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Characterization of $*$- semimultipliers in the prime rings

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Abstract Let R be an associative ring. A $*$- semimultiplier is an additive map $F : R \to R$ such that $F(xy) = F(x)g(y^*) = g(x^*)F(y)$ where g is some additive map and $F(g(x)) = g(F(x))$ for all $x \in R$. We make extensive use of functional identities defined in prime ring R of the forms $xE_1(y) + yE_2(x) = 0$ or $xE_1(y) + yE_2(x) \in Z(R) \subseteq C$ where E_1, E_2 are any arbitrary functions on the prime ring R and $Z(R)$, C are the center and the extended centroid of R respectively. We have proved that in a prime ring R under some additional conditions, a $*$ -semimultiplier $F : R \to R$ is a map given by $F(x) = \lambda x + \mu(x)$, where $\lambda \in C$ and $\mu : R \to C$. We have also shown that a prime ring admitting the $*$-semimultiplier satisfies S_4 , the standard identity of degree 4 under some suitable conditions. Further, some other important results are also incorporated.

1 Introduction

In the entire paper, R will denote an associative prime ring with an involution $*$ and $Z(R)$ its center. We first recall a prime ring R that is whenever $aRb = (0)$, then either $a = 0$ or $b = 0$. An additive map $* : R \to R$ is called an involution, if $(xy)^* = y^*x^*$ for all $x, y \in R$ and $(x^*)^* = x$ for all $x \in R$. A derivation is an additive map $d : R \to R$ satisfying $d(xy) = xd(y) + d(x)y$ for all $x,\ y \in R$. An additive map $G : R \to R$ satisfying $G(xy) = xd(y) + G(x)y$ for all $x,\ y \in R$ is called a generalized derivation associated with derivation d.

Bergen [8] first gave the definition of semiderivation that is an additive map $H : R \to R$ with an associated function $g : R \to R$ such that $H(xy) = g(x)H(y) + H(x)y = xH(y) + H(x)g(y)$ for all $x,\ y \in R$ and $H(g(x)) = g(H(x))$ for all $x \in R$. If g is an identity map then a semiderivation is just a derivation. A lot of work has been done in this direction. See ([13],[7], [8], [12]).

An additive map $T : R \to R$ is called a left (resp. right) centralizer map or left (resp. right) multiplier map if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$), holds for all $x, y \in R$. A centralizer is an additive map which is both a right as well as a left centralizer. An ample of work has been done on left (resp. right) centralizers in prime and semiprime rings during the last few decades. See ([18] ,[17] [19]).

In a parallel fashion, an additive map $T : R \to R$ is said to be a left $*$-centralizer (resp. reverse left $*$-centralizer) if $T(xy) = T(x)y^*$ (resp. $T(xy) = T(y)x^*$) holds for all $x, y \in R$ and the definition of a right $*$- centralizer (resp. reverse right $*$-centralizer) should be self explanatory. An additive mapping $T : R \to R$ is called a $*$-centralizer if T is both a left and right $*$-centralizer. An additive map $T : R \to R$ is said to be a Jordan left $*$-centralizer if $T(x^2) = T(x)x^*$ is satisfied for all $x \in R$. We emphasize that for some fixed element $a \in R$, the mapping $x \to ax^*$ is a reverse left $*$-centralizer and $x \to x^*a$ is a reverse right $*$-centralizer on R. Finally, α-centralizer also have been studied, where $\alpha : R \to R$ is an endomorphism of R. See [1].

Deriving motivation from centralizers like α-centralizers K.H. Kim [15], after a simple adaptation of definition of a semiderivation, gave the definition of a semimultiplier. An additive map $F : R \to R$ is called a semimultiplier with an associated additive surjective map $g : R \to R$ if $F(xy) = F(x)g(y) = g(x)F(y)$ for all $x,\ y \in R$ and $F(g(x)) = g(F(x))$ for all $x \in R$. Further, an additive map $F : R \to R$ is called a $*$-semimultiplier with associated surjective map $g : R \to R$ if $F(xy) = F(x)g(y^*) = g(x^*)F(y)$ for all $x,\ y \in R$ and $F(g(x)) = g(F(x))$ for all $x \in R$. K.H. Kim [16] gave the definition of $*$-semimultiplier and studied the commutativity of prime ring admitting a $*$-semimultiplier. We have introduced a generalized form of a $*$-semimultiplier by considering the associated map $g : R \to R$ to be an arbitrary function instead of surjective map. We now give an example for a $*$-semimultiplier given as below.

Example 1.1. Consider $\mathbb{Z}[i]$, the ring of Gaussian integer and $F : \mathbb{Z}[i] \to \mathbb{Z}[i]$ which is defined as follows,

$$F(a + ib) = \lambda(a + ib) \text{ where } \lambda \text{ is a fixed element of } \mathbb{Z} \text{ and } a, b \in \mathbb{Z}.$$

The associated surjective map $g : \mathbb{Z}[i] \to \mathbb{Z}[i]$ is defined as follows $g(a + ib) = a - ib$ and involution map $* : \mathbb{Z}[i] \to \mathbb{Z}[i]$ is given by $(a + ib)^* = a - ib$. Then F along with surjective map g and involution $*$ is a $*$-semimultiplier .

We have also obtained some results on $*$-semimultipliers by extending their codomains. In this case we have defined a $*$-semimultiplier as: a $*$-semimultiplier $F : R \to Q_r(R)$ (or $Q_{ml}(R)$) is a map associated with an additive map $g : R \to Q_r(R)$ (or $Q_{ml}(R)$) such that $g(R) = R$ and is given as follows $F(xy) = F(x)g(y^*) = g(x^*)F(y)$ for all $x, y \in R$ and $F(g(x)) = g(F(x))$ for all $x \in R$.

If $S \subseteq R$, then an addditive map $F : R \to R$ is called a centralizing map on S if $[F(x), x] \in Z(R)$ for all $x \in S$ and a commuting on S if $[F(x), x] = 0$ for all $x \in S$. We take $C(r) = \{x \in R|\ xr = rx\}$ and call it as the centralizer of the element r. It is well known that $Z(R) = \cap C(r)$.

We write $Q_{mr}(R)$ (resp. $Q_{ml}(R)$), $Q_r(R)$ and $Q_{ms}(R)$ for maximal right ring of quotients of R (resp. maximal left ring of quotients of R), two sided ring of quotients of R and for symmetric right ring of quotients of R respectively. By [4] it is known that $R \subseteq Q_{ms}(R) \subseteq Q_r(R) \subseteq Q_{mr}(R)$ where all the overrings $Q_{ms}(R), Q_r(R)$ and $Q_{mr}(R)$ are prime rings with the same center C. Prompted by primeness of R, C is a field called the extended centroid of R. For further references browse [4]. In view of [[4], Proposition 2.2.1], we state some properties of $Q_r(R)$ as follows:

(i) $R \subseteq Q_r(R)$;

(ii) For every $q \in Q_r(R)$ there exists a nonzero ideal I of R such that $qI \subseteq R$;

(iii) If $q \in Q_r(R)$ and I is a nonzero ideal of R such that $qI = 0$, then $q = 0$;

(iv) If I is an ideal of R and $f : I \longrightarrow R$ is a right R-module map, then there exists $q \in Q_r(R)$ such that $f(x) = qx$ for all $x \in I$.

These are the characterizing properties of $Q_r(R)$.

For $p, q \in R$, let $[p, q] = pq - qp$ be the commutator . When R satisfies S_4 it means, R satisfies the standard polynomial identity of degree four. Further references can be taken from [14]. For $t \in R$, we define deg(t) to be the minimal algebraic degree over C if t is algebraic over C and deg$(t) = \infty$, otherwise. For a subset T of R, we define deg$(T) = \sup\{\text{deg}(t) \mid t \in T\}$. We refer the reader to [11] for details.

We will make an extensive use of functional identities (F.I.) of special types in an attempt to completely characterize a $*$-semimultiplier in the setting of prime ring with involution $*$. The F.I. in use are of the form $xE_1(y) + yE_2(x) = 0$ or $xE_1(y) + yE_2(x) \in Z(R) \subseteq C$. For further references see [11].

Precisely, we have shown that a $*$-semimultiplier under some suitable conditions will be of the form $F(x) = \lambda x + \mu(x)$ where $\lambda \in C$ and $\mu : R \to C$ is an additive map. We have also paved a way for a prime ring admitting a $*$-semimultiplier, to satisfy S_4, the standard polynomial identity of degree four.

We have obtained important results pertaining to a $*$-semimultipliers, taking motivation from the results proved in the context of derivations. For instance, what will happen if range of a $*$-semimultiplier is in the center of ring $Z(R)$ after motivation from [[9], in Lemma 4.2]. Some results which were studied in [[3], Theorem 2.2] in the sense of generalized derivations have been also studied in the scenario of $*$-semimultipliers in Theorem 3.5. In the last section we have worked on the situation when two or more than two $*$-semimultipliers are connected via the special type of identities of M. Bresar in [[9] Lemma 2.2-2.3 and Theorem 2.1] and what happens if square of a $*$-semimultiplier is reduced to zero? From the condition used in [[2], Theorem 2.1 and 2.2] we characterized a $*$-semimultiplier which satisfy the F.I. from [6],[5] and [11]. We will make frequent use of following identities associated with commutators and anti-commutators without mentioning specifically each time .
That is,

$$xo(yz) = (xoy)z - y[x,z],$$

$$xo(yz) = y(xoz) + [x,y]z,$$

$$[xy,z] = [x,z]y + x[y,z],$$

$$[x,yz] = [x,y]z + y[x,z].$$

Lemma 1.2. *[[10], lemma 2.1] Suppose that non zero elements $a_i, b_i \in Q_r(R)$, $i = 1,2,...m$, satisfy $\sum_{i=1}^{m} a_i x b_i = 0$, for all $x \in R$, then a_i as well as b_i's are $C-$dependent.*

Lemma 1.3. *In a prime ring R if a and ac are in center of R and if c is not in center then $a = 0$.*

Lemma 1.4. *Let R be a prime ring and $a,b \in R$ such that $axb = bxa$ for all $x \in R$. If $a \neq 0$ then $b = \lambda a$ where $\lambda \in C$, the extended centroid of R.*

Theorem 1.5. *[[10], Theorem 4.18] Let I be an ideal of a prime ring R which is non- commutative. Let $f_1, f_2, f_3, f_4 : I \to R$ be the additive maps and set $\pi(x,y) = f_1(x)y + xf_2(y) + f_3(y)x + yf_4(x)$. If $\pi(x,y) \in Z(R)$ for all $x,y \in R$ and characteristic of $R \neq 2,3$, then R satisfies S_4, the standard polynomial identity of degree four.*

Theorem 1.6. *[[9], Theorem 3.2] Let R be a non-commutative prime ring and if an additive map F of R is commuting map, then there exists $\lambda \in C$ and an additive map $\xi : R \to C$ such that $F(x) = \lambda x + \xi(x)$, for all $x \in R$.*

2 Main Results

Theorem 2.1. *Let R be a non-commutative prime ring with involution $*$ and char$(R) \neq 2,3$ and F be a $*$-semimultiplier such that $F : R \to R$ and $g : R \to R$ be an associated onto map. If $F(x^2) \in Z(R)$ for all $x \in R$, then R satisfies S_4 the standard polynomial identity of degree four.*

Proof. Since F is additive map and $F(x^2) \in Z(R)$. On linearizing F, we get $F(xy + yx) \in Z(R)$, for all $x,y \in R$. Thus

$$F(x)g(y^*) + g(y^*)F(x) \in Z(R), \text{ for all } x,\ y \in R.$$

Let $g(y^*) = w$. As g is onto, then we have

$$F(x)w + wF(x) \in Z(R), \text{ for all } x,\ w \in R. \tag{2.1}$$

Interchange x and w in relation (2.1), we have

$$F(w)x + xF(w) \in Z(R). \tag{2.2}$$

From (2.1) and (2.2), we have

$$F(x)w + wF(x) + F(w)x + xF(w) \in Z(R) \text{ for all } x,\ y \in R. \tag{2.3}$$

$$\text{Let} \quad \mu(x,w) = F(x)w + wF(x) + F(w)x + xF(w).$$

By Theorem 1.5 we have, R satisfies S_4. □

Theorem 2.2. *Let R be a non-commutative prime ring with involution $*$ and F be a $*$-semimultiplier such that $F : R \to Q_r(R)$ and $g : R \to Q_r(R)$ be an associated additive map such that $g(R) = R$. If $F(x^2) \in Z(R)$ for all $x \in R$, then $F = 0$.*

Proof. Since $F(x^2) \in Z(R) \subseteq Q_r(R)$. Now for given $F(x)$ and the relation (2.1), let $\xi(w) = F(x)w + wF(x) \in Z(R)$.

$$\xi(wr) - \xi(w)r = F(x)wr + wrF(x) - F(x)wr - wF(x)r, \text{ for every } w,\ r \in R.$$

Since we see that $[\xi(wr) - \xi(w)r, r] = 0$, for every $w, r \in R$. This implies that,

$$\xi(wr) - \xi(w)r = w[r, F(x)] \in C(r).$$

That is,

$$[w[r, F(x)], r] = 0.$$

This implies that,

$$rw[r, F(x)] = w[r, F(x)]r.$$

By Lemma 1.2, either r is $C-$dependent with 1 which amounts to say $R \subseteq Z(R)$ which is contradictory to our assumption. Thus we have,

$$[r, F(x)] = 0, \text{ that is } F(x) \in C \text{ for given } x \in R.$$

We can repeat this process for each $x \in R$, to conclude that $F(x) \in C$, holds for all $x \in R$. Replace x by xt where $t \in R$ in the relation $F(x) \in C$ we have,

$$F(xt) = F(x)g(t^*) \in C \text{ for all } x,\ t \in R.$$

Thus,

$$F(x)R \subseteq C, \quad \text{since} \quad g(R) = R.$$

By Lemma 1.3 and since R is non-commutative, $F(x) = 0$ for all $x \in R$. Hence $F = 0$. □

Theorem 2.3. *Let R be a non-commutative prime ring with involution $*$ and F be a $*$-semimultiplier and $g : R \to R$ be an associated additive surjective map. If $F([x, y]) = \pm yx$, then there exists $\lambda \in C$ and $\mu : R \to C$ such that $F(x) = \lambda x + \mu(x)$.*

Proof. From assumption,

$$F([x, y]) = \pm yx \text{ for all } x,\ y \in R.$$

Here use yx instead of y where $x \in R$, we get

$$F([x, y]x) = \pm yx^2 \text{ for all } x,\ y \in R.$$

By definition of $*$-semimultiplier,

$$F([x, y])g(x^*) = \pm yx^2 \text{ for all } x,\ y \in R.$$

By hypothesis, we have

$$\pm yx(g(x^*) - x) = 0 \text{ for all } x,\ y \in R.$$

This implies that

$$yx(g(x^*) - x) = 0 \text{ for all } x,\ y \in R.$$

Replace y by yc where $c \in R$ in above relation, we get

$$ycx(g(x^*) - x) = 0 \text{ for all } c,\ x,\ y \in R.$$

That is,

$$yR\{x(g(x^*) - x)\} = (0) \text{ for all } x,\ y \in R.$$

By primeness of R, we get

$$x(g(x^*) - x) = 0 \text{ for all } x \in R. \tag{2.4}$$

Since g is an additive map and $*$ is an involution, therefore on linearizing relation (2.4) we have

$$x(g(z^*) - z) + z(g(x^*) - x) = 0 \text{ for all } x,\ z \in R. \tag{2.5}$$

Since relation (2.5) is a functional identity. Rewriting relation (2.5) as

$$xE(z) + zH(x) = 0, \text{ for all } x,\ z \in R,$$

where $E(z) = g(z^*) - z$ and $H(x) = g(x^*) - x$. Using [[5], Theorem 2.5], we have $E(z) = 0$, for all $z \in R$. Thus in all, $g(x^*) = x$, for all $x \in R$. So we get the following result,

$$F(xy) = F(x)y = xF(y), \text{ for all } x,\ y \in R. \tag{2.6}$$

Put $x = y$, in (2.6), we get

$$[F(x), x] = 0, \text{ for all } x \in R.$$

That is, F is commuting.
By Theorem 1.6, there exists $\lambda \in C$ and $\mu : R \to C$ such that

$$F(x) = \lambda x + \mu(x) \text{ for all } x \in R.$$

□

Theorem 2.4. *Let R be a non-commutative prime ring with involution $*$ and F be a $*$-semimultiplier associated with an additive surjective map $g : R \to R$. If $F([x, y]) = \pm\alpha yx$, where $0 \neq \alpha$ a fixed element of R, then there exists $\lambda \in C$ and $\mu : R \to C$ such that F is given by $F(x) = \lambda x + \mu(x)$.*

Proof. From assumption,

$$F([x, y]) = \pm\alpha yx \text{ for all } x,\ y \in R.$$

Here use yx instead of y where $x \in R$, we get

$$F([x, y]x) = \pm\alpha yx^2 \text{ for all } x,\ y \in R.$$

By the definition of $*$-semimultiplier,

$$F([x, y])g(x^*) = \pm\alpha yx^2 \text{ for all } x,\ y \in R.$$

By assumption, we have

$$\pm\alpha yx(g(x^*) - x) = 0 \text{ for all } x,\ y \in R.$$

This implies that,

$$\alpha yx(g(x^*) - x) = 0 \text{ for all } x,\ y \in R.$$

Since $\alpha \neq 0$, by primeness of R, we get

$$x(g(x^*) - x) = 0 \text{ for all } x \in R,$$

which is relation (2.4). Thus we get the desired result. □

Theorem 2.5. *Let R be a non-commutative prime ring with involution $*$ and F be a $*$-semimultip lier associated with an additive surjective map $g : R \to R$. If $F([x, y]) = \pm xy$, then there exists $\lambda \in C$ and $\mu : R \to C$ such that F is given by $F(x) = \lambda x + \mu(x)$.*

Proof. From assumption,

$$F([x, y]) = \pm xy \text{ for all } x,\ y \in R.$$

Here use xy instead of x where $y \in R$, we get

$$F([xy, y]) = \pm xy^2 \text{ for all } x,\ y \in R.$$

This implies that,

$$F([x, y]y) = \pm xy^2.$$

By definition of $*$-semimultiplier,

$$F([x, y])g(y^*) = \pm xy^2.$$

$$\pm xy(g(y^*) - y) = 0 \text{ for all } x,\ y \in R.$$

This implies that,

$$xy(g(y^*) - y) = 0 \text{ for all } x,\ y \in R.$$

Replace x by xq where $q \in R$ in above relation, we get

$$xqy(g(y^*) - y) = 0, \text{ for all } q,\ x,\ y \in R.$$

This implies that,

$$xR\{y(g(y^*) - y)\} = (0), \text{ for all } x,\ y \in R$$

By primeness of R, we get

$$y(g(y^*) - y) = 0 \text{ for all } y \in R \text{ , which is relation (2.4).} \tag{2.7}$$

Thus above relation initiates the desired result. □

Theorem 2.6. *Let R be a non-commutative prime ring with involution $*$ and F be a $*$-semimultip lier associated with an additive surjective map $g : R \to R$. If $F([x, y]) = \pm\alpha xy$, where $0 \neq \alpha$ a fixed element of R , then there exists $\lambda \in C$ and $\mu : R \to C$ such that F is given by $F(x) = \lambda x + \mu(x)$.*

Proof. From assumption,

$$F([x, y]) = \pm\alpha xy, \text{ for all } x,\ y \in R.$$

Here use xy instead of x where $y \in R$, we get

$$F([xy, y]) = \pm\alpha xy^2, \text{ for all } x,\ y \in R.$$

From above we get the following relation,

$$F([x, y]y) = \pm\alpha xy^2, \text{ for all } x,\ y \in R.$$

By the definition of $*$-semimultiplier ,

$$F([x, y])g(y^*) = \pm\alpha xy^2, \text{ for all } x,\ y \in R.$$

This implies that,

$$\pm\alpha xyg(y^*) = \pm\alpha xy^2, \text{ for all } x,\ y \in R.$$

That is,

$$\pm\alpha xy(g(y^*) - y) = 0, \text{ for all } x,\ y \in R.$$

Above relation can be written as,

$$\alpha xy(g(y^*) - y) = 0, \text{ for all } x,\ y \in R.$$

This implies that,

$$\alpha R\{y(g(y^*) - y)\} = (0), \text{ for all } y \in R.$$

By primeness of R and since $\alpha \neq 0$, we get,

$$y(g(y^*) - y) = 0 \text{ for all } y \in R, \text{ which is relation (2.4)}. \tag{2.8}$$

Thus we get the desired result. □

Theorem 2.7. *Let R be a non-commutative prime ring with involution $*$ and F be a $*$-semimultiplier associated with an additive surjective map $g : R \to R$. If $F([x, y]) = \pm[x, y]$ then there exists $\lambda \in C$ and $\mu : R \to C$ such that F is given by $F(x) = \lambda x + \mu(x)$.*

Proof. By assumption,

$$F([x, y]) = \pm[x, y], \text{ for all } x,\ y \in R.$$

Use yx instead of y where $x \in R$ in above relation, we get

$$F([x, yx]) = \pm[x, yx] \text{ for all } x,\ y \in R.$$

This implies that,

$$F([x, y]x) = \pm[x, y]x \text{ for all } x,\ y \in R.$$

Utilize the definition of $*$- semimultiplier, we get the following relation,

$$F([x, y])g(x^*) = \pm[x, y]x \text{ for all } x,\ y \in R.$$

From assumption,

$$\pm[x, y]g(x^*) = \pm[x, y]x \text{ for all } x,\ y \in R.$$

We finally gain the following relation,

$$[x, y](g(x^*) - x) = 0 \text{ for all } x,\ y \in R. \tag{2.9}$$

Use yz instead of y where $y \in R$ in (2.9) to get,

$$[x, yz](g(x^*) - x) = 0 \text{ for all } x,\ y,\ z \in R.$$

This implies that,

$$([x, y]z + y[x, z])(g(x^*) - x) = 0 \text{ for all } x,\ y,\ z \in R.$$

This results in following relation,

$$[x, y]z(g(x^*) - x) = 0 \text{ for all } x,\ y,\ z \in R. \tag{2.10}$$

From above relation (2.10) we have following observation, if $A = \{x \in R |\ [x, y] = 0 \text{ for all } y \in R\}$ and $B = \{x \in R |\ g(x^*) = x\}$. Then A and B are additive subgroups of R whose union is R, but R being an additive group it cannot be the union of its two proper subgroups. Thus, either $A = R$ or $B = R$. Let $A = R$ then R is commutative, which leads to a contradiction. Therefore we assume $B = R$. Hence $g(x^*) = x$ for all $x \in R$. From the definition of a $*$-semimultiplier, F is a two sided centralizer that is ,

$$F(xy) = F(x)y = xF(y). \tag{2.11}$$

Put $x = y$ in (2.11) we conclude that F is commuting that is $[F(x), x] = 0$ for all $x \in R$. By Theorem 1.6 there exists $\lambda \in C$ and $\mu : R \to C$ such that,

$$F(x) = \lambda x + \mu(x) \text{ for all } x \in R.$$

□

Theorem 2.8. *Let R be a non-commutative prime ring with involution $*$ and F be a $*$-semimultip lier associated with an additive surjective map $g : R \to R$. If $F([x, y]) = \pm\alpha[x, y]$, where $0 \neq \alpha$ a fixed central element of R, then there exists $\lambda \in C$ and $\mu : R \to C$ such that F is given by $F(x) = \lambda x + \mu(x)$.*

Proof. By assumption,

$$F([x, y]) = \pm\alpha[x, y], \text{ for all } x, \ y \in R.$$

Use yx instead of y where $x \in R$ in above relation, we get

$$F([x, yx]) = \pm\alpha[x, yx], \text{ for all } x, \ y \in R.$$

This implies that,

$$F([x, y]x) = \pm\alpha[x, y]x, \text{ for all } x, \ y \in R.$$

From the definition of $*$-semimultiplier, we have

$$F([x, y])g(x^*) = \pm\alpha[x, y]x, \text{ for all } x, \ y \in R.$$

From given assumption,

$$\pm\alpha[x, y]g(x^*) = \pm\alpha[x, y]x, \text{ for all } x, \ y \in R.$$

Finally, we establish the following relation,

$$\pm\alpha[x, y](g(x^*) - x) = 0, \text{ for all } x, \ y \in R.$$

Above relation can be rewritten as following relation,

$$\alpha[x, y](g(x^*) - x) = 0, \text{ for all } x, \ y \in R. \tag{2.12}$$

Use yz instead of y where $z \in R$ in (2.12) to get,

$$\alpha[x, yz](g(x^*) - x) = 0, \text{ for all } x, \ y, \ z \in R.$$

This implies that,

$$\alpha([x, y]z + y[x, z])(g(x^*) - x) = 0, \text{ for all } x, \ y, \ z \in R.$$

In above relation since $0 \neq \alpha \in Z(R)$, therefore above relation together with relation (2.12), gives the following result,

$$\alpha[x, y]z(g(x^*) - x) = 0 \text{ for all } x, \ y, \ z \in R. \tag{2.13}$$

From above relation (2.13), we have the following observation if $A = \{x \in R| \ \alpha[x, y] = 0\}$ and $B = \{x \in R| \ g(x^*) = x\}$. Then A and B are additive subgroups of R whose union is R, but R being an additive group it cannot be the union of its two proper subgroups. Thus, either $A = R$ or $B = R$. Let $A = R$. This implies that $\alpha[x, y] = 0$, for all $x, y \in R$. Now replace y by zy where $z \in R$, then $\alpha z[x, y] = 0$. Since R is not commutative, there exists $x_0, y_0 \in R$ such that $[x_0, y_0] \neq 0$. As R is prime so we conclude that $\alpha = 0$ which leads to a contradiction. Therefore we now assume that $B = R$. Hence $g(x^*) = x$ for all $x \in R$ which gives relation (2.11) and hence the required result follows. □

Theorem 2.9. *Let R be a non-commutative prime ring with involution $*$. Let F be a $*$-semimultip lier on R associated with a function $g : R \to R$ such that $F([x, y]) = \pm(xoy)$, then F is a map given by $F(x) = \lambda x + \mu(x)$, where $\lambda \in C$ and $\mu : R \to C$.*

Proof. By assumption,

$$F([x, y]) = \pm(xoy) \text{ for all } x, \ y \in R$$

Use yx instead of y where $x \in R$ in above relation, we get

$$F([x, yx]) = \pm(xo(yx)) \text{ for all } x, \ y \in R.$$

From above we get the following relation,

$$F([x,y]x) = \pm(xoy)x \text{ for all } x,\ y \in R.$$

From the definition of $*$-semimultiplier, we have

$$F([x,y])g(x^*) = \pm(xoy)x \text{ for all } x,\ y \in R.$$

By given hypothesis,

$$\pm(xoy)g(x^*) = \pm(xoy)x \text{ for all } x,\ y \in R.$$

This implies that,

$$\pm(xoy)(g(x^*) - x) = 0 \text{ for all } x,\ y \in R.$$

Above relation can be written as,

$$(xoy)(g(x^*) - x) = 0 \text{ for all } x,\ y \in R. \tag{2.14}$$

Use yz instead of y where $z \in R$ in (2.14) to get

$$(xo(yz))(g(x^*) - x) = 0 \text{ for all } x,\ y,\ z \in R.$$

This implies that

$$(y(xoz) + [x,y]z)(g(x^*) - x) = 0 \text{ for all } x,\ y,\ z \in R.$$

This implies that

$$[x,y]z(g(x^*) - x) = 0 \text{ for all } x,\ y,\ z \in R \tag{2.15}$$

which leads to (2.10) and hence we get the required result. □

Theorem 2.10. *Let R be a non-commutative prime ring with involution $*$ and F be a $*$-semimultiplier on R associated with a function $g : R \to R$. If $F([x,y]) = \pm\alpha(xoy)$, where $\alpha \neq 0$ fixed central element of R, then there exists $\lambda \in C$ and $\mu : R \to C$ such that F is a map given by $F(x) = \lambda x + \mu(x)$.*

Proof. By assumption,

$$F([x,y]) = \pm\alpha(xoy), \text{ for all } x,\ y \in R.$$

Use yx instead of y where $x \in R$ in above relation we get,

$$F([x,yx]) = \pm\alpha(xo(yx)) \text{ for all } x,\ y \in R.$$

This implies that,

$$F([x,y]x) = \pm\alpha(xoy)x \text{ for all } x,\ y \in R.$$

From the definition of $*$-semimultiplier, we have

$$F([x,y])g(x^*) = \pm\alpha(xoy)x \text{ for all } x,\ y \in R.$$

This implies that,

$$\pm\alpha(xoy)g(x^*) = \pm\alpha(xoy)x \text{ for all } x,\ y \in R.$$

Thus we arrive at following relation,

$$\pm\alpha(xoy)(g(x^*) - x) = 0 \text{ for all } x,\ y \in R.$$

Above relation can be rewritten as,

$$\alpha(xoy)(g(x^*) - x) = 0 \text{ for all } x,\ y \in R. \tag{2.16}$$

Use yz instead of y where $z \in R$ in above relation, we get

$$\alpha(xo(yz))(g(x^*) - x) = 0 \text{ for all } x,\ y,\ z \in R.$$

This implies that,

$$\alpha(y(xoz)+[x,y]z)(g(x^*)-x)=0 \text{ for all } x,\ y,\ z\in R.$$

In above relation since $0\neq\alpha\in Z(R)$, therefore above relation together with relation (2.16), gives the following result,

$$\alpha[x,y]z(g(x^*)-x)=0 \text{ for all } x,\ y,\ z\in R.$$

From above relation we are in the receipt of (2.13) which gives the required result. □

Theorem 2.11. *Let R be a non-commutative prime ring with involution $*$ and I be a non-zero ideal and F be a $*$-semimultiplier associated with a function $g:R\to R$. If $F(x)F(y)=\pm xy$ for all $x,y\in I$ then there exists $\lambda\in C$ and $\mu:I\to C$ such that $F(x)=\lambda x+\mu(x)$ for all $x\in I$.*

Proof. By assumption,

$$F(x)F(y)=\pm xy \text{ for all } x,\ y\in I.$$

Use yz instead of y where $z\in R$,

$$F(x)F(yz)=\pm x(yz) \text{ for all } x,\ y,\ z\in I.$$

This implies that,

$$F(x)F(y)g(z^*)=\pm(xy)z \text{ for all } x,\ y,\ z\in I.$$

From the assumption,

$$\pm xyg(z^*)=\pm(xy)z \text{ for all } x,\ y,\ z\in I.$$

Above relation gives the follwing result,

$$\pm xy(g(z^*)-z)=0 \text{ for all } x,\ y,\ z\in I.$$

Use yq instead of q where $q\in R$ in above relation, we get

$$\pm xyq(g(z^*)-z)=0 \text{ for all } x,\ y,\ z,\ q\in I.$$

The above relation can be rewritten as,

$$xyR(g(z^*)-z)=0 \text{ for all } x,\ y,\ z,\ q\in I.$$

By primeness of R,

$$xy=0 \text{ for all } x,\ y\in I.$$

This implies that,

$$x\in I\cap l(I)=\{0\}.$$

A contradiction since $I\neq\{0\}$. Hence we find that,

$$g(z^*)=z, \text{ for all } z\in I.$$

Utilizing the definition of $*$-semimultiplier and above relation, we have

$$F(xy)=F(x)g(y^*)=g(x^*)F(y), \text{ for all } x,\ y\in I.$$

This implies that,

$$F(xy)=F(x)y=xF(y), \text{ for all } x,\ y\in I.$$

Put $x=y$ in above relation we get,

$$[F(x),x]=0, \text{ for all } x\in I.$$

By [[10], Theorem 4.2] , there exists $\lambda\in C$ and $\mu:I\to C$ such that,

$$F(x)=\lambda x+\mu(x), \text{ for all } x\in I.$$

□

Theorem 2.12. *Let I be a non-zero right ideal of a non-commutative prime ring R with involution $*$. Further let $F : R \to R$ be a $*$-semimultiplier and g be an associated surjective map such that $F(x) \in Z(R)$ for all $x \in I$. Then $F = 0$ or g vanishes on I.*

Proof. Let $v \in I$ and $u \in R$. Then by assumption we have

$$F(v), F(vu) \in Z(R), vu \in I \Rightarrow [F(vu), u] = 0, \text{ for all } v \in I, \text{ for all } u \in R.$$

Utilizing the definition of $*$-semimultiplier, we have

$$[F(v)g(u^*), u] = 0, \text{ for all } v \in I, \text{ for all } u \in R,$$

which implies that

$$[F(v), u]g(u^*) + F(v)[g(u^*), u] = 0, \text{ for all } v \in I, \text{ for all } u \in R. \tag{2.17}$$

Since $F : I \to Z(R)$, therefore from (2.17), we have

$$F(v)[g(u^*), u] = 0, \text{ for all } v \in I, \text{ for all } u \in R. \tag{2.18}$$

From (2.18), we have

$$F(v)[w, u] = 0, \text{ for all } v \in I, \text{ for all } u, w \in R.$$

$$F(v)p[w, u] = 0 \text{ for all } v \in I, \text{ for all } u, w, p \in R.$$

From the primeness and non-commutativity of R, we conclude that

$$F(v) = 0, \text{ for all } v \in I.$$

Replace v by ve, where $e \in R$ in above relation, we have

$$F(ve) = 0 \text{ for all } v \in I, \text{ for all } e \in R.$$

That is,

$$g(v^*)F(e) = 0 \text{ for all } v \in I, \text{ for all } e \in R.$$

We now replace e by et where $t \in R$ to get the following relation,

$$g(v^*)g(e^*)F(t) = 0 \text{ for all } v \in I, \text{ for all } e \in R.$$

Since g is surjective,

$$g(v^*)RF(t) = (0), \text{ for all } v \in I, \text{ for all } t \in R.$$

By primeness of R, we have either

$$F(t) = 0 \text{ for all } t \in R.$$

Thence we conclude,

$$F = 0.$$

or,

$$g(v^*) = 0, \text{ for all } v \in I.$$

□

Theorem 2.13. *Let R be a non-commutative prime ring with involution $*$ and $deg(R) > 2$. Further let I be a non-zero ideal, where $F : R \to Q_{ml}(R)$ be a $*$-semimultiplier associated with an additive surjective map $g : R \to Q_{ml}(R)$. If $F(x)F(y) = \pm yx\alpha$ for all $x, y \in I$ and α be a fixed element of R, then there exists $\lambda \in C$ and $\mu : I \to C$ such that $F(x) = \lambda x + \mu(x)$, for all $x \in I$ or $F = 0$.*

Proof. By assumption,

$$F(x)F(y) = \pm yx\alpha, \text{ for all } x,\ y \in I.$$

Use yx instead of x where $y \in R$ in above relation,

$$F(yx)F(y) = \pm y(yx)\alpha, \text{ for all } x,\ y \in I.$$

Utilize the definition of a $*$-semimultiplier in above relation we obtain the following results,

$$g(y^*)F(x)F(y) = \pm y(yx)\alpha, \text{ for all } x,\ y \in I.$$

From given assumption,

$$\pm g(y^*)(yx)\alpha = \pm y(yx)\alpha, \text{ for all } x,\ y \in I.$$

This implies that,

$$(g(y^*) - y)yx\alpha = 0, \text{ for all } x,\ y \in I.$$

From above we get the following relation,

$$(g(y^*) - y)yRx\alpha = (0), \text{ for all } x,\ y \in I.$$

Thus by primeness of R, either

$$x\alpha = 0, \text{ for all } x \in I,$$

or

$$(g(y^*) - y)y = 0, \text{ for all } y \in I.$$

By implementing primeness of R in $x\alpha = 0$, we find that $\alpha = 0$. Thus,

$$F(x)F(y) = 0, \text{ for all } x,\ y \in I,$$

which has simple consequence as,

$$F(x)oF(y) = 0, \text{ for all } x,\ y \in I.$$

Use yt in place of y where $t \in R$ in above relation,

$$F(x)oF(yt) = 0, \text{ for all } x,\ y \in I \text{ and for all } t \in R.$$

This implies that,

$$F(x)o(F(y)g(t^*)) = 0, \text{ for all } x,\ y \in I \text{ and for all } t \in R.$$

Making use of the definition of a $*$-semimultiplier, we obtain the following relation,

$$(F(x)oF(y))g(t^*) - F(y)[F(x), g(t^*)] = 0, \quad \text{for all } x,\ y \in I \text{ and for all } t \in R.$$

$$F(y)[F(x), g(t^*)] = 0, \text{ for all } x,\ y \in I \text{ and for all } t \in R.$$

From above relation, we have

$$F(y)[F(x), w] = 0, \text{ for all } x,\ y \in I \text{ and for all } w \in R.$$

Use wp instead of w where $p \in R$ in above relation, we obtain that

$$F(y)w[F(x), p] = 0, \text{ for all } x,\ y \in I \text{ and for all } w,\ p \in R$$

Since R is prime either $F(y) = 0$, for all $y \in I$ or $F(x) \in Z(R)$, for all $x \in I$.
In the both case $F = 0$, following argument from Theorem 2.12.
Further if we have,

$$(g(y^*) - y)y = 0, \text{ for all } y \in I.$$

Using $x + z$ in place of y where $x, z \in I$ we get a functional identity on the ideal I of ring R of the form,

$$E_1(z)x + E_2(x)z = 0, \text{ for all } x,\ z \in I.$$

where

$$E_1(z) = g(z^*) - z, \text{ for all } z \in I.$$

and

$$E_2(x) = g(x^*) - x \text{ for all } x \in I.$$

Since deg(R) > 2, we have from [[6] Theorem 2.2]

$$g(z^*) = z \text{ for all } z \in I.$$

Hence from the definition of $*$-semimultiplier, we have

$$F(xy) = F(x)g(y^*) = g(x^*)F(y) \text{ for all } x, \ y \in I.$$

We obtain that,

$$F(xy) = F(x)y = xF(y) \text{ for all } x, \ y \in I.$$

Thus, we get the following,

$$[F(x), x] = 0 \text{ for all } x \in I.$$

Above relation prompts the desired result following [[10],Theorem 4.2]. □

3 $*$-SEMIMULTIPLIERS CONNECTED VIA SOME SPECIAL TYPE OF IDENTITIES

Theorem 3.1. *Let R be a non-commutative prime ring with involution $*$. Let F and G be two $*$-semimultipliers on R and f and g be the associated surjective maps respectively. If $F(x)G(y) = G(x)F(y)$, for all $x, y \in R$ and $F \neq 0$, then there exists $\lambda \in C$ such that $G(x) = \lambda F(x)$ for all $x \in R$.*

Proof. We are given that ,

$$F(x)G(y) = G(x)F(y), \text{ for all } x, \ y \in R. \tag{3.1}$$

Use yz instead of y where $z \in R$ in (3.1), we have

$$F(x)G(yz) = G(x)F(yz), \text{ for all } x, \ y, \ z \in R.$$

Utilizing the definition of $*$-semimultipliers, we have

$$F(x)g(y^*)G(z) = G(x)f(y^*)F(z), \text{ for all } x, \ y, \ z \in R.$$

Since g and f are surjective maps, we have

$$F(x)wG(z) = G(x)qF(z), \text{ for all } x, \ w, \ q, \ z \in R. \tag{3.2}$$

Use w instead of q in (3.2), we have

$$F(x)wG(z) = G(x)wF(z), \text{ for all } x, \ w, \ z \in R. \tag{3.3}$$

Replace z by x in above relation, we have

$$F(x)wG(x) = G(x)wF(x), \text{ for all } x, \ w \in R.$$

Hence if $F(x) \neq 0$ for some $x \in R$, then from Lemma 1.4 there exists $\lambda(x) \in C$ such that

$$G(x) = \lambda(x)F(x), \text{ for some } x \in R.$$

We now tend to show that this $\lambda(x)$ is independent of x and for which we proceed as follows. If $F(x) \neq 0$ and $F(z) \neq 0$ then it follows from (3.3) that,

$$F(x)w\lambda(z)F(z) = \lambda(x)F(x)wF(z), \text{ for all } w \in R.$$

Thus we have established the following result,

$$(\lambda(z) - \lambda(x))F(x)wF(z) = 0, \text{ for all } w \in R.$$

Since R is prime we have,

$$(\lambda(z) - \lambda(x)) = 0.$$

This implies that,

$$\lambda(x) = \lambda(z), \text{ that is the value } \lambda(x) \text{ is independent of } x$$

Thus we have proved that there exists $\lambda \in C$, such that

$$G(x) = \lambda F(x), \text{ holds for all } x \in R.$$

□

Theorem 3.2. *Let R be a non-commutative prime ring with involution $*$. Let D, F, G and H be the $*$-semimultipliers on R and d, f, g and h be the associated surjective maps respectively. If $D(x)G(y) = H(x)F(y)$ for all $x, y \in R$ and $F \neq 0$, $D \neq 0$, then there exists $\lambda \in C$ such that $G(x) = \lambda F(x)$ and $H(x) = \lambda D(x)$ for all $x \in R$.*

Proof. We are given that,

$$D(x)G(y) = H(x)F(y) \text{ for all } x,\ y \in R. \tag{3.4}$$

Use yz instead of y where $z \in R$ in (3.4), we have

$$D(x)G(yz) = H(x)F(yz) \text{ for all } x,\ y,\ z \in R.$$

Utilizing the definition of $*$-semimultiplier and since associated maps d, g, h and f are surjective, we have

$$D(x)g(y^*)G(z) = H(x)f(y^*)F(z) \text{ for all } x,\ y,\ z \in R. \tag{3.5}$$

Put $g(y^*) = w$ and $f(y^*) = q$ in (3.5) we have,

$$D(x)wG(z) = H(x)qF(z) \text{ for all } x,\ w,\ z \in R. \tag{3.6}$$

Use w instead of q in (3.6), we have

$$D(x)wG(z) = H(x)wF(z) \text{ for all } x,\ w,\ z \in R. \tag{3.7}$$

Use $wF(p)$ instead of w where $p \in R$ in (3.7),

$$D(x)wF(p)G(z) = H(x)wF(p)F(z) \text{ for all } x,\ w,\ z,\ p \in R.$$

From (3.7), we have

$$D(x)wF(p)G(z) = D(x)wG(p)F(z) \text{ for all } x,\ w,\ p,\ z \in R.$$

From above relation,we have

$$D(x)w(F(p)G(z) - G(p)F(z)) = 0 \text{ for all } x,\ w,\ p,\ z \in R.$$

By primeness of R and since $D \neq 0$, we have

$$F(p)G(z) - G(p)F(z) = 0 \text{ for all } p,\ z \in R.$$

From Theorem 3.1, we have

$$G(x) = \lambda F(x), \text{ for some } \lambda \in C \text{ and for all } x \in R.$$

Using above relation in (3.7), we have

$$D(x)w\lambda F(z) = H(x)wF(z) \text{ for all } x,\ w,\ z \in R.$$

We have the following relation,

$$D(x)w\lambda F(z) = H(x)wF(z) \text{ for all } x,\ w,\ z \in R.$$

This implies that,

$$(D(x)\lambda - H(x))wF(z) = 0 \text{ for all } x,\ w,\ z \in R.$$

Owing to primeness of ring R and since $F \neq 0$, we have

$$H(x) = \lambda D(x) \text{ for all } x \in R.$$

□

Theorem 3.3. *Let R be a non-commutative prime ring with involution $*$ where F is a $*$-semimultip lier on R and g is the associated onto maps. If $(F(x))^2 = 0$ for all $x \in R$, then $F = 0$.*

Proof. Since,

$$(F(x))^2 = 0 \text{ for all } x \in R. \tag{3.8}$$

$$F(x)oF(y) = 0 \text{ for all } x,\ y \in R. \tag{3.9}$$

Use yt instead of y where $y \in R$ in above relation (3.9) and the definition of $*$-semimultiplier, we obtain that,

$$F(x)oF(yt) = 0 \text{ for all } x,\ y,\ t \in R.$$

This implies that,

$$F(x)o(F(y)g(t^*)) = 0 \text{ for all } x,\ y,\ t \in R.$$

Utilizing the definition of a $*$-semimultiplier, we obtain that

$$(F(x)oF(y))g(t^*) - F(y)[F(x), g(t^*)] = 0 \text{ for all } x,\ y,\ t \in R.$$

$$F(y)[F(x), g(t^*)] = 0 \text{ for all } x,\ y,\ t \in R. \tag{3.10}$$

In (3.10), since g is surjective map, we have the following result

$$F(y)[F(x), w] = 0 \text{ for all } x,\ y,\ w \in R. \tag{3.11}$$

Use wp instead of w where $p \in R$ in (3.11), we obtain that,

$$F(y)w[F(x), p] = 0 \text{ for all } x,\ y,\ w,\ p \in R.$$

Since R is prime either $F(y) = 0$ or $F(x) \in Z(R)$. In the latter case $F = 0$, following argument from Theorem 2.12 . □

Theorem 3.4. *Let R be a non-commutative prime ring with involution $*$ where D, G and H are $*$-semimultipliers on R and d, g and h be the associated surjective maps respectively. If $D(x) = aG(x) + H(x)b$, for all $x \in R$, where $a \notin Z(R),\ \ b \notin Z(R)$, then $D = G = H = 0$.*

Proof. We are given that,

$$D(x) = aG(x) + H(x)b, \text{ for all } x \in R, \text{ where } a \notin Z(R),\ \ b \notin Z(R). \tag{3.12}$$

Use xy instead of x where $y \in R$ in (3.12), we have ,

$$D(xy) = aG(xy) + H(xy)b, \text{ for all } x,\ y \in R.$$

Utilizing the definition of $*$-semimultiplier and since associated maps d, g and h are surjective , we have

$$d(x^*)D(y) = ag(x^*)G(y) + h(x^*)H(y)b, \text{ for all } x,\ y \in R.$$

$$wD(y) = acG(y) + tH(y)b, \text{ for all } w,\ y,\ c,\ t \in R.$$

From (3.12), above relation becomes,

$$waG(y) + wH(y)b = acG(y) + tH(y)b, \text{ for all } c,\ y,\ w,\ t \in R.$$

Use w instead of c in above relation, we have

$$waG(y) + wH(y)b = awG(y) + tH(y)b, \text{ for all } t,\ y,\ w \in R.$$

This implies that,

$$[w, a]G(y) + (w - t)H(y)b = 0, \text{ for all } w,\ y,\ t \in R.$$

Use w instead of t in above relation, we have

$$[w, a]G(y) = 0, \text{ for all } w,\ y \in R.$$

Use we instead of w where $e \in R$ in above relation, we get

$$[w, a]eG(y) = 0, \text{ for all } w,\ y,\ e \in R.$$

Using the fact that R is prime and $a \notin Z(R)$, we get,

$$G(y) = 0, \text{ for all } y \in R.$$

That is,

$$G = 0.$$

Now using above relation in (3.12), we obtain that

$$D(x) = H(x)b, \text{ for all } x \in R.$$

Again replacing x with xy where $y \in R$ in above relation, we have

$$D(xy) = H(xy)b, \text{ for all } x,\ y \in R.$$

$$D(x)d(y^*) = H(x)h(y^*)b, \text{ for all } x,\ y \in R.$$

Since d and h are surjective functions, we get

$$D(x)w = H(x)qb, \text{ for all } x,\ w,\ q \in R.$$

$$H(x)bw = H(x)qb, \text{ for all } x,\ w,\ q \in R.$$

Use w instead of q in above relation, we get

$$H(x)[b, w] = 0, \text{ for all } x,\ w \in R.$$

Replace w by wr, where $r \in R$ we have,

$$H(x)w[b, r] = 0, \text{ for all } x,\ w,\ r \in R.$$

By primeness of R and since $b \notin Z(R)$ we have, from above relation

$$H(x) = 0, \text{ for all } x \in R. \text{ That is } H = 0.$$

Thus we infer the following,

$$H(x) = 0 = G(x), \text{ for all } x \in R.$$

From (3.12), we have $D(x) = 0, \text{ for all } x \in R$. That is $D = 0$. □

Theorem 3.5. *Let I be a non-zero ideal of a non-commutative prime ring R with involution $*$. Further let F be a $*$-semimultiplier associated with a surjective map $g : R \to R$. If $F(x^2) = \pm x^2$, for all $x \in R$ then there exists $\lambda \in C$ and $\mu : R \to C$ such that $F(x) = \lambda x + \mu(x)$, for all $x \in R$.*

Proof. Suppose, $F(x^2) = \pm x^2,$ for all $x \in R$.
Use $x + y$ instead of x where $y \in R$ in above relation we obtain that,

$$F(xoy) = \pm(xoy), \text{ for all } x,\ y \in R. \tag{3.13}$$

Use yx instead of y, in (3.13) we see,

$$F(xo(yx)) = \pm(xo(yx)), \text{ for all } x,\ y \in R.$$

From the definition of $*$-semimultiplier,

$$F(xoy)g(x^*) = \pm(xoy)x, \text{ for all } x\ y \in R.$$

From given assumption,

$$\pm(xoy)g(x^*) = \pm(xoy)x, \text{ for all } x,\ y \in R.$$

This implies that,

$$\pm(xoy)(g(x^*) - x) = 0, \text{ for all } x,\ y \in R. \tag{3.14}$$

Use yq instead of y where $q \in R$ in above relation (3.14) we find that,

$$\pm(y(xoq) + [x, y]q)(g(x^*) - x) = 0, \text{ for all } x,\ y \in R.$$

$$[x, y]q(g(x^*) - x) = 0, \text{ for all } x,\ y \in R. \tag{3.15}$$

From above relation we are in the receipt of (2.10) and hence we establish our result. □

Theorem 3.6. *Let R be a non-commutative prime ring and $0 \neq F$ be a $*$-semimultiplier and g be an associated surjective map. If $[a, F(x)] = 0$, for all $x \in R$ for some fixed element $a \in R$, then $a \in Z(R)$.*

Proof. Above result follows immediately by simple replacement of x by xy where $y \in R$ in $[a, F(x)] = 0$ and utilizing primeness. □

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