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Primal Hyperideals Of Multiplicative Hyperring

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Abstract In this paper, we introduce the concepts of adjoint, n-adjoint of a hyperideals and primal and n-primly hyperideals of a commutative multiplicative hyperrings. Many results concerning prime, n-primly, primary and primal hyperideals of a commutative multiplicative hyperrings are given, illustrated by several examples. Also we characterise all prime, primary, n-primly and primal C_u hyperideals of quotient hyperring.

1 Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians. Marty introduced hypergroups as a generalization of groups. He published some notes on hypergroups, using them in different contexts as algebraic functions, rational fractions, non-commutative groups and then many researchers have been worked on this new field of modern algebra and developed it. It was later observed that the theory of hyperstuctures has many applications in both pure and applied sciences, for example, semi-hypergroups are the simplest algebraic hyperstructures that posses the properties of closure and associativity. In algebraic hyperstructures, the product of two elements is not an element but a set, while in classical algebraic structures, the binary operation of two elements of a set is a gain an element of the set. Marty Krasner was the first researcher who gave the idea of hyperstructure theory in 1983, [8]. Hyperstructures have various application in applied and pure sciences such as Latices, Geometry, Cryptography, Automata and Artificial Intelligence. In the sence of Matry, a hypergroup is a nonempty set H endowed by hyperstructure $\star : H \times H \longrightarrow P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H, which satisfy associative law and product axioms. The hyperrings were introduced by Marty Krasner. Krasner hyperrings are a generalization of classical rings in which the multiplicative operation is a binary operation while the addition operation is a hyperoperation. The theory of hyperrings has been developed by many researchers see [1], [11], [7], [16]. There are various types of hyperrings and one of the important classes of hyperrings, called multiplicative hyperring, was introduced in [7]. Primal ideals in a commutative ring with nonzero identity have been introduced and studied by L.Fuchs in [10], and continued to primal ideals over strong co-ideal in semirings, [14]. And continued to primary hyperideals of multiplicative hyperrings, [7]. This paper is concerned with introducing the concepts of n-primly and primal hyperideals on commutative multiplicative hyperrings. These concepts were introduced and studied in commutative rings, see [2], [9], [6], [10], [13]. Also, we introduce some results on n-primly and primal hyperideals, and investigate the relations between n-primly, primal, prime, primary and irreducible hyperideals. We also study the effect of good homomorphisms on these hyperideals and characterize all prime, n-primly, primal and C_u-hyperideal of any quotient hyperring. We illustrate the results by several examples.

1.1 Multiplicative hyperrings

In algebraic hyperstructures, the product of two elements is not an element but a set, while in classical algebraic structures, the binary operation of two elements of a set is again an element of the set. More exactly, a map $\star$:$H \times H \longrightarrow P^*(H)$ is called a hyperoperation, where $P^*(H)$

is the set of all nonempty subsets of H. If $A, B \in P^*(H)$ and $x \in H$, then we define

$$A \star B = \cup_{a\in A,\ b\in B} a \star b,\ A \star x = A \star \{x\}.$$

A semihypergroup $(H, \star)$ is a nonempty set with the associative hyperoperation, i.e., that is $(a \star b) \star c = a \star (b \star c),\ for\ all\ a,\ b,\ c \in\ H$. A semihypergroup H is called a hypergroup if for every $a \in H,\ a \star H = H = H \star a$, which is quasihypergroup. Similar to hypergroups, hyperrings are algebraic structures more general than rings, subsitutiting both or only of the binary operations of addition and multiplication by hyperoperations, see in [15]. In this paper we give some definitions and results of hyperstructures that we need to develop.

Definition 1.1. [7]A triple $(R, +, \star)$ is called a multiplicative hyperring if,

(i) $(R, +)$ is an abelian group;

(ii) $(R,\ \star)$ is a semihypergroup;

(iii) $\forall\ a,\ b,\ c \in R$: $a \star (b + c) \subseteq a \star b + a \star c$ and $(b + c) \star a \subseteq b \star a + c \star a$;

(iv) $\forall\ a,\ b \in R : a \star (-b) = (-a) \star b = -(a \star b)$.

If in (iii) we have equalities instead of inclusions, then we say that the multiplicative hyperring is a strongly distributive.

A multiplicative hyperring $(R, +, \star)$ is said to be commutative if R is commutative with respect to operation $+$ and hyperoperation $\star$. Throughout this paper $(R, +, \star)$ denotes a multiplicative hyperring, and all hyperrings are assumed to be commutative with identity, see [7].

Example 1.2. [12] Let $(R,\ +,\ \cdot)$ be a ring and I be an ideal of it. We define the following hyperoperation on R. For all $a,\ b \in R$, $a \star b = a \cdot b + I$. Then $(R, +, \star)$ is a strongly distributive hyperring. Indeed, first of all, $(R,\ +)$ is an abelian group. Then, for all $a,\ b,\ c \in R$, we have $a \star (b \star c) = a \star (b \cdot c + I) = \bigcup_{h\in I} a \star (b \cdot c + h) = \bigcup_{h\in I} a \cdot (b \cdot c + h) + I = a \cdot b \cdot c + I$ and similarly, we have $(a \star b) \star c = a \cdot b \cdot c + I$. Moreover, for all $a,\ b,\ c \in R$, we have $a \star (b + c) = a \cdot (b + c) + I = a \cdot b + a \cdot c + I = a \star b + a \star c$ and similarly, we have $(b + c) \star a = b \star a + c \star a$. Finally, for all $a,\ b \in R$, we have $a \star (-b) = a \cdot (-b) + I = (-a) \cdot b + I = (-a) \star b$ and $-(a \star b) = (-a \cdot b) + I = a \cdot (-b) + I = a \star (-b)$.

Definition 1.3. [16] Let R be a multiplicative hyperring. We called $a \in R$ is a regular if there exists $x \in R$ such that $a \in a \star x \star a$. So, we can define that R is a regular multiplicative hyperring, if all of elements in R are regular elements. The set of all regular elements in R is denoted by $V(R)$.

Example 1.4. [16] Let $(R,\ +,\ \star)$ be the regular commutative ring with an unitary element. For every subset $A \in P^*(R)$, $|A| \geq 2$, and $1 \in A$, define a multiplicative hyperring $(R_A,\ +,\ \star)$, where $R_A = R$ and for all $x,\ y \in R_A$, $x \star y = \{xay | a \in A\}$. Then $(R_A,\ +,\ \star)$ is a regular multiplicative hyperring. Since, for all $a \in R$, there exists $r \in R$ such that $a = ara$. Now, by setting $x = r$ we have, $a \star x \star a = \{asx | s \in A\} \star a = \{asxta | s,\ t \in A\} = \{axast | s,\ t \in A\} = \{ast | s,\ t \in A\}$, since $1 \in A$, we have $a \in a \star x \star a$. Hence $(R_A,\ +,\ \star)$ is a regular.

Definition 1.5. [4] Let R be a multiplicative hyperring. Then

(i) An element $e \in R$ is said to be **a left (resp. right) identity** if $a \in e \star a$ (resp. $a \in a \star e$) for $a \in R$. An element e is called an identity element if it is both left and right identity element.

(ii) An element $e \in R$ is said to be **a left (resp. right) scalar identity** if $\{a\} = e \star a$ (resp. $\{a\} = a \star e$) for $a \in R$. An element e is called an scalar identity element if it is both left and right scalar identity element.

(iii) An element a is called **a left (right) invertible** (with respect to e), if there exists $x \in R$, such that $e \in x \star a$ $(e \in a \star x)$ and a is called an invertible if it is both a left and right invertible.

A multiplicative hyperring R is called a left (right) invertible if every element of R has a left (right) invertible and R is called an invertible if it is both a left and a right invertible. Denote the set of all invertible elements in R by $U(R)$ (with respect to the identity e by $U_e(R)$), see [4].

Theorem 1.6. *[12] For a strongly distributive hyperring $(R, +, \star)$, the following statements are equivallent:*

(i) *there exists $a \in R$ such that $|0 \star a| = 1$,*

(ii) *there exists $a \in R$ such that $|a \star 0| = 1$,*

(iii) $|0 \star 0| = 1$,

(iv) $\forall a,\ b \in R$ *such that* $|a \star b| = 1$,

(v) $(R, +, \star)$ *is a ring.*

Definition 1.7. [12] A hyperring $(R, +, \star)$ is called unitary if it contains an element u, such that $a \star u = u \star a = \{a\}$ for all $a \in R$.

Definition 1.8. [12] A nonempty subset I of a multiplicative hyperring $(R,\ +,\ \star)$ is said to be a hyperideal of R if $I - I \subseteq I$ and for all $a,\ b \in I$, and $r \in R$, $r \star a \cup a \star r \subseteq I$.

Remark 1.9. [15] We say that a nonempty subset I is a hyperideal of a commutative multiplicative hyperring R if $a - b \in I$ and $r \star a \subseteq I$ for any $a,\ b \in I$, and $r \in R$.

Definition 1.10. [12] Let $(R,\ +,\ \star)$ be a multiplicative hyperring and H be a nonempty subset of R. We say that H is a subhyperring of $(R,\ +,\ \star)$ if $(H,\ +,\ \star)$ is a multiplicative hyperring. In other words, H is a subhyperring of $(R,\ +,\ \star)$ if $H - H \subseteq H$ and for all $x,\ y \in H,\ x \star y \subseteq H$.

Remark 1.11.
(i) The intersection of two subhyperrings of a multiplicative hyperring $(R, +, \star)$ is a subhyperring of R. The intersection of two hyperideals of a multiplicative hyperring $(R, +, \star)$ is a hyperideal of R. Moreover, any intersection of subhyperrings of a multiplicative hyperring is a subhyperring, while any intersection of hyperideals of a multiplicative hyperring is a hyperideal, see [12].
(ii) Let $(R, +, \cdot)$ be a multiplicative hyperring. The principal hyperideal of R generated by a is given by $< a >= \{pa : p \in Z\} + \{\Sigma_{i=1}^{n} x_i + \Sigma_{j=1}^{m} y_j + \Sigma_{k=1}^{l} z_k : \forall i,\ j,\ k,\ \exists r_i,\ s_j,\ u_k \in\ R,\ x_i \in r_i \cdot a,\ y_j \in a \cdot s_j,\ z_k \in t_k \cdot a \cdot u_k\}$.
The zero hyperideal is the hyperideal generated by the additive identity 0, $< 0 >= \{\Sigma_{i=1}^{n} x_i + \Sigma_{j=1}^{m} y_j + \Sigma_{k=1}^{l} z_k : \forall i,\ j,\ k,\ \exists r_i,\ s_j,\ t_k,\ u_k \in\ R,\ x_i \in r_i \cdot 0,\ y_j \in 0 \cdot s_j,\ z_k \in t_k \cdot 0 \cdot u_k\}$, see [4].

Definition 1.12. [2] Let I be a proper hyperideal of a hyperring R. The hyperideal I is called an irreducible hyperideal of R if $I = J \cap K$, where $J,\ K$ are hyperideals of R, implies $I = J$ *or* $I = K$.

Definition 1.13. [4] A hyperideal $I\ (\neq R)$ of a multiplicative hyperring R is a maximal hyperideal in R if for any hyperideal J of R, $I \subset J \subseteq R$, then $J = R$.

Definition 1.14. [15] A proper hyperideal P of a hypering R is called a prime hyperideal of R if for every pair of elements $a,\ b \in R$ whenever $a \cdot b \subseteq P$, then either $a \in P$ or $b \in P$.

Definition 1.15. [7] Let Q be a proper hyperideal of a multiplicative hyperring R. The hyperideal Q is called a primary hyperideal of R if for each $a, b \in R$ whenever $a \star b \subseteq Q$, then either $a \in Q$ or $b^n \subseteq Q$ for some $n \in \mathbf{N}$.

Example 1.16. [4] Every prime hyperideal of a commutative multiplicative hyperring is a primary hyperideal. The set E of all even integers, is not a prime hyperideal, but is a primary hyperideal of a multiplicative hyperring Z_A over the ring of integers Z, induced by the set A of all positive even integers.

Definition 1.17. [15] Let C be the class of all finite hyperproducts of elements of a multiplicative hyperring R.
i.e $C = \{r_1 \star r_2 \star r_3 \star \ldots r_n, r_i \in R, i = 1, 2, 3, \ldots n,$ n $is\ finite\}$. Let I be a hyperideal of R. If for any $A_J \subseteq C$, where A_J is the class of all J hyperproducts of elements of R, $(\cup_{J=1}^{n} A_J) \cap I \neq +\emptyset$ implies $(\cup_{J=1}^{n} A_J) \subseteq I$, then I is said to be C-union hyperideal of R and denoted by C_u-hyperideal.

Example 1.18. [15] Let $(Z, +, \cdot)$ be the ring of integers. We define the hyperoperation $x \star y = \{2xy, 4xy\}$, for all $x,\ y \in Z$, then $(Z, +, \star)$ is a multiplicative hyperring. In $R = (Z, +, \star)$, since $x \star y = \{2xy, 4xy\}$ for any $x,\ y \in R$. Then all finite products of elements are subsets of a hyperideal $2Z = \{2n,\ n \in Z\}$. Since for all finite products A_j. $(\cup_j A_j) \cap I \neq \emptyset$ and $\cup_j A_j \subseteq 2Z$, then the hyperideal $2Z$ is a C_u-hyperideal of R.

Definition 1.19. [4] A multiplicative hyperring R is a C_u-Noetherian if every ascending chain of C_u hyperideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ there exists $n \in N$ such that $I_i = I_n$, for all $i \geq N$.

Definition 1.20. [4] Let I be a hyperideal of a multiplicative hyperring $(R,\ +,\ \star)$. The intersection of all prime hyperideals of R containing I, is called the prime radical of I, being denoted by $Rad(I)$, $\sqrt{I} \subseteq Rad(I)$ where

$$\sqrt{I} = \{x, x^n \subseteq I,\ for\ some\ n \in \mathbf{N}\}.$$

The equality holds when I is a C_u-hyperideal of R. If the multiplicative hyperring R does not have any prime hyperideal containing I, we define $Rad(I) = R$.

Example 1.21. By Example 1.18, we will prove that $\sqrt{8Z} = Z$. Note that $1^2 = \{2,\ 4\} \nsubseteq 8Z$, $1^3 = \{4,\ 8,\ 16\} \nsubseteq 8Z$, but $1^4 = \{8,\ 16,\ 32,\ 64\} \subseteq 8Z$. So $1 \in \sqrt{8Z}$.

Recall that we can define quotient multiplicative hyperrings similar to quotient rings in classical algebra.

Definition 1.22. [15] Let $(R,\ +,\ \cdot)$ be a multiplicative hyperring and I be a hyperideal of R. We consider the usual addition of cosets and multiplication defined as:

$$(a + I) \star (b + I) = \{c + I | c \in a \cdot b\},$$

on the set $R/I = \{a+I|\ a \in R\}$ of all cosets of I. Then $(R/I, +, \star)$ is a multiplicative hyperring.

Definition 1.23. [5] A homomorphism (resp. good homomorphism) between two multiplicative hyperrings $(R_1, +, \circ)$ and $(R_2, +, \circ_2)$ is a map $f : R_1 \to R_1$ such that for all $x,\ y \in R$, we have $f(x+y) = f(x) +_2 f(y)$ and $f(x \circ y) \subseteq f(x) \circ_2 f(y)$ ($f(x \circ y) = f(x) \circ_2 f(y)$ respectively).

Definition 1.24. [12] $f : R \longrightarrow S$ is an isomorphism if it is homomorphism, and its inverse f^{-1} is homomorphism, too.

Let $f : R_1 \longrightarrow R_2$ be a good homomorphism of multiplicative hyperrings. The kernel of f is the inverse image of $< 0 >$, the hyperideal generated by the zero in R_2, and it is denoted by $Ker(f)$. Since the inverse images of hyperideals are hyperideals, it follows that the kernel is a hyperideal. Similarly as in ring theory, we have $f(< 0 >) \subseteq < 0 >$, which means that $< 0 > \subseteq Kerf$, [12].

Theorem 1.25. *[7] Let $f : R \longrightarrow S$ be a good homorphism and I, J be hyperideals of R and S, respectively. Then the followings are satisfied:*

(i) *If I is a C_u hyperideal containing $Ker(f)$ and f is an epimorphism, then $f(I)$ is a C_u hyperideal of S.*

(ii) *If J is a C_u hyperideal of S, then $f^{-1}(J)$ is a C_u hyperideal of R.*

2 n-adjoint sets of multiplicative hyperring

Definition 2.1. Let n be a positive integer. Let I be a hyperideal of R. The set of all elements that are not n-primary to I is called the n-adjoint set for I and is denoted by $n - adj(I)$. That is,

$$n - adj(I) = \{a \in R : a^n \star b \subseteq I\ \ for\ \ some\ \ b \in R - \sqrt{I}\}.$$

Remark 2.2. In a commutative multiplicative hyperring R with scalar identity e with hyperoperation $\star$. If I is a hyperideal of R, then

$$n - adj(I) \neq R, \forall n > 0, n \in N.$$

Proof. If $n - adj(I) = R$, then $e \in n - adj(I)$. Thus $\exists b \in R - \sqrt{I}$ such that $\{b\} = (e)^n \star b \subseteq I$, which is a contradiction.

Remark 2.3. Let $I =< p >$ be a prime hyperideal of Z_A for some $A \in P^*(Z)$ with $|A| \geq 2$, p is prime integer, then it is clear that I is prime ideal in Z, however the converse is not true as shown in the following example.

Example 2.4. In the multiplicative hyperring of the integer Z_A with $A = \{6,\ 9\}$, the principal hyperideal $< 3 >= \{3n, n \in Z\}$ is not a prime hyperideal of Z_A. In fact, $1 \star 1 = \{6, 9\} \subseteq < 3 >$, but $1 \notin < 3 >$.

However, in some cases the converse of Remark 2.3 is true as shown in the following example.

Example 2.5. In the multiplicative hyperring of the integer Z_A with $A = \{2, 3\}$, every principal hyperideal generated by a prime integer p is a prime hyperideal of Z_A.
Proof.
(i) Let $p = 2$. If $a \star b \subseteq I =< 2 >$, then $2 \mid ab$. Thus $2 \mid a$ or $2 \mid b$. Hence either $a \in < 2 >$ or $b \in < 2 >$.
(ii) Let $p = 3$. If $a \star b \subseteq I =< 3 >$, then $3 \mid ab$. Thus $3 \mid a$ or $3 \mid b$. Hence either $a \in < 3 >$ or $b \in < 3 >$.
(iii) Let p be any prime number such that $p \notin \{2, 3\}$. If $a \star b \subseteq I =< p >$, then $p \mid 2ab$ and $p \mid 3ab$. Thus, $p \mid ab$ (since p is prime number different than 2 and 3). Therefore, $p \mid a$ or $p \mid b$. Hence either $a \in < p >$ or $b \in < p >$. In fact, as a generalization of the previous example it is easy to prove the following remark.

Remark 2.6. If $I =< p >$ is a principal hyperideal of Z generated by the prime integer p and A is a set of prime integers with $|A| \geq 2$, then I is a prime hyperideal of Z_A

Example 2.7. Let $(Z,\ +,\ \cdot)$ be the ring of integers. For all $x,\ y \in Z$. We define the hyperoperation $x \star y = \{2xy,\ 3xy\}$. Then $R = (Z,\ +,\ \star)$ is a multiplicative hyperring.
$n - adj(2Z) = 2Z$ for every positive integer $n \geq 1$.
$1 - adj(4Z) = 4Z$.
$n - adj(4Z) = 2Z$ for every positive integer $n \geq 2$.
$1 - adj(8Z) = 8Z$.
$2 - adj(8Z) = 4Z$.
$n - adj(8Z) = 2Z$ for every positive integer $n \geq 3$.
$1 - adj(9Z) = 9Z$.
$n - adj(9Z) = 3Z$ for every positive integer $n \geq 2$.
$n - adj(6Z) = 2Z \cup 3Z$ for every positive integer n.
$1 - adj(12Z) = 4Z \cup 3Z$.
$n - adj(12Z) = 2Z \cup 3Z$ for every positive integer $n \geq 2$.

Example 2.8. In Example 1.18.
$n - adj(2Z) = \{\quad\}$ for every positive integer $n \geq 1$. Since $\sqrt{2Z} = Z$.
$n - adj(4Z) = 2Z$. for every positive integer $n \geq 1$. Since $\sqrt{4Z} = 2Z$.
$1 - adj(8Z) = 4Z$.
$2 - adj(8Z) = 2Z$.
$n - adj(8Z) = 2Z$ for every positive integer $n \geq 2$.

Theorem 2.9. *Let I be a hyperideal of R with $\sqrt{I} \neq R$, then*

$$I \subseteq 1 - adj(I) \subseteq 2 - adj(I) \subseteq 3 - adj(I) \subseteq ... \subseteq n - adj(I).$$

Proof. *Let $a \in I$, then $a \star 1 \subseteq I$ with $1 \in R - \sqrt{I}$. So $a \in 1 - adj(I)$. Now, If $a \in 1 - adj(I)$, then $\exists\ b \in R - \sqrt{I}$ such that $a \star b \subseteq I$. Hence $a^2 \star b \subseteq I$, because I is a hyperideal. Similarly, for any m if $a \in m - adj(I)$, then $a^m \star b \subseteq I$, for some $b \in R - \sqrt{I}$ implies $a^{m+1} \star b \subseteq I$.*

Corollary 2.10. *If I is a prime hyperideal of R, then*

$$I \subseteq 1 - adj(I) \subseteq 2 - adj(I) \subseteq ... \subseteq n - adj(I).$$

Proof. *Since I is a prime hyperideal of R, then I is a proper, also $\sqrt{I}$ is a proper. Note that if $1 \in \sqrt{I}$, then $1^n \subseteq I$ for some $n \in Z^+$. Since I is a prime, then $1 \in I$, which is a contradiction.*

Remark 2.11. Let n be a positive integer.

(i) $2Z$ in Example 2.7, is a prime hyperideal of R. But $2Z$ in Example 2.8, is not a prime hyperideal of R. Also from Examples 2.7, 2.8, we can see that $n - adj(I)$ is not necessarily a hyperideal of R. Note that $n - adj(6Z) = 2Z \cup 3Z$ is not a hyperideal of R.

(ii) For the proper hyperideals I of R, in Example 2.7, we have $I \subseteq 1 - adj(I) \subseteq 2 - adj(I) \subseteq 3 - adj(I) \subseteq ... \subseteq n - adj(I)$. But For a proper hyperideal $I = 2Z$ of R, in Example 2.8, we have $I \nsubseteq n - adj(I)$. Since $\sqrt{2Z} = Z$ is not a proper hyperideal of R.

Definition 2.12. Let I be a hyperideal of R. The adjoint set of I, which is denoted as

$$adj(I) = \{a \in R : a \star b \subseteq I \ for \ some \ b \in R - I\}.$$

i.e. $adj(I)$ is the set of all elements that are not prime to I.

Example 2.13. Let $(Z, +)$ be an abelian group and $2Z$ be a subgroup of Z. $\forall\, x,\ y \in Z$, we define $x \circ y = 2Z$. Then $R = (Z, +, \circ)$ is a multiplicative hyperring.
In fact, $adj(2Z) = Z$, because $1 \in Z$ and $\exists 1 \in R - 2Z$, satisfies $1 \circ 1 \subseteq 2Z$, which implies that $1 \in adj(2Z)$ *and hence* $adj(2Z) = Z$.

Example 2.14. Let Z_4 be abelian group and $\star$ be the hyperoperation on Z_4 defined by: $x \cdot y =< x,\ y >= xZ_4 + yZ_4$ (the subgroup of $(Z_4, +)$ generated by x and y for all $x,\ y \in Z_4$.) Then $(Z_4, +, \star)$ is a multiplicative hyperring. The addition and the hypermultiplication as in the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\star$	0	1	2	3
0	$\{0\}$	Z_4	$\{0,2\}$	Z_4
1	Z_4	Z_4	Z_4	Z_4
2	$\{0,2\}$	Z_4	$\{0,2\}$	Z_4
3	Z_4	Z_4	Z_4	Z_4

The only hyperideal is Z_4 with $\gamma(Z_4) = \{\ \}$.

Theorem 2.15. *For any hyperideal I of R, $1 - adj(I) \subseteq adj(I)$.*
Proof. *Let $a \in 1 - adj(I)$, then $\exists b \in R - \sqrt{I}$ such that $a \star b \subseteq I$. Hence $b \in R - I$. Therefore, $a \in adj(I)$.*

Theorem 2.16. *If I is a prime hyperideal of R, then*

$$1 - adj(I) = adj(I).$$

Proof. *Let $a \in adj(I)$, then $\exists b \in R - I$ such that $a \star b \subseteq I$. Since I is prime hyperideal, then $b^m \subseteq R - I$, for any positive integer m. Hence $b \in R - \sqrt{I}$. Therefore, $a \in 1 - adj(I)$. By Theorem 2.15, the equality holds.*
By Corollary 2.10 and Theorem 2.16, we have the following corollary.

Corollary 2.17. *If I is a prime hyperideal of R, then*

$$1 - adj(I) = adj(I) \subseteq 2 - adj(I) \subseteq 3 - adj(I) \subseteq ... \subseteq n - adj(I).$$

Thus $adj(I) \subseteq n - adj(I)$, for every positive integer n.

Theorem 2.18. *Let n be a positive integer. Let I be a proper hyperideal of R, with $\sqrt{I} \neq R$. Then*

$$I \subseteq \sqrt[n]{I} \subseteq n - adj(I), \text{ where } \sqrt[n]{I} = \{a \in R,\ \ a^n \subseteq I\}.$$

Proof. *Let n be any positive integer. It is clear that $I \subseteq \sqrt[n]{I}$. Now, let $r \in \sqrt[n]{I}$, then $r^n \subseteq I$. Thus $r^n \star 1 \subseteq I$. Since $1 \in R - \sqrt{I}$, then $r \in n - adj(I)$, and the proof is complete.*

3 n-primly hyperideals of multiplicative hyperring

We noticed in the previous section that the n-adjoint sets of a hyperideal I of R are not necessarily hyperideals of R. In this section, we will study the hyperideals whose n-adjoint sets are hyperideals. We call these Kinds of hyperideals n-primly hyperideals as in the following definition.

Definition 3.1. Let n be a positive integer. A hyperideal I of a commutative multiplicative hyperring R with $\sqrt{I} \neq R$ is called n-primly hyperideal of R if
$n - adj(I)$ is a closed under addition.

Remark 3.2. From previous definition, we get $n-adj(I)$ is a hyperideal of R. Since $n-adj(I)$ is closed under addition, it is enough to show that for every $r \in R$ and every $a \in n-adj(I)$, $r \star a \subseteq n - adj(I)$. Let $r \in R$ and $a \in n - adj(I)$, then $\exists b \in R - \sqrt{I}$ such that $a^n \star b \subseteq I$. Thus $r^n \star a^n \star b \subseteq I$, which implies that $(r \star a)^n \star b \subseteq I$. Therefore, $r \star a \subseteq n - adj(I)$.

Example 3.3. From Example 2.7, we have $4Z$, $8Z$ and $9Z$ are n-primly hyperideals of R, while $6Z$ and $12Z$ are not n-primly hyperideals of R, for every positive integer n.

Theorem 3.4. *Let n be a positive integer. If I is an n-primly hyperideal of R, then $n - adj(I)$ is a primary hyperideal of R.*
Proof. *Since $1 \notin n - adj(I)$, and I is n-primly hyperideal of R, then $n - adj(I)$ is a proper hyperideal of R. Let $a \star b \subseteq n - adj(I)$ with $a \notin n - adj(I)$, then $\exists\ c \in R - \sqrt{I}$ such that $(a \star b)^n \star c \subseteq I$. Thus $a^n \star (b^n \star c) \subseteq I$. Since $a \notin n - adj(I)$, then $b^n \star c \subseteq \sqrt{I}$, so $(b^n \star c)^m = b^{nm} \star c^m \subseteq I$ for some positive integer m. Since $c \notin \sqrt{I}$, then $c^m \notin \sqrt{I}$. Thus $(b^m)^n \star c^m \subseteq I$ implies $b^m \subseteq n - adj(I)$. Therefore, $n - adj(I)$ is a primary hyperideal of R.*

4 On Primal Hyperideal Of Multiplicative Hyperring

In this section, we introduce primal hyperideal of a hyperring R and give some properties and examples. A hyperideal I of a hyperring R is called primal if the set of all elements of R that are not prime to I form hyperideal of R.
Here an element $r \in R$ is called prime to I if $r \circ s \subseteq I \Rightarrow s \in I$,that is, the residual

$$(I : r) = \{s \in R, r \circ s \subseteq I\} = I.$$

Note that $I \subseteq (I : r)$, for any hyperideal I. Thus r is prime to I if $(I : r) \subseteq I$.

Lemma 4.1. *In the commutative multiplicative hyperring of integers Z_A. Let I be a proper hyperideal of Z_A, let $\gamma(I)$ be the set of elements of Z_A that are not prime to I, let p be a prime number. Then the following hold:*

(i) $I \subseteq \gamma(I) = \{r \in R,\ r \star s \subseteq I, for\ \ some\ \ s \in R - I\}$.

(ii) *If $\gamma(I) =< p >$ is a principal hyperideal of Z_A generated by a positive integer p and $A \nsubseteq< p >$, then $\gamma(I)$ is a prime hyperideal of R.*

(iii) *If $\gamma(I) =< a >$ is a principal hyperideal of Z_A and $A \subseteq< a >\neq Z$, then $\gamma(I)$ is a non prime C_u hyperideal of Z_A.*

Proof. ***(i)*** *Let $r \in I$. We can assume that $r \neq 0$ (since $0 \in \gamma(I)$). As $0 \neq r = 1{\star}r \subseteq I$ with $1 \notin I$, we must have r is not prime hyperideal to I, then $r \in \gamma(I)$), then $I \subseteq \gamma(I)$.*
(ii) *Let p be a prime integer and $A \nsubseteq< p >$. Then there exists $c \in A\backslash < p >$. Now suppose that $a \star b \subseteq< p >$ and $a \notin< p >$, then $\{acb :\ c \in A\} \subseteq< p >$ and hence $b \in< p >$, since $< p >$ is prime ideal in Z, $c \notin< p >$ and $a \notin< p >$. Thus $< p >$ is a prime hyperideal of the multiplicative hyperring Z_A.*
(iii) *Let $A \subseteq< a >$, so for any $r_i \in Z, i = 1, 2, 3, ..., n$, $n \in N$, we have $r_1 \star r_2 \star ... \star r_n = \{r_1c_1r_2c_2...r_{n-1}c_{n-1}r_n : c_i \in A, i = 1, 2, 3, ..., n,\ n \in N\} \subseteq< a >$. Hence, $\gamma(I)$=$< a >$ is a C_u hyperideal of Z_A. Since $< a >\neq Z$, then $1 \notin< a >$, $1 \star 1 \subseteq< a >$. Thus $< a >$ is not a prime hyperideal of Z_A.*

Remark 4.2. If I is a primal hyperideal of Z_A and $A \not\subseteq \gamma(I) =< p >$, where p is a prime number, then by Lemma 4.1 (ii), $\gamma(I)$ is a prime hyperideal called the adjoint prime hyperideal $\gamma(I)$ of I. In this case we also say that I is a primal hyperideal with $\gamma(I)$ is a prime hyperideal.

Theorem 4.3. *Let $(Z,\ +,\ \cdot)$ be the ring of integers. Let $R = (Z,\ +,\ \star)$ is a multiplicative hyperring, where $\star$ is defined $for\ all$ $x,\ y \in Z$ as:*
$x \star y = \{pxy,\ qxy\}$, with p and q are two fixed different prime. Then
(i) $\forall\ n \geq 1$, $I = p^n Z$, $J = q^n Z$ are primal hyperideals of R with $\gamma(p^n Z) = pZ$, $\gamma(q^n Z) = qZ$.
(ii) $J = pqZ$, is not a primal hyperideal of R with $\gamma(pqZ) = pZ \cup qZ$.
Proof. ***(i)*** *Let $a \in \gamma(pZ)$, then $\exists\, b \in R - pZ$ with $a \star b = \{pab,\ qab\} \subseteq pZ$. So p divides any elements in $a \star b$ implies that $p \mid ab$. Thus $p \mid a$. So $a \in pZ$. Now, let $a \in pZ$, then $b = q$ satisfies $a \star b = \{pab,\ qab\} \subseteq pZ$. Hence, $\exists\, b = q \in R - pZ$ with $a \star b \subseteq pZ$. So $a \in \gamma(pZ)$, and hence $\gamma(pZ) = pZ$. Suppose $a \in \gamma(p^2 Z)$, then $\exists\, b = p \in R - p^2 Z$ with $a \star b = \{p^2 a,\ qap\} \subseteq p^2 Z$. So p^2 divides any elements in $a \star b$ implies that $p \mid a$. Thus $a \in pZ$. Suppose $a \in pZ$, then $b = p$ satisfies $a \star b = \{pab,\ qab\} \subseteq p^2 Z$ Hence $\exists\, b = p \in R - p^2 Z$ with $a \star b \subseteq p^2 Z$. Thus $a \in \gamma(p^2 Z)$ and hence $\gamma(p^2 Z) = pZ$. Let $a \in \gamma(p^3 Z)$, then $\exists\, b = p^2 \in R - p^3 Z$ with $a \star b = \{p^2 a,\ qap^2\} \subseteq p^3 Z$. So p^3 divides any elements in $a \star b$ implies that $p \mid a$. Thus $a \in pZ$. Now, let $a \in pZ$, then $b = p^2$ satisfies $a \star b = \{p^2 a,\ qap^2\} \subseteq p^3 Z$. Hence $\exists\, b = p^2 \in R - p^3 Z$ with $a \star b \subseteq p^3 Z$. Thus $a \in \gamma(p^3 Z)$, and hence $\gamma(p^3 Z) = pZ$, where pZ is a hyperideal of R. Similarly, $\forall\ n > 3$, the hyperideals $I = p^n Z$ and $J = q^n Z$ are primal hyperideals of R with $\gamma(p^n Z) = pZ$, $\gamma(q^n Z) = qZ$, where pZ and qZ are hyperideals of R.*
(ii) *Let $a \in pZ$, then $b = q$ satisfy that $a \star b = \{p^2 q,\ pq^2\} \subseteq pqZ$. Thus $\exists\, b = q \in R - pqZ$ such that $a \star b = \{p^2 q,\ pq^2\} \subseteq pqZ$ implies that $a \in \gamma(pqZ)$, and hence $pZ \subseteq \gamma(pqZ)$. Now, let $a \in qZ$, then $b = p$ satisfy that $a \star b = \{p^2 q,\ pq^2\} \subseteq pqZ$. Then $\exists\, b = p \in R - pqZ$ such that $a \star b = \{p^2 q,\ pq^2\} \subseteq pqZ$. Therefore, $a \in \gamma(pqZ)$, and hence $qZ \subseteq \gamma(pqZ)$. Therefore, $pZ \cup qZ \subseteq \gamma(pqZ)$. Let $a \in \gamma(pqZ)$. Then $\exists\, b \in R - pqZ$ such that $a \star b \subseteq pqZ$. Thus pq divides any elements in $a \star b$. But $pq \nmid b$, implies that $pq \mid a$. Hence we have $p \mid a$ or $q \mid a$. Thus $a \in pZ \cup qZ$ and hence $\gamma(pqZ) = pZ \cup qZ$ is not a hyperideal of R. So $J = pqZ$ is not a primal hyperideal of R.*

Example 4.4. Consider the hyperring $(Z,\ +,\ \star)$ in Example 2.7. The hyperideal $16Z$ is a primal hyperideal of R, with $\gamma(16Z) = 2Z$, which is a hyperideal of R, by Theorem 4.3 (i). But, the hyperideal $6Z$ is not a primal hyperideal of R, with $\gamma(10Z) = 2Z \cup 3Z$, which is not a hyperideal of R, by Theorem 4.3 (ii).

Theorem 4.5. *If Q is a primary hyperideal of R, then Q is a primal hyperideal of R.*
Proof. *To show that Q is a primal hyperideal of R, it is enough to show that $\gamma(Q) = \sqrt{Q}$. Let $a \in \sqrt{Q}$, then there exists $n > 0$, such that $a^n \subseteq Q$.*
Let n be the smallest positive integer such that $a^n \subseteq Q$. By induction, if $n = 1$, then $a \in Q \subseteq \gamma(Q)$, because Q is a proper i.e., $1 \notin Q$ with $a \star 1 \subseteq Q$. If $n > 1$, then $a^{n-1} \star a \subseteq Q$ with $a^{n-1} \not\subseteq Q$, we get that $a \in \gamma(Q)$. Thus, $\sqrt{Q} \subseteq \gamma(Q)$.
Conversely, let $a \in \gamma(Q)$, then there exists $c \in R - Q$ with $a \star c \subseteq Q$, thus Q is primary gives $a^n \subseteq Q$, for some $n \in \boldsymbol{N}$. which implies that $a \in \sqrt{Q}$. Therefore, $\gamma(Q) = \sqrt{Q}$ and so $\gamma(Q)$ is a hyperideal of R.

Since by definition every prime hyperideal of R is primary, then we can conclude the following result.

Corollary 4.6. *If I is a prime hyperideal of R, then I is a primal hyperideal of R.*

The converse of Corollary 4.6, need not be true.

Example 4.7. Consider the hyperring $(Z,\ +,\ \star)$ in Example 2.7. $8Z$ is a primary hyperideal of R, let $a \star b \subseteq 8Z$, for any $a,\ b \in R$, then $8 \mid ab$. Therefore $2 \mid a$ or $4 \mid b$. Hence $a^2 \subseteq 8Z$ or $b^2 \subseteq 8Z$. Which implies by Theorems 4.5 and 4.3 (i), $8Z$ is a primal hyperideal of R, with $\gamma(8Z) = 2Z$, which is a hyperideal of R. But $8Z$ is not a prime hyperideal of R, because $2 \star 4 = \{16,\ 24\} \subseteq 8Z$, but neither $2 \in 8Z$ nor $4 \in 8Z$.

Example 4.8. Suppose the set of all congruence classes of integers modulo 4, i.e $Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ and its multiplicative subgroup of units $G = \{\bar{1}, \bar{3}\}$ and construct R as Z_4/G, i.e.

$$R=\{\overline{r}G,\overline{r}\in Z_4\}=\{\overline{\overline{r}},\overline{r}\in Z_4\}$$

There is: $\overline{\overline{0}}=\{\overline{0}\}$, $\overline{\overline{1}}=\{\overline{1},\overline{3}\}=\overline{\overline{3}}$, $\overline{\overline{2}}=\{\overline{2}\}$.
Now on R define the hyperaddition $\oplus$ and multiplication $\circ$ by

$$\overline{\overline{r}}\oplus\overline{\overline{s}}=\{\overline{\overline{t}},\overline{\overline{t}}\bigcap(\overline{\overline{r}}+\overline{\overline{s}})\neq\emptyset\}$$
$$\overline{\overline{r}}\star\overline{\overline{s}}=\overline{\overline{rs}}.$$

When computing e.g

$\overline{\overline{1}}+\overline{\overline{1}}=\{\overline{1},\overline{3}\}+\{\overline{1},\overline{3}\}=\{\overline{2},\overline{0}\}=\overline{\overline{2}},\overline{\overline{0}}.$
$\overline{\overline{1}}+\overline{\overline{2}}=\{\overline{1},\overline{3}\}+\{\overline{2}\}=\{\overline{1},\overline{3}\}=\overline{\overline{1}}.$
$\overline{\overline{2}}+\overline{\overline{2}}=\{\overline{2}\}+\{\overline{2}\}=\overline{\overline{0}}.$
For simplicity in the table, we will omit the set prackets.

Table 1: The hyperaddition $\oplus$ and the hypermultiplication $\star$ as in the following tables:

$\oplus$	$\overline{\overline{0}}$	$\overline{\overline{1}}$	$\overline{\overline{2}}$
$\overline{\overline{0}}$	$\overline{\overline{0}}$	$\overline{\overline{1}}$	$\overline{\overline{2}}$
$\overline{\overline{1}}$	$\overline{\overline{1}}$	$\overline{\overline{0}},\overline{\overline{2}}$	$\overline{\overline{1}}$
$\overline{\overline{2}}$	$\overline{\overline{2}}$	$\overline{\overline{1}}$	$\overline{\overline{0}}$

$\star$	$\overline{\overline{0}}$	$\overline{\overline{1}}$	$\overline{\overline{2}}$
$\overline{\overline{0}}$	$\overline{\overline{0}}$	$\overline{\overline{0}}$	$\overline{\overline{0}}$
$\overline{\overline{1}}$	$\overline{\overline{0}}$	$\overline{\overline{1}}$	$\overline{\overline{2}}$
$\overline{\overline{2}}$	$\overline{\overline{0}}$	$\overline{\overline{2}}$	$\overline{\overline{0}}$

The hyperideals: $I_0=\{\overline{\overline{0}}\}, I_1=\{\overline{\overline{0}},\overline{\overline{2}}\}$, where I_1 is a maximal, an irreducible and a prime hyperideal. By Corollary 4.6, I_1 is a primal hyperideal. On the other hand, I_0 is not a prime hyperideal. Because $\overline{\overline{2}}\star\overline{\overline{2}}=\overline{\overline{0}}$. But $\overline{\overline{2}}\notin I_0$. Also I_0 is primary hyperideal in a hyperring R, let $\overline{\overline{r}}\star\overline{\overline{s}}\subseteq I_0$, for any $\overline{\overline{r}},\overline{\overline{s}}\in R$, then we have the following cases (noting that R is a commutative)

(i) $\overline{\overline{r}}=\overline{\overline{0}}$ and $\overline{\overline{s}}\in\{\overline{\overline{0}},\overline{\overline{1}},\overline{\overline{2}}\}$ $implies$ $\overline{\overline{r}}^2\subseteq I_0$, i.e.$n=2$.

(ii) $\overline{\overline{r}}=\overline{\overline{s}}=\overline{\overline{2}}$ $implies$ $\overline{\overline{2}}^2\subseteq I_0$, i.e.$n=2$.

Note that, $I_0=\{\overline{\overline{0}}\}$ is primal hyperideal of R. Since $\gamma(I_0)=\sqrt{I_0}=\{\overline{\overline{0}},\overline{\overline{2}}\}$, which is the hyperideal I_1.

Theorem 4.9. *[4] If Q is a primary C_u hyperideal of a multiplicative hyperring $(R,+,\star)$, then $\sqrt{Q}$ is a C_u prime hyperideal of R.*

Since by Corollary 4.6, every prime hyperideal of R is a primal we can conclude by Theorem 4.9, the following results.

Corollary 4.10. *Suppose that Q is a primary C_u hyperideal of R. Then $\sqrt{Q}$ is a primal C_u hyperideal of R.*
Proof. *By Theorem 4.9, $\sqrt{Q}$ is a prime C_u hyperideal of R, then by Corollary 4.6, $\sqrt{Q}$ is a primal C_u hyperideal of R.*

Theorem 4.11. *Let I be an irreducible hyperideal of R, then I is a primal hyperideal of R.*
Proof. *Assume that I is irreducible hyperideal of R, we need to show that $\gamma(I)$ is a hyperideal of R. Let $a,\ b\in\gamma(I)$, then $I\subset(I:a)$ and $I\subset(I:b)$, I is irreducible hyperideal of R gives, $I\subset(I:a)\cap(I:b)\subseteq(I:a+b)$, hence $a+b\notin I$, then $a+b\in\gamma(I)$. Finally, if $r\in R$, then $I\subset(I:a)\subseteq(I:r\star a)$ show that $r\star a$ is not prime to I which implies that $r\star a\subseteq\gamma(I)$, and the proof is complete.*

The converse of Theorem 4.11, need not be true.

Example 4.12. Consider the ring $(Z_6,\ \oplus,\ \odot)$, that for all $\overline{a},\ \overline{b}\in Z_6$, $\overline{a}\oplus\overline{b}$ and $\overline{a}\odot\overline{b}$ are remainder of $\frac{a+b}{6}$ and $\frac{a.b}{6}$ where " $+$ " and "." are ordinary addition and multiplication for all $a,\ b\in Z_6$. For all $\overline{a},\ \overline{b}\in Z_6$, we define the hyperoperation
$\overline{a}\star\overline{b}=\{\overline{0},\ \overline{ab},\ \overline{2ab},\ \overline{3ab},\ \overline{4ab},\ \overline{5ab}\}$. Then $R=(Z_6,\ \oplus,\ \star)$ is a commutative multiplicative hyperring. The addition and the hypermultiplication as in the following tables:

$\oplus$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{1}$
$\overline{3}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{4}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{5}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$

$\star$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\{\overline{0}\}$	$\{\overline{0}\}$	$\{\overline{0}\}$	$\{\overline{0}\}$	$\{\overline{0}\}$	$\{\overline{0}\}$
$\overline{1}$	$\{\overline{0}\}$	Z_6	$\{\overline{0},\overline{2},\overline{4}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{2},\overline{4}\}$	Z_6
$\overline{2}$	$\{\overline{0}\}$	$\{\overline{0},\overline{2},\overline{4}\}$	$\{\overline{0},\overline{2},\overline{4}\}$	$\{\overline{0}\}$	$\{\overline{0},\overline{2},\overline{4}\}$	$\{\overline{0},\overline{2},\overline{4}\}$
$\overline{3}$	$\{\overline{0}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0}\}$	$\{\overline{0},\overline{3}\}$
$\overline{4}$	$\{\overline{0}\}$	$\{\overline{0},\overline{2},\overline{4}\}$	$\{\overline{0},\overline{2},\overline{4}\}$	$\{\overline{0}\}$	$\{\overline{0},\overline{2},\overline{4}\}$	$\{\overline{0},\overline{2},\overline{4}\}$
$\overline{5}$	$\{\overline{0}\}$	Z_6	$\{\overline{0},\overline{2},\overline{4}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{2},\overline{4},\overline{0}\}$	Z_6

The all hyperideals: $I_0 = \{\overline{0}\}$, $I_1 = \{\overline{0},\ \overline{2},\ \overline{4}\}$, $I_2 = \{\overline{0},\ \overline{3}\}$. I_1, I_2 are maximal, prime, primary, irreducible and primal hyperideals. Note that, I_0 is neither prime nor primary hyperideal, because $\overline{2} \star \overline{3} = \{\overline{0}\} \subseteq I_0$, but $\overline{2} \notin I_0$ and $\overline{3} \notin I_0$. Also I_0 is not a primary hyperideal of R, since $\overline{2} \star \overline{3} = \{\overline{0}\} \subseteq I_0$, $\overline{2} \notin I_0$, and $\forall\ n > 0$ we have $\overline{3}^n = \{\overline{0},\ \overline{3}\} \nsubseteq I_0$, and also $\overline{2} \star \overline{3} = \{\overline{0}\} \subseteq I_0$, $\overline{3} \notin I_0$, and $\forall\ n > 0$ we have $\overline{2}^n = \{\overline{0},\ \overline{2},\ \overline{4}\} \nsubseteq I_0$. Moreover, I_0 is primal hyperideal of R with $\gamma(I_0) = \{\overline{0},\ \overline{2},\ \overline{4}\}$ is a hyperideal of R, but it is reducible hyperideal of R.

Every hyperideal is contained in a prime hyperideal, hence a primal hyperideal. Now, we can conclude the following result.

Theorem 4.13. *Let R be a multiplicative hyperring. Then every hyperideal I of R is an intersection of all primal hyperideals of R, which contains I.*
Proof. *Let I be a hyperideal of R, and $\{P_\alpha\}_{\alpha\in\Gamma}$ be collection of all primal hyperideals of R, which contains I. We show that $I = \cap_{\alpha\in\Gamma}\{P_\alpha\}$. Clearly $I \subseteq \cap_{\alpha\in\Gamma}\{P_\alpha\}$. For reverse of inclusion, let $x \notin I$. Set*

$$\Sigma = \{J{:}J\ \ hyperideal,\ I \subseteq J,\ x \notin J\}.\ Then\ I \in \Sigma,\ so\ \Sigma \neq \phi.$$

It is clear that $(\Sigma, \subseteq)$ is a poset. By Zorn Lemma, Σ has a maximal element. Let K be a maximal element of Σ, we claim that Σ is irreducible. If $K = K_1 \cap K_2$ where $K_1 \subset K$ and $K_2 \subset K$, maximality of K implies that $x \in K_1$ and $x \in K_2$. Therefore $x \in K$, a contradiction. This shows that K is irreducible, and so K is primal by Theorem 4.11. Hence $x \notin K$ implies that $x \notin \cap_{\alpha\in\Gamma}P_\alpha$. Therefore $\cap_{\alpha\in\Gamma}P_\alpha \subseteq I$ and so $\cap_{\alpha\in\Gamma}P_\alpha = I$.

Theorem 4.14. *Let R be a Noetherian multiplicative hyperring. Then every proper hyperideal is an intersection of finitly many primal hyperideals of R.*
Proof. *Let Σ denotes the sets of all proper hyperideal of R, such that I is not a finite intersection of primal hyperideals, we claim that $\Sigma = \phi$. For if not, Σ has a maximal element K. But K is not primal. Thus K is not irreducible by Theorem 4.11 and so $K = K_1 \cap K_2$, where K_1 and K_2 are finite intersction of primal hyperideals of R, and so is K, a contradiction. Hence every hyperideal is an intersection of finitly many primal hyperideals of R.*

Definition 4.15. Let I be a hyperideal of a hyperring R and P be prime hyperideal of R that contains I. The isolated P-component of I, $U(I,\ P)$ is the intersection of all hyperideals which contains I and are such that every element not in P is prime to them. That is, $U(I,\ P) = \cap J$, where J is a hyperideal with property that $I \subseteq J$ and $\forall\ x \notin P$, x is prime to J

Theorem 4.16. *If I is a primal hyperideal of R with adjoint prime hyperideal P, then $I = U(I,\ P)$.*
Proof. *Clearly, $I \subseteq U(I,\ P)$. Since $U(I,\ P)$ is the intersection of all hyperideals J with $I \subseteq J$*

and $x \notin P$, then x is prime to J and I is itself such a hyperideal, $U(I,\ P) \subseteq I$. Hence $I = U(I,\ P)$.

Theorem 4.17. *Let R be a Noetherian multiplicative hyperring. Then every hyperideal is the intersection of its upper isolated components $U(I_1, P_1),\ U(I_2, P_2),$
$U(I_3, P_3),\ ...,\ U(I_n, P_n)$, where $P_1, P_2, P_3, ..., P_n$ are the adjoint prime hyperideals of all primal hyperideals $I_1,\ I_2,\ I_3,\ ...,\ I_n$, n is finite that contains I.*
Proof. *By Theorem 4.13, $I = I_1 \cap I_2 \cap I_3 \cap ... \cap I_n$, where I_i is a primal hyperideal with P_i is adjoint prime hyperideal, so by Theorem 4.16, $I_i = U(I_i, P_i)$. Thus $I = \cap_{i=1}^{n} I_i = \cap_{i=1}^{n} U(I_i, P_i)$.*

The following example shows that the intersection of finite primal hyperideal need not primal hyperideal.

Example 4.18. If we continue with Example 2.7 and use its notation, then the subsets $I = 2Z,\ J = 3Z$ are prime hyperideals of R, then by Corollary 4.6, each I, J is a primal hyperideal of R. But $I \cap J = 6Z$ is not a primal hyperideal of R, see Example 4.4.

5 Primal hyperideals for multiplicative hyperring good homomorphism

Recall that if Q is a hyperideal of hyperring S and $f : R \longrightarrow S$ is a good homomorphism, then $f^{-1}(Q)$ is always a hyperideal of hyperring R. However, if I is a hyperideal of the hyperring R and $f : R \longrightarrow S$ be a good homomorphism, then $f(I)$ need not to be a hyperideal of S.

Theorem 5.1. *Let R and S be multiplicative hyperrings. If $f : R \longrightarrow S$ is a good homomorphism and $I,\ J$ be proper hyperideals of R and S, respectively. Then the followings are satisfied:*

(i) *If I is a primal hyperideal containing $Ker(f)$ and f is an epimorphism, then $f(I)$ is a primal hyperideal of S.*

(ii) *If J is a primal hyperideal of S, then $f^{-1}(J)$ is a primal hyperideal of R.*

Proof. ***(i)*** *It easy to see that $f(I)$ is a hyperideal of S. It is enough to show that $f(\gamma(I)) = \gamma(f(I))$ is a hyperideal of S. Let $y_1,\ y_2 \in f(\gamma(I))$, $s \in S$ and since f is onto, then there exist $x_1,\ x_2 \in \gamma(I),\ r \in R$ such that $f(x_1) = y_1,\ f(x_2) = y_2$ and $f(r) = s$. Since I is primal hyperideal of R. Then $\gamma(I)$ is a hyperideal of R, then $x_1 - x_2 \in \gamma(I),\ r \star x_1 \subseteq \gamma(I)$. So that*

$$y_1 - y_2 = f(x_1) - f(x_2) = f(x_1 - x_2) \in f(\gamma(I)),\ \ s \star y_1 = f(r) \star f(x_1) = f(r \star x_1) \subseteq f(\gamma(I)).$$

So $f(\gamma(I))$ is a hyperideal of S. Finally, let $a \in \gamma(f(I))$, then $\exists\ b \in R - f(I)$ such that $a \star b \subseteq f(I)$. Now $I \subseteq \gamma(I)$, then $f(I) \subseteq f(\gamma(I))$ and also $f(I) \subseteq \gamma(f(I))$, and so $\exists\ b \in R - f(\gamma(I))$ such that $a \star b \subseteq f(\gamma(I))$, which implies that $a \in f(\gamma(I))$. Conversly, let $y \in f(\gamma(I))$, implies $y = f(r) \in S$ for some $r \in \gamma(I)$. Then $r \in R$ and $\exists c \in R - I$ such that $r \star c \subseteq I$. Hence $f(r \star c) = f(r) \star f(c) \subseteq f(I)$ for some $f(c) \in S - f(I)$. Hence, $y = f(r) \in \gamma(f(I))$. Therefore I is primal hyperideal of S.
(ii) *It easy to see that $f^{-1}(J)$ is a hyperideal of R. It enough to show that $f^{-1}(\gamma(J)) = \gamma(f^{-1}(J))$ is a hyperideal of R. Let $a_1,\ a_2 \in f^{-1}(\gamma(J)),\ r \in R$, then $f(a_1),\ f(a_2) \in \gamma(J),\ f(r) \in S$. Since J is a primal hyperideal of S. Then $\gamma(J)$ is a hyperideal of S. So that $f(a_1) - f(a_2) = f(a_1 - a_2) \in \gamma(J)$, and also $f(r) \star f(a_1) = f(r \star a_1) \subseteq \gamma(J)$. Therefore, $a_1 - a_2 \in f^{-1}(\gamma(J))$, and also $r \star a_1 \subseteq f^{-1}(\gamma(J))$. Hence $f^{-1}(\gamma(J))$ is a hyperideal of R. Finally, let $a \in \gamma(f^{-1}(J))$, then $\exists\ b \in R - (f^{-1}(J))$ such that $a \star b \subseteq f^{-1}(J)$. Now $J \subset \gamma(J)$, then $f^{-1}(J) \subseteq f^{-1}(\gamma(J))$ and also $f^{-1}(J) \subseteq \gamma(f^{-1}(J))$, then $\exists\ b \in R - f^{-1}(\gamma(J))$ such that $a \star b \subseteq f^{-1}(\gamma(J))$, which implies that $a \in f^{-1}(\gamma(J))$. Conversly, let $x \in f^{-1}(\gamma(J))$, implies $x = f^{-1}(s) \in R$ for some $s \in \gamma(J)$. Then, $s \in S$ and $\exists b \in S - J$ such that $s \star b \subseteq J$. Hence $f^{-1}(s \star b) \subseteq f^{-1}(s) \star f^{-1}(b) \subseteq f^{-1}(J)$ for some $f^{-1}(s) \in R - f^{-1}(J)$, by definition 1.24. Hence $x = f^{-1}(s) \in \gamma(f^{-1}(J))$. Therefore, $f^{-1}(\gamma(J)) = \gamma(f^{-1}(J))$. Hence $f^{-1}(J)$ is a primal hyperideal of R.*

One can show that all hyperideal of R/I is of the form J/I where J is a hyperideal of R containing I, since the natural homomorphism $\phi : R \longrightarrow R/I,\ \phi(r) = r + I$ is a good epimorphism, [7]. The next theorem investigate the relation between the primal hyperideal of R and R/I, for some hyperideal I of R containing J.

Theorem 5.2. *Let I, J be proper hyperideals of R, with $J \subseteq I$, I is a primal hyperideal of R iff I/J is a primal hyperideal of R/J.*

Proof. *To prove this result, we must show that $\gamma(I/J) = \gamma(I)/J$.*
Let $a+J \in \gamma(I/J)$, then there exist $b+J \in R-(I/J)$ such that $(a+J)\star(b+J) = (a\star b)+J \subseteq I/J$. So there exist $b \in R-I$ such that $a \star b \subseteq I$, which implies that $a \in \gamma(I)$. Therefore $a+J \in \gamma(I)/J$. Conversely, let $a+J \in \gamma(I)/J$, then $a \in \gamma(I)$, then there exist $b \in R-I$ such $a \star b \subseteq I$. Therefore, there exist $b+J \in R-(I/J)$ such that $(a+J)\star(b+J) = (a\star b)+J \subseteq I/J$ and so $a+J \in \gamma(I/J)$.

Corollary 5.3. *Let R and S be multiplicative hyperrings. If $f : R \longrightarrow S$ be a good homorphism and I, J be hyperideals of R and S, respectively. Then the followings are satisfied:*

(i) *If I is a primary hyperideal containing $Ker(f)$ and f is an epimorphism, then $f(I)$ is a primal hyperideal of S.*

(ii) *If J is a primary hyperideal of S, then $f^{-1}(J)$ is a primal hyperideal of R.*

Proof. *(i) and (ii) Follows from Theorem 4.5, 5.1.*

Corollary 5.4. *Let $I \subseteq Q$ be hyperideals of R, then*

(i) *If Q is a primary hyperideal of R. Then Q/I is a primal hyperideal of R/I.*

(ii) *If Q is a C_u primary hyperideal of R containing I. Then $\sqrt{Q}/I$ is a C_u primal hyperideal of R/I.*

Proof. (i) *Follows from Theorem 4.5, 5.2.*
(ii) *Follows from Corollary 4.10, Theorem 5.2.*

6 Conclusion

In this paper, we introduce the concepts of adjoint, n-adjoint, n-primly and primal hyperideals over commutative multiplicative hyperrings.
Many results concerning these concepts are proved. Also, we discuss primal hyperideals on hyperring homomorphism under special conditions. We also study primal hyperideals over quotient hyperrings. Many interested examples are given. We recommend that this study can be done on non commutative hyperrings.

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