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Binomial Sums with Skew-Harmonic Numbers

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Abstract We derive a new expression for binomial sums with skew-harmonic numbers. Our derivation is based on elementary analysis of the Euler transform of these sums. The main result may be regarded as a companion formula for binomial sums with harmonic numbers that we proved recently. We provide some examples to demonstrate the attractiveness of our approach. In particular, we state some new identities involving skew-harmonic numbers, and Fibonacci and Lucas numbers.

1 Motivation

Harmonic numbers $(H_n)_{n\geq 0}$ are defined by $H_0 = 0$ and for all $n \geq 1$

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

In contrast, alternating or skew-harmonic numbers ([2, 3, 16]) $(H_n^-)_{n\geq 0}$ are given by $H_0^- = 0$ and for all $n \geq 1$

$$H_n^- = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}.$$

Skew-harmonic numbers are partial sums of the expansion of $\ln(2)$:

$$\ln(2) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k},$$

which itself is the special case $x = 1$ of the Newton-Mercator series

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \quad -1 < x \leq 1.$$

There is also a relation between skew-harmonic numbers and the digamma function $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ (see [2]):

$$\psi\Big(\frac{n+1}{2}\Big) - \psi\Big(\frac{n}{2}\Big) = 2(-1)^{n-1}(\ln(2) - H_{n-1}^-).$$

Harmonic numbers and generalized harmonic numbers are interesting research objects. They have been studied by Euler and many other mathematicians. A historical account is given in the first part of the article [16]. They appear in many beautiful combinatorial identities. The very recent research on the topic has produced a considerable amount of new results concerning finite and infinite series with (generalized) harmonic numbers (see [1], [4]-[10] and [13]-[20]). Some identities for the skew-harmonic numbers are listed in [2] and [3], among others.

In 2009, Boyadzhiev [1] studied binomial sums with harmonic numbers using the Euler transform. He proved the following identity valid for $n \geq 1$

$$\sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k} H_k = (a+b)^n H_n - \Big(b(a+b)^{n-1} + \frac{b^2}{2}(a+b)^{n-2} + \cdots + \frac{b^n}{n}\Big), \tag{1.1}$$

where a and b are arbitrary complex numbers. Frontczak [10] modified the arguments of Boyadzhiev slightly and derived an alternative expression for these sums as

$$\sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k} H_k = \Big((a+b)^n - b^n\Big) H_n - a \sum_{k=0}^{n-1} (a+b)^k b^{n-1-k} H_{n-1-k}. \tag{1.2}$$

The formula (1.2) allows to derive the following identity

$$H_n = \sum_{k=1}^{n} \Big(\frac{1}{k} 2^{n-k} - 2^{k-1} H_{n-k}\Big), \tag{1.3}$$

which can be used as a new defining equation for harmonic numbers.

In this article, we continue the work from [10]. We derive a new expression for binomial sums with skew-harmonic numbers by analyzing the Euler transform of these sums. The main result may be regarded as an analogue formula for the above identity (1.2). We also provide some examples to demonstrate the attractiveness of our approach. In particular, we state some new identities involving skew-harmonic numbers, and Fibonacci and Lucas numbers.

2 The Main Result.

Let $A(z)$ be the ordinary generating function for the skew-harmonic numbers. It is known that ([16])

$$A(z) = \sum_{n=0}^{\infty} H_n^- z^n = \frac{\ln(1+z)}{1-z},$$

where the series converges for $|z| < 1$. For $a, b \in \mathbb{C}$ let further $S_n(a,b)$ be defined as

$$S_n(a,b) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H_k^-. \tag{2.1}$$

Then we have the following theorem.

Theorem 2.1. *For all $n \geq 1$ it holds that*

$$\begin{aligned} S_n(a,b) &= (a-b)^n H_n^- + b^n H_n + 2b \sum_{k=0}^{n-1} (a+b)^k (a-b)^{n-1-k} H_{n-1-k}^- \\ &\quad + a \sum_{k=0}^{n-1} (a+b)^k b^{n-1-k} H_{n-1-k}. \end{aligned} \tag{2.2}$$

Proof. Let $S(z)$ be the ordinary generating function for the sum $S_n(a,b)$. Then, by Euler's transform

$$\begin{aligned} S(z) &= \sum_{n=0}^{\infty} S_n(a,b) z^n \\ &= \frac{1}{1-bz} A\Big(\frac{az}{1-bz}\Big) \\ &= \frac{\ln(1+(a-b)z)}{1-(a+b)z} - \frac{\ln(1-bz)}{1-(a+b)z}. \end{aligned}$$

Now, we have that

$$\frac{\ln(1-bz)}{1-(a+b)z} = \frac{az}{1-(a+b)z}\frac{\ln(1-bz)}{1-bz} + \frac{\ln(1-bz)}{1-bz},$$

and

$$\frac{\ln(1+(a-b)z)}{1-(a+b)z} = \frac{\ln(1+(a-b)z)}{1-(a-b)z} + \frac{2bz}{1-(a+b)z}\frac{\ln(1+(a-b)z)}{1-(a-b)z}.$$

Hence,

$$\begin{aligned} S(z) &= \frac{\ln(1+(a-b)z)}{1-(a-b)z} + \frac{2bz}{1-(a+b)z}\frac{\ln(1+(a-b)z)}{1-(a-b)z} \\ &\quad + \Big(-\frac{\ln(1-bz)}{1-bz}\Big) + \frac{az}{1-(a+b)z}\Big(-\frac{\ln(1-bz)}{1-bz}\Big) \\ &= \sum_{n=0}^{\infty}(a-b)^n H_n^- z^n + \sum_{n=0}^{\infty} b^n H_n z^n \\ &\quad +2bz\Big(\sum_{n=0}^{\infty}(a+b)^n z^n\Big)\Big(\sum_{n=0}^{\infty}(a-b)^n H_n^- z^n\Big) \\ &\quad +az\Big(\sum_{n=0}^{\infty}(a+b)^n z^n\Big)\Big(\sum_{n=0}^{\infty} b^n H_n z^n\Big). \end{aligned}$$

Using Cauchy's product rule for power series and comparing the coefficients of z^n completes the proof. □

We proceed with some examples. For $(a,b)=(a,a)$ we get the identity

$$\sum_{k=0}^{n}\binom{n}{k}H_k^- = H_n + \sum_{k=0}^{n-1} 2^k H_{n-1-k}. \tag{2.3}$$

Combining ([10])

$$\sum_{k=0}^{n}\binom{n}{k}H_k = (2^n-1)H_n - \sum_{k=0}^{n-1} 2^k H_{n-1-k},$$

with ([1])

$$\sum_{k=0}^{n}\binom{n}{k}H_k = 2^n\Big(H_n - \sum_{k=1}^{n}\frac{1}{k2^k}\Big),$$

we can easily derive the following identity valid for all $n \geq 1$:

$$\sum_{k=1}^{n}\binom{n}{k}H_k^- = 2^n \sum_{k=1}^{n}\frac{1}{k2^k}. \tag{2.4}$$

Although not appearing in the references, it is most likely that identity (2.4) is known. A charming proof using induction can be given as follows: Since the identity is trivially true for $n=1$, the inductive step is

$$\begin{aligned}
\sum_{k=1}^{n+1}\binom{n+1}{k}H_k^- &= \sum_{k=1}^{n}\binom{n+1}{k}H_k^- + H_{n+1}^- \\
&= \sum_{k=1}^{n}\left(\binom{n}{k}+\binom{n}{k-1}\right)H_k^- + H_{n+1}^- \\
&= \sum_{k=1}^{n}\binom{n}{k}H_k^- + \sum_{k=0}^{n-1}\binom{n}{k}H_{k+1}^- + H_{n+1}^- \\
&= \sum_{k=1}^{n}\binom{n}{k}H_k^- + \sum_{k=0}^{n-1}\binom{n}{k}\left(H_k^- + (-1)^{k+2}\frac{1}{k+1}\right) + H_{n+1}^- \\
&= 2\sum_{k=1}^{n}\binom{n}{k}H_k^- + \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{k+1} \\
&= 2^{n+1}\sum_{k=1}^{n}\frac{1}{k2^k} + \frac{1}{n+1} \\
&= 2^{n+1}\sum_{k=1}^{n+1}\frac{1}{k2^k}.
\end{aligned}$$

Note that at the end of the proof we have used the relation

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{k+1} = \int_0^1 (1-x)^n dx.$$

Finally, as an interesting by-product, we point out that

$$\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k}H_k^- = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k2^k} = \ln(2). \tag{2.5}$$

For $(a,b)=(-1,1)$ we deduce that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k H_k^- = (-1)^n 2^n (H_n^- - H_{n-1}^-) + H_n - H_{n-1} = \frac{1}{n}(1-2^n), \tag{2.6}$$

which means that $\frac{1}{n}(1-2^n)$ is the binomial transform of $(-1)^n H_n^-$. The inverse transform gives immediately

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k\frac{1}{k}(1-2^k) = H_n^-. \tag{2.7}$$

The last two identities are known. They appear as equations (9.20) and (9.21) with a different proof in the textbook [3]. For $(a,b)=(2,1)$ the theorem gives

$$\sum_{k=0}^{n}\binom{n}{k}2^k H_k^- = H_n^- + H_n + 2\sum_{k=0}^{n-1}3^k(H_{n-1-k}^- + H_{n-1-k}).$$

Using the connection $H_n^- = H_n - H_{\lfloor \frac{n}{2}\rfloor}$ with $\lfloor x\rfloor$ being the floor function at x (see [15] for instance), we obtain

$$\sum_{k=0}^{n}\binom{n}{k}2^k H_k^- = 2H_n - H_{\lfloor \frac{n}{2}\rfloor} + 2\sum_{k=0}^{n-1}3^k\left(2H_{n-1-k} - H_{\lfloor \frac{n-1-k}{2}\rfloor}\right). \tag{2.8}$$

Another expression for the sum can be derived using

$$H_n^- + H_n = 2\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2j+1} = 2h_{\lfloor \frac{n+1}{2} \rfloor},$$

where we have adopted the notation from [16]

$$h_n = \sum_{j=1}^{n} \frac{1}{2j-1}. \tag{2.9}$$

This allows to express the sum as

$$\sum_{k=0}^{n} \binom{n}{k} 2^k H_k^- = 2h_{\lfloor \frac{n+1}{2} \rfloor} + 4\sum_{k=0}^{n-1} 3^k h_{\lfloor \frac{n-k}{2} \rfloor}. \tag{2.10}$$

Our final example is an identity involving hyperbolic functions. With $(a, b) = (e^x, e^{-x})$ it follows that for all $x \neq 0$:

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} e^{2kx} H_k^- \quad &= \quad H_n + e^{nx} 2^n \sinh^n(x) H_n^- + \sum_{k=0}^{n-1} 2^k \cosh^k(x) e^{(k+2)x} H_{n-1-k} \\ &\quad + e^{(n-1)x} 2^n \sinh^{n-1}(x) \sum_{k=0}^{n-1} \coth^k(x) H_{n-1-k}^-.\end{aligned}$$

3 Connections with Fibonacci numbers.

From Theorem 2.1 it is also possible to deduce some identities involving skew-harmonic numbers and Fibonacci (Lucas) numbers (see [11] and [12] for more information).

Proposition 3.1. *Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Then, for all $n \geq 1$, we have the relations*

$$\sum_{k=0}^{n} \binom{n}{k} F_k H_k^- = (-1)^{n+1} F_n H_n^- + \sum_{k=0}^{n-1} (F_{2k+1} H_{n-1-k} + 2F_{3k+1-n} H_{n-1-k}^-), \tag{3.1}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} L_k H_k^- = 2H_n + (-1)^n L_n H_n^- + \sum_{k=0}^{n-1} (L_{2k+1} H_{n-1-k} + 2L_{3k+1-n} H_{n-1-k}^-). \tag{3.2}$$

Proof. Evaluate (2.2) at $(a, b) = (\alpha, 1)$ and $(a, b) = (\beta, 1)$, respectively. This gives

$$S_n(\alpha, 1) = (-1)^n \beta^n H_n^- + H_n + 2\sum_{k=0}^{n-1} \alpha^{3k+1-n} H_{n-1-k}^- + \sum_{k=0}^{n-1} \alpha^{2k+1} H_{n-1-k}$$

and

$$S_n(\beta, 1) = (-1)^n \alpha^n H_n^- + H_n + 2\sum_{k=0}^{n-1} \beta^{3k+1-n} H_{n-1-k}^- + \sum_{k=0}^{n-1} \beta^{2k+1} H_{n-1-k},$$

where we have used the additional relations $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. Now, calculate $S_n(\alpha, 1) \pm S_n(\beta, 1)$ and use the Binet forms for F_n and L_n, respectively. □

Comparing the above equations with their harmonic number analogues from [10]

$$\sum_{k=0}^{n} \binom{n}{k} F_k H_k = F_{2n} H_n - \sum_{k=0}^{n-1} F_{2k+1} H_{n-1-k}, \tag{3.3}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}L_kH_k=(L_{2n}-2)H_n-\sum_{k=0}^{n-1}L_{2k+1}H_{n-1-k}, \tag{3.4}$$

then, with h_n being defined in (2.9), we can obtain

$$\sum_{k=0}^{n}\binom{n}{k}F_kh_{\lfloor\frac{k+1}{2}\rfloor}=\frac{1}{2}\Big(F_{2n}H_n-(-1)^nF_nH_n^-\Big)+\sum_{k=0}^{n-1}F_{3k+1-n}H_{n-1-k}^-, \tag{3.5}$$

as well as

$$\sum_{k=0}^{n}\binom{n}{k}L_kh_{\lfloor\frac{k+1}{2}\rfloor}=\frac{1}{2}\Big(L_{2n}H_n+(-1)^nL_nH_n^-\Big)+\sum_{k=0}^{n-1}L_{3k+1-n}H_{n-1-k}^-. \tag{3.6}$$

Proposition 3.2. *For all $n\geq 1$, the following relations hold:*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}F_kH_k^- &= F_{2n}H_n^- -2F_{2n-2}H_{n-1}^- -(-1)^nH_{n-1}\\ &\quad -2\sum_{k=1}^{n-1}F_{2n-2-3k}H_{n-1-k}^- -(-1)^n\sum_{k=1}^{n-1}F_{k-1}H_{n-1-k},\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}L_kH_k^- &= L_{2n}H_n^- +2(-1)^nH_n-2L_{2n-2}H_{n-1}^- -(-1)^nH_{n-1}\\ &\quad -2\sum_{k=1}^{n-1}L_{2n-2-3k}H_{n-1-k}^- +(-1)^n\sum_{k=1}^{n-1}L_{k-1}H_{n-1-k}.\end{aligned} \tag{3.8}$$

Proof. Evaluate (2.2) at $(a,b)=(\alpha,-1)$ and $(a,b)=(\beta,-1)$, respectively. Combine the results as in the previous proof. □

Binomial sums with even indexed Fibonacci (Lucas) numbers can be expressed as follows:

Proposition 3.3. *For all $n\geq 1$, the following expressions are valid:*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}F_{2k}H_k^- &= F_nH_n^- +\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}5^k\Big(2F_{n-1}H_{n-1-2k}^- +F_{2k+2}H_{n-1-2k}\Big)\\ &\quad +\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}5^k\Big(2L_{n-1}H_{n-2-2k}^- +L_{2k+3}H_{n-2-2k}\Big),\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}L_{2k}H_k^- &= L_nH_n^- +2H_n+\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}5^k\Big(2L_{n-1}H_{n-1-2k}^- +L_{2k+2}H_{n-1-2k}\Big)\\ &\quad +\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}5^{k+1}\Big(2F_{n-1}H_{n-2-2k}^- +F_{2k+3}H_{n-2-2k}\Big).\end{aligned} \tag{3.10}$$

Proof. Evaluating (2.2) at $(a,b)=(\alpha^2,1)$ yields

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\alpha^{2k}H_k^- &= \alpha^nH_n^- +H_n+2\alpha^{n-1}\sum_{k=0}^{n-1}5^{k/2}H_{n-1-k}^-\\ &\quad +\sum_{k=0}^{n-1}5^{k/2}\alpha^{k+2}H_{n-1-k},\end{aligned}$$

where we have used that $a-b=\alpha$ and $a+b=\sqrt{5}\alpha$. Similarly, with $(a,b)=(\beta^2,1)$,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\beta^{2k}H_k^- &= \beta^n H_n^- + H_n + 2\beta^{n-1}\sum_{k=0}^{n-1}(-1)^k 5^{k/2}H_{n-1-k}^- \\ &\quad + \sum_{k=0}^{n-1}(-1)^k 5^{k/2}\beta^{k+2}H_{n-1-k}.\end{aligned}$$

Now, we can combine the two sums according to the Binet forms and

$$\alpha^{k+2}-(-1)^k\beta^{k+2}=\begin{cases}\sqrt{5}F_{k+2}, & k \text{ even};\\ L_{k+2} & k \text{ odd}.\end{cases}$$

□

It is possible to state more identities of this kind. We have found similar expressions for the sums

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}F_{2k}H_k^-, \qquad \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}L_{2k}H_k^-,$$
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}F_{2n-3k}H_k^-, \qquad \sum_{k=0}^{n}\binom{n}{k}(-1)^{k}L_{2n-3k}H_k^-,$$

and others. All of them are left for a personal study. Connections of skew-harmonic numbers to other important number sequences, such as Mersenne or Pell numbers, are also easily deducible from Theorem 2.1.

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