of consecutive numbers. The following are the few identities involving Fibonacci and Lucas numbers.

$$\sum_{j=0}^{4i+3} F_{k+j} = F_{2i+2}L_{k+2i+3} \quad \text{and} \quad \sum_{j=0}^{4i+3}(-1)^j F_{k+j} = F_{2i+2}L_{k+2i} \tag{1.1}$$

$$\sum_{j=0}^{4i+1} F_{k+j} = L_{2i+1}F_{k+2i+2} \quad \text{and} \quad \sum_{j=0}^{4i+1}(-1)^j F_{k+j} = L_{2i+1}F_{k+2i-1} \tag{1.2}$$

$$\sum_{j=0}^{4i} F_{k+j} = F_{2i}L_{k+2i} + L_{2i+1}F_{k+2i} \quad \text{and} \quad \sum_{j=0}^{4i}(-1)^j F_{k+j} = F_{k+2i}L_{2i+1} - L_{k+2i}F_{2i} \tag{1.3}$$

and for other identities interested readers may refer [1].

Definition 1.1. [8] For any two positive integers *j and k*, the generalized Fibonacci sequence $\{D_j^k\}$ is defined as;

$$D_j^k = F_j + F_{j+1} + F_{j+2} + \ldots + F_{j+k-1} + F_{j+k} = \sum_{i=j}^{j+k} F_i \tag{1.4}$$

$$D_{j+2}^k = D_j^k + D_{j+1}^k, \quad \text{for all} \quad j, k = 0, 1, 2, 3, \ldots \tag{1.5}$$

Definition 1.2. [8] For any two positive integers *j and k*, the generalized Lucas sequence $\{E_j^k\}$ is defined as;

$$E_j^k = L_j + L_{j+1} + L_{j+2} + \ldots + L_{j+k-1} + L_{j+k} = \sum_{i=j}^{j+k} L_i \tag{1.6}$$

$$E_{j+2}^k = E_j^k + E_{j+1}^k, \quad \text{for all} \quad j,\, k = 0, 1, 2, 3, \ldots \tag{1.7}$$

In [2, 3], the D'Ocagne's identity for Fibonacci numbers is given as follows:

$$F_m F_{n+1} - F_n F_{m+1} = (-1)^n F_{m-n}. \tag{1.8}$$

The objective of this article is to develop D'Ocagne's identity for the generalized Fibonacci and Lucas numbers $\{D_j^k\}$ and $\{E_j^k\}$. Using strong mathematical induction, some interesting results are developed in [7].

2 D'Ocagne's identity for generalized of Fibonacci numbers

For $j = 0, 1, 2, \cdots$ and $k = 0, 1, 2, \cdots$, the identity of generalized Fibonacci sequence is developed in [8] and is stated as:

$$\Delta_j^k = (D_{j+1}^k)^2 - D_j^k D_{j+2}^k = (-1)^j \left[(-1)^k + \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} + \left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - 1 \right] \tag{2.1}$$

Theorem 2.1. *Let $\{D_j^k\}$ be a sequence of generalized Fibonacci numbers and for fixed k, the D'Ocagne's identity is*

$$D_m^k D_{n+1}^k - D_n^k D_{m+1}^k = F_j \Delta_n^k, \text{ if } m = n + j,\ j \geq 0, \tag{2.2}$$

where F_j is the j-th Fibonacci number.

Proof. We prove the result by induction on j. For $j = 0$, we have $F_0 = 0$ and

$$D_m^k D_{n+1}^k - D_n^k D_{m+1}^k = D_n^k D_{n+1}^k - D_n^k D_{n+1}^k = 0 = F_0 \Delta_n^k.$$

For $j = 1$, we have $F_1 = 1$ and

$$\begin{aligned} [(D_{n+1}^k)^2 - D_n^k D_{n+2}^k] &= F_1[(D_{n+1}^k)^2 - D_n^k D_{n+2}^k] \\ &= F_1 \Delta_n^k. \end{aligned}$$

Hence the result (2.2) holds for $j = 0, 1$.

Assume that the result (2.2) holds for $m = n + j$, $j > 1$. Using (1.5), for $m = n + j - 1$, we have

$$D_{n+j-1}^k D_{n+1}^k - D_n^k D_{n+j}^k = F_{j-1}[(D_{n+1}^k)^2 - D_n^k D_{n+2}^k] \tag{2.3}$$

and for $m = n + j$, we have

$$D_{n+j}^k D_{n+1}^k - D_n^k D_{n+j+1}^k = F_j[(D_{n+1}^k)^2 - D_n^k D_{n+2}^k] \tag{2.4}$$

Now, for $m = n + (j + 1)$, using (2.3) and (2.4),we have

$$\begin{aligned} D_{n+j+1}^k D_{n+1}^k - D_n^k D_{n+j+2}^k &= D_{n+j}^k D_{n+1}^k + D_{n+j-1}^k D_{n+1}^k - D_n^k D_{n+j+1}^k - D_n^k D_{n+j}^k \\ &= (D_{n+j-1}^k D_{n+1}^k - D_n^k D_{n+j}^k) - (D_{n+j}^k D_{n+1}^k - D_n^k D_{n+j+1}^k) \\ &= F_{j-1}[(D_{n+1}^k)^2 - D_n^k D_{n+2}^k] + F_j[(D_{n+1}^k)^2 - D_n^k D_{n+2}^k] \\ &= (F_{j-1} + F_j)[(D_{n+1}^k)^2 - D_n^k D_{n+2}^k] \\ &= F_{j+1}[(D_{n+1}^k)^2 - D_n^k D_{n+2}^k] \\ &= F_{j+1} \Delta_n^k. \end{aligned}$$

Thus, the result (2.2) holds for $j + 1$. Hence, by the principle of induction (2.2) holds for all $j \geq 1$. □

3 D'Ocagne's identity for generalized of Lucas numbers

For $j = 0, 1, 2, \cdots$ and $k = 0, 1, 2, \cdots$, the identity of generalized Lucas sequence is developed in [8] and is stated as:

$$\nabla_j^k = (E_{j+1}^k)^2 - E_j^k E_{j+2}^k = 5(-1)^j \left[(-1)^k + \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} + \left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - 1 \right] \tag{3.1}$$

Theorem 3.1. *Let $\{E_j^k\}$ be a sequence of generalized Lucas number, then for fixed k the D'Ocagne's identity is defined as*

$$E_m^k E_{n+1}^k - E_n^k E_{m+1}^k = F_j \nabla_n^k, \text{ if } m = n + j, \ j \geq 0, \tag{3.2}$$

where F_j is the j-th Fibonacci number.

Proof. We prove the result by induction on j. For $j = 0$, we have $F_0 = 0$ and

$$E_m^k E_{n+1}^k - E_n^k E_{m+1}^k = E_n^k E_{n+1}^k - E_n^k E_{n+1}^k = 0 = F_0 \nabla_n^k.$$

For $j = 1$, we have $F_1 = 1$ and

$$\begin{aligned} [(E_{n+1}^k)^2 - E_n^k E_{n+2}^k] &= F_1 [(E_{n+1}^k)^2 - E_n^k E_{n+2}^k] \\ &= F_1 \nabla_n^k. \end{aligned}$$

Hence the result (3.2) holds for $j = 0, 1$.

Assume that the result (3.2) holds for $m = n + j$, $j > 1$. Using (1.7), for $m = n + j - 1$, we have

$$E_{n+j-1}^k E_{n+1}^k - E_n^k E_{n+j}^k = F_{j-1} [(E_{n+1}^k)^2 - E_n^k E_{n+2}^k] \tag{3.3}$$

and for $m = n + j$, we have

$$E_{n+j}^k E_{n+1}^k - E_n^k E_{n+j+1}^k = F_j [(E_{n+1}^k)^2 - E_n^k E_{n+2}^k] \tag{3.4}$$

Now, for $m = n + (j + 1)$, using (3.3) and (3.4),we have

$$\begin{aligned} E_{n+j+1}^k E_{n+1}^k - E_n^k E_{n+j+2}^k &= E_{n+j}^k E_{n+1}^k + E_{n+j-1}^k E_{n+1}^k - E_n^k E_{n+j+1}^k - E_n^k E_{n+j}^k \\ &= (E_{n+j-1}^k E_{n+1}^k - E_n^k E_{n+j}^k) - (E_{n+j}^k E_{n+1}^k - E_n^k E_{n+j+1}^k) \\ &= F_{j-1} [(E_{n+1}^k)^2 - E_n^k E_{n+2}^k] + F_j [(E_{n+1}^k)^2 - E_n^k E_{n+2}^k] \\ &= (F_{j-1} + F_j) [(E_{n+1}^k)^2 - E_n^k E_{n+2}^k] \\ &= F_{j+1} [(E_{n+1}^k)^2 - E_n^k E_{n+2}^k] \\ &= F_{j+1} \nabla_n^k. \end{aligned}$$

Thus, the result (3.2) holds for $j + 1$. Hence, by the principle of induction (3.2) holds for all $j \geq 1$. □

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