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Submanifolds of $N(\kappa)$-contact metric manifolds

H. G. Nagaraja and Dipansha Kumari

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Abstract. Anti-invariant submanifolds of $N(\kappa)$-contact metric manifolds are studied. We prove the condition for an anti-invariant submanifold of $N(\kappa)$-contact metric manifold to be Sasakian, flat or constant ξ-sectional and ϕ-sectional curvature.

1 Introduction

Chen and Ogiue [6], Kon [10] studied anti-invariant submanifolds of Kaehlerian manifolds and Yano [15], Yano and Kon [16] studied anti-invariant submanifolds of Sasakian manifolds. Later Ishihara [9], Hasan Sahid ([11] and [12]), Sular et al [13] made an extensive study of anti-invariant submanifolds of almost contact metric manifolds. In [4], it has been shown by Blair that if in a contact metric manifold M, the Ricci operator Q and a $(1,1)$ tensor ϕ commute then M is either Sasakian or flat or of constant ξ-sectional curvature and constant ϕ-sectional curvature. In this paper, we prove the conditions for commutativity of Q and ϕ in an anti-invariant contact submanifold of $N(\kappa)$-contact metric manifold.

2 Preliminaries

Let M be an m dimensional almost contact metric manifold with almost contact metric structure (ϕ,ξ,η,g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g on M satisfying [2]:

$$\begin{aligned}&\phi^2=-I+\eta\otimes\xi,\quad \eta(\xi)=1,\quad \phi\xi=0,\ \ \eta\cdot\phi=0,\\ &g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\quad g(X,\xi)=\eta(X),\end{aligned}\tag{2.1}$$

for any $X,Y\in\Gamma(TM)$.
Let Φ denote the 2-form in M given by $\Phi(X,Y)=g(X,\phi Y)$. The κ-nullity distribution on a contact metric manifold M for a real number κ is a distribution [14]

$$N(\kappa):p\longrightarrow N_p(\kappa)=\{Z\in T_pM:R(X,Y)Z=\kappa(g(Y,Z)X-g(X,Z)Y)\}\tag{2.2}$$

for any $X,Y\in T_pM$, where R denotes the Riemannian curvature tensor of M and T_pM denotes the tangent vector space of M at any point $p\in M$.
If the characteristic vector field ξ of a contact metric manifold belongs to the κ-nullity distribution, then

$$R(X,Y)\xi=\kappa(\eta(Y)X-\eta(X)Y).\tag{2.3}$$

A contact metric manifold with $\xi\in N(\kappa)$ is called a $N(\kappa)$-contact metric manifold. In an $N(\kappa)$ contact metric manifold the following relations hold:

$$(\nabla_X\phi)Y=g(X+hX,Y)\xi-\eta(Y)(X+hX),\tag{2.4}$$

$$h\xi=0,\tag{2.5}$$

$$h^2 = (\kappa - 1)\phi^2, \tag{2.6}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.7}$$

$$\begin{aligned}(\nabla_Y h)X - (\nabla_X h)Y &= 2(\kappa - 1)g(Y, \phi X)\xi + (1 - \kappa)(\eta(Y)\phi X - \eta(X)\phi Y) \\ &\quad + \eta(Y)\phi h X - \eta(X)\phi h Y,\end{aligned} \tag{2.8}$$

for all $X, Y \in \Gamma(TM)$, where h is a symmetric tensor and ∇ is the Levi-Civita connection on the manifold M.
In 1983, Chuman [7] defined D-conformal curvature tensor B as a tensor field on an n-dimensional $(n > 4)$ Riemannian manifold (M, g) by

$$\begin{aligned}B(X, Y)Z = R(X, Y)Z &+ \frac{1}{n-3}[S(X, Z)Y - S(Y, Z)X + g(X, Z)QY \\ &- g(Y, Z)QX + S(Y, Z)\eta(X)\xi - S(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)QX \\ &- \eta(X)\eta(Z)QY] - \frac{K-2}{n-3}[g(X, Z)Y - g(Y, Z)X] \\ &+ \frac{K}{n-3}[g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X],\end{aligned} \tag{2.9}$$

where $K = \frac{r+2(n-1)}{n-2}$, Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of M.
Let N be an $(2n+1)$- dimensional immersed submanifold of M. Then the Gauss and Weingarten formulas are respectively given by

$$\nabla_X Y = \tilde{\nabla}_X Y + \sigma(X, Y) \tag{2.10}$$

and

$$\nabla_X V = -A_V X + \tilde{\nabla}^{\perp}_X V \tag{2.11}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$, where σ, $\tilde{\nabla}$, $\tilde{\nabla}^{\perp}$ and A denote respectively the second fundamental form, Levi-civita connection, the normal connection and the shape operator on the submanifold N. The second fundamental form and shape operator are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V), \tag{2.12}$$

where g denotes the induced metric on N as well as the Riemannian metric g on M.
The covariant derivative of σ is given by

$$(\nabla_X \sigma)(Y, Z) = \tilde{\nabla}^{\perp}_X \sigma(Y, Z) - \sigma(\tilde{\nabla}_X Y, Z) - \sigma(Y, \tilde{\nabla}_X Z), \tag{2.13}$$

for any $X, Y, Z \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$.
Let $R_N(X, Y)Z$ and $R_M(X, Y)Z$ denote the Riemannian curvature tensors of the submanifold N and the ambient manifold M respectively. Then we have

$$R_M(X, Y)Z = R_N(X, Y)Z + (\nabla_X \sigma)(Y, Z) - (\nabla_Y \sigma)(X, Z) + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X \tag{2.14}$$

for $X, Y, Z \in \Gamma(TN)$ [5].
If $X, Y \in \Gamma(TN)$ and $U, V \in \Gamma(TN^{\perp})$ then we have

$$\begin{aligned}g(R_M(X, Y)U, V) &= g(R_N(X, Y)U, V) + g(\sigma(X, U), \sigma(Y, V)) \\ &\quad - g(\sigma(Y, U), \sigma(X, V))\end{aligned} \tag{2.15}$$

and

$$g(R_M(X,Y)U,V) = g(R_N^{\perp}(X,Y)U,V) - g([A_U,A_V]X,Y). \tag{2.16}$$

Here we recall few results which we will use in section 3.
Lemma A. [8] Let $M^{2m+1}(\phi,\xi,\eta,g)$ with $m \geq 2$ be a $N(\kappa)$-contact metric manifold. Then the following relation holds:

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z = {} & \{(1-\kappa)(\eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z)) \\ & + \eta(X)g(\phi hY,Z) - \eta(Y)g(\phi hX,Z)\}\xi \\ & - g(Y+hY,Z)(\phi X + \phi hX) \\ & + g(X+hX,Z)(\phi Y + \phi hY) \\ & - g(\phi Y + \phi hY,Z)(X+hX) \\ & + g(\phi X + \phi hX,Z)(Y+hY) \\ & - \eta(Z)\{(1-\kappa)(\eta(X)\phi Y - \eta(Y)\phi X) \\ & + \eta(X)\phi hY - \eta(Y)\phi hX\}. \end{aligned} \tag{2.17}$$

Theorem A. [3]Let M^{2n+1} be a contact metric manifold and suppose that $R(X,Y)\xi = 0$ for all vector fields X and Y. Then M^{2n+1} is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant curvature 4.
Theorem B.[4] Let M^3 be a contact metric manifold on which $Q\phi = \phi Q$. Then M^3 is either Sasakian, flat or of constant ξ-sectional curvature $r < 1$ and constant ϕ-sectional curvature $-r$.

3 Anti-invariant submanifold of $N(\kappa)$-contact metric manifold

Let N be an $(2n+1)$ dimensional anti-invariant submanifold of m dimensional $N(\kappa)$-contact metric manifold M. Then we have $\phi(TN) \subset \Gamma(TN^{\perp})$ for $X \in \Gamma(TN)$ [1].

First of all, we shall prove the following result.

Theorem 3.1. *There does not exist a totally umbilical anti-invariant submanifold of an $N(\kappa)$-contact metric manifold.*

Proof. The equation (2.7) in (2.10), for $Y = \xi$ gives

$$-\phi X - \phi hX = \tilde{\nabla}_X \xi + \sigma(X,\xi). \tag{3.1}$$

Equating the tangential and normal components in (3.1), we obtain

$$\tilde{\nabla}_X \xi = 0 \tag{3.2}$$

and

$$\sigma(X,\xi) = \phi X - \phi hX. \tag{3.3}$$

For an anti-invariant submanifold N of $N(\kappa)$-contact metric manifold M a straightforward computation gives the following:

$$A_{\phi Y}X = -g(X,Y)\xi + \eta(Y)X, \tag{3.4}$$

$$\tilde{\nabla}_X^{\perp}\phi Y = \phi(\nabla_X Y) - \eta(Y)hX, \tag{3.5}$$

$$\tilde{\nabla}_X^{\perp}\xi = -\phi X - \phi hX \tag{3.6}$$

and

$$A_\xi X = 0. \tag{3.7}$$

Next suppose that N is totally umbilical submanifold of M. Then we have

$$\sigma(X,Y) = g(X,Y)H. \tag{3.8}$$

From (3.3), we get

$$\sigma(\xi,\xi)=0. \tag{3.9}$$

Using (3.9) in (3.8), we get

$$H=0.$$

This contradiction proves the Theorem. □

Theorem 3.2. *An anti-invariant submanifold N of an $N(\kappa)$-contact metric manifold M is locally isometric to $E^{(n+1)}(0)\times S^n(4)$ for $n>1$.*

Proof. Setting $Z=\xi$ in (2.14), we have

$$R_M(X,Y)\xi = R_N(X,Y)\xi + (\nabla_X\sigma)(Y,\xi) - (\nabla_Y\sigma)(X,\xi) + A_{\sigma(X,\xi)}Y - A_{\sigma(Y,\xi)}X. \tag{3.10}$$

Using (2.11) , (2.13) and (3.2) in (3.10), we get

$$R_M(X,Y)\xi = R_N(X,Y)\xi - \sigma(\tilde{\nabla}_X Y,\xi) + \sigma(\tilde{\nabla}_Y X,\xi) - \nabla_Y\sigma(X,\xi) + \nabla_X\sigma(Y,\xi). \tag{3.11}$$

Employing (2.3), (2.10), (2.5), (2.6) and (3.2) in (3.11), we obtain

$$R_N(X,Y)\xi = 0. \tag{3.12}$$

By Theorem A and (3.11), Theorem is proved. □

We now prove the main result of the paper:

Theorem 3.3. *Let N be a 3-dimensional anti-invariant submanifold of $N(\kappa)$-contact metric manifold M. If the normal curvature tensor vanishes then N is either Sasakian, flat or of constant ξ-sectional curvature $r<1$ and constant ϕ-sectional curvature $-r$.*

Proof. Using (2.16), we write

$$g(R_M(X,Y)\phi Z,\phi W) = g(R_N^{\perp}(X,Y)\phi Z,\phi W) - g([A_{\phi Z},A_{\phi W}]X,Y), \tag{3.13}$$

for any $X,Y,Z,W\in\Gamma(TN)$.
Using (3.4) in (3.13), a simple computation gives

$$\begin{aligned} g(R_M(X,Y)\phi Z,\phi W) = g(R_N^{\perp}(X,Y)\phi Z,\phi W) - \{&g(X,W)g(\phi Y,\phi Z)\\ &- g(X,Z)g(\phi Y,\phi W) - (g(Y,Z)\eta(W)\\ &- g(Y,W)\eta(Z))\eta(X)\}. \end{aligned} \tag{3.14}$$

Since N is an anti-invariant submanifold of M, (2.17) reduces to

$$\begin{aligned} R_M(X,Y)\phi Z = \phi R_M(X,Y)Z - g(Y,Z)(\phi X + \phi hX)&\\ + g(X,Z)(\phi Y+\phi hY) - \eta(Z)\{(1-\kappa)(\eta(X)\phi Y&\\ - \eta(Y)\phi X) + \eta(X)\phi hY - \eta(Y)\phi hX\},& \end{aligned} \tag{3.15}$$

for any $X,Y,Z\in\Gamma(TN)$. From (3.14) and (3.15), we get

$$\begin{aligned} g(R_N^{\perp}(X,Y)\phi Z,\phi W) = g(\phi R_M(X,Y)Z,\phi W) - g(Y,Z)g(\phi X+\phi hX,\phi W)&\\ + g(X,Z)g(\phi Y+\phi hY,\phi W) - \eta(Z)\{(1-\kappa)(\eta(X)&\\ g(\phi Y,\phi W) - \eta(Y)g(\phi X,\phi W)) + \eta(X)g(\phi hY,\phi W)&\\ - \eta(Y)g(\phi hX,\phi W)\} + g(X,W)g(\phi Y,\phi Z)&\\ - g(X,Z)g(\phi Y,\phi W) - (g(Y,Z)\eta(W)&\\ - g(Y,W)\eta(Z))\eta(X).& \end{aligned} \tag{3.16}$$

Using (2.15), (3.16) reduces to

$$\begin{aligned} g(R_N^{\perp}(X,Y)\phi Z,\phi W) &= g(R_N(X,Y)Z,W) + g(\sigma(X,Z),\sigma(Y,W)) \\ &- g(\sigma(Y,Z),\sigma(X,W)) - \eta(R_M(X,Y)Z)\xi \\ &- g(Y,Z)g(\phi X + \phi hX,\phi W) \\ &+ g(X,Z)g(\phi Y + \phi hY,\phi W) - \eta(Z)\{(1-\kappa)(\eta(X) \\ & g(\phi Y,\phi W) - \eta(Y)g(\phi X,\phi W)) + \eta(X)g(\phi hY,\phi W) \\ &- \eta(Y)g(\phi hX,\phi W)\} + g(X,W)g(\phi Y,\phi Z) \\ &- g(X,Z)g(\phi Y,\phi W) - (g(Y,Z)\eta(W) \\ &- g(Y,W)\eta(Z))\eta(X). \end{aligned} \tag{3.17}$$

Using (2.1) and (2.2) in (3.17), we obtain

$$\begin{aligned} g(R_N^{\perp}(X,Y)\phi Z,\phi W) &= g(R_N(X,Y)Z,W) + g(\sigma(X,Z),\sigma(Y,W)) \\ &- g(\sigma(Y,Z),\sigma(X,W)) + \kappa\{-g(Y,Z)\eta(X)\eta(W) \\ &+ g(X,Z)\eta(Y)\eta(W) - g(X,W)\eta(Y)\eta(Z) \\ &+ g(Y,W)\eta(X)\eta(Z)\}. \end{aligned} \tag{3.18}$$

Now we calculate $g(\sigma(X,Z),\sigma(Y,W))$ and $g(\sigma(Y,Z),\sigma(X,W))$.
We can write

$$g(\sigma(X,Z),\sigma(Y,W)) = g(A_{\sigma(X,Z)}Y,W). \tag{3.19}$$

From (2.11), we have

$$\nabla_Y\sigma(X,Z) = -A_{\sigma(X,Z)}Y - \tilde{\nabla}_Y^{\perp}\sigma(X,Z). \tag{3.20}$$

Applying ϕ on both sides of (3.20), we get

$$\nabla_Y\phi\sigma(X,Z) - (\nabla_Y\phi)\sigma(X,Z) = -\phi A_{\sigma(X,Z)}Y + \phi\tilde{\nabla}_Y^{\perp}\sigma(X,Z). \tag{3.21}$$

Using (2.4) and (2.11) in (3.21), we obtain

$$-A_{\phi\sigma(X,Z)}Y + \tilde{\nabla}_Y^{\perp}\phi\sigma(X,Z) = -\phi A_{\sigma(X,Z)}Y + \phi\tilde{\nabla}_Y^{\perp}\sigma(X,Z). \tag{3.22}$$

Again applying ϕ on both sides of (3.22) and equating the tangential parts, we obtain

$$A_{\sigma(X,Z)}Y - \eta(A_{\sigma(X,Z)}Y)\xi = 0. \tag{3.23}$$

Using (2.12) in (3.23), we obtain

$$A_{\sigma(X,Z)}Y = 0. \tag{3.24}$$

In view of (3.19) and (3.24), (3.18) reduces to

$$\begin{aligned} -\phi R_N^{\perp}(X,Y)\phi Z &= R_N(X,Y)Z - \kappa\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \end{aligned} \tag{3.25}$$

From (3.25) it follows that the normal curvature of N vanishes if and only if

$$R_N(X,Y)Z = \kappa\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \tag{3.26}$$

Using (3.26), we obtain

$$S_N(X,Y) = \kappa\{g(X,Y) + (2n-1)\eta(X)\eta(Y)\} \tag{3.27}$$

and

$$QX = \kappa X + \kappa(2n-1)\eta(X)\xi. \tag{3.28}$$

From (3.28), it follows that

$$Q\phi = \phi Q. \tag{3.29}$$

From Theorem B and equation (3.29), the Theorem is proved. □

Theorem 3.4. *An anti-invariant submanifold of $N(-1)$-contact metric manifold with vanishing normal curvature tensor is D-conformally flat.*

Proof. From (2.9), for $n \geqslant 2$, we write

$$\begin{aligned} B(X,Y)Z = R_N(X,Y)Z &+ \frac{1}{2(n-1)}[S_N(X,Z)Y - S_N(Y,Z)X + g(X,Z)QY \\ &- g(Y,Z)QX + S_N(Y,Z)\eta(X)\xi - S_N(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)QX \\ &- \eta(X)\eta(Z)QY] - \frac{K-2}{n-3}[g(X,Z)Y - g(Y,Z)X] \\ &+ \frac{K}{n-3}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X], \end{aligned} \tag{3.30}$$

where $r = 4\kappa n$ and $K = \frac{4n(\kappa+1)}{2n-1}$.
Substituting (3.26) and (3.27) in (3.30), we obtain

$$\begin{aligned} B(X,Y)Z = &-\frac{\kappa+1}{(n-1)(2n-1)}[g(X,Z)Y - g(Y,Z)X] \\ &+ \frac{2n(\kappa+1)}{(n-1)(2n-1)}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X], \end{aligned} \tag{3.31}$$

completing the proof. □

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Author information

H. G. Nagaraja and Dipansha Kumari, Department of Mathematics,Bangalore University, Jnana Bharathi, Bengaluru-560056, INDIA.
E-mail: hgnraj@yahoo.com, dipanshakumari@gmail.com

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