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SOME PROPERTIES OF MONOMIAL IDEALS ON REGULAR SEQUENCES IN A NOETHERIAN RING

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Abstract In this paper, we continue the study of monomial ideals with respect to a regular sequence on a commutative Noetherian ring R. Let $\mathbf{x} := x_1, \ldots, x_d$ be a regular R-sequence contained in the Jacobson radical of R. We first show that each monomial ideal $\mathfrak{a}$ of R with respect to $\mathbf{x}$ has a unique decomposition as an irredundant finite intersection of ideals of the form $x^{e_1}_{\sigma(1)}R + \cdots + x^{e_s}_{\sigma(s)}R$, where σ is a permutation of $\{1, \ldots, d\}$, $s \in \{1, \ldots, d\}$ and $e_1, \ldots, e_s$ are positive integers. As a consequence, it follows that $\mathfrak{a}$ is an irreducible ideal if and only if it is a generalized-parameter ideal. In addition, it is shown that if $\mathbf{x}R$ is a prime ideal, then the radical of $\mathfrak{a}$ and the symbolic powers of $\mathfrak{a}$ are monomials. Finally, we prove that if $\mathfrak{a}$ is a square-free monomial ideal such that $\mathrm{Ass}_R\, R/\mathfrak{a}^k \subseteq \mathrm{Ass}_R\, R/\mathfrak{a}$, for all integers $k \geq 1$, then $\mathfrak{a}$ is normal and $\mathfrak{a}^{(k)} = \mathfrak{a}^k$, where $\mathfrak{a}^{(k)}$ denotes the kth symbolic power of $\mathfrak{a}$.

1 Introduction

Let R denote a commutative Noetherian ring with the identity 1_R, and let $\mathbf{x} := x_1, \ldots, x_d$ be a regular R-sequence. A *monomial* with respect to $\mathbf{x}$ is a power product $x_1^{e_1} \ldots x_d^{e_d}$, where $e_1, \ldots, e_d$ are non-negative integers (so a monomial is either a non-unit or the identity element 1_R), and a *monomial ideal* with respect to $\mathbf{x}$ is a proper ideal generated by monomials.

A monomial ideal $\mathfrak{a}$ of R with respect to $\mathbf{x}$ is called *reducible* if there exist two monomial ideals $\mathfrak{b}$, $\mathfrak{c}$ of R with respect to $\mathbf{x}$ such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ and $\mathfrak{a} \neq \mathfrak{b}, \mathfrak{c}$. It is called *irreducible* if it is not reducible.

Monomial ideals are important in several areas of current research in commutative Noetherian rings, and they have been studied in their own right in several papers (for example see [2, 3, 4, 7, 12, 13]), so many interesting results are proved about such ideals.

Recall that an element $x \in R$ is said to be *integrally dependent* on an ideal $\mathfrak{b}$ of R if there exists an integer $n \geq 1$ and an equation of the form

$$x^n + c_1x^{n-1} + \cdots + c_n = 0,$$

where $c_i \in \mathfrak{b}^i$ for $i = 1, \ldots, n$. The set of all elements that are integrally dependent on $\mathfrak{b}$ is called the *integral closure* of $\mathfrak{b}$, denoted by $\bar{\mathfrak{b}}$. It is well known that $\bar{\mathfrak{b}}$ is an ideal of R and that $\mathfrak{b} \subseteq \bar{\mathfrak{b}}$. If $\mathfrak{b} = \bar{\mathfrak{b}}$, then $\mathfrak{b}$ is called *integrally closed*; and we say that $\mathfrak{b}$ is *normal* if for every integer $n \geq 1$, $\mathfrak{b}^n$ is integrally closed.

We refer the reader to [9] and [11] for more detailed information about integral dependence on ideals.

For a positive integer n, $\mathfrak{a}^{(n)}$ denotes the nth symbolic power of $\mathfrak{a}$, which is defined as the intersection of the primary components of $\mathfrak{a}^n$ corresponding to the minimal associated primes of $\mathfrak{a}$.

The aim of the present paper is to prove various results concerning monomial ideals with respect to a regular sequence on Noetherian rings. The first main result provides a new and short proof of the main results of Heinzer et al. (see [3, Corollary 4.10] and [4, Theorems 4.1 and

4.10]). Namely, for a regular R-sequence $\mathbf{x} := x_1, \dots, x_d$ which is contained in the Jacobson radical of R, it is shown that each monomial ideal $\mathfrak{a}$ of R with respect to $\mathbf{x}$ has a unique decomposition as an irredundant finite intersection of ideals of the form $x_{\sigma(1)}^{e_1}R + \cdots + x_{\sigma(s)}^{e_s}R$, where σ is a permutation of $\{1, \dots, d\}$, $s \in \{1, \dots, d\}$ and $e_1, \dots, e_s$ are positive integers.

Several corollaries of this result are proved. First, we start with the following result.

Theorem 1.1. *Let R denote a Noetherian ring and let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence contained in the Jacobson radical of R. Suppose that $\mathfrak{a}$ is a monomial ideal of R with respect to $\mathbf{x}$. Then the following conditions are hold:*

(i) *$\mathfrak{a}$ is an irreducible ideal if and only if it is a generalized-parameter ideal.*

(ii) (Cf. [5, Theorem 1(A)].) *If $\mathfrak{a}$ is a prime ideal, then for all integers $n \geq 1$, $\mathfrak{a}^{(n)} = \mathfrak{a}^n$. Moreover, if $\mathbf{x}R$ is a prime ideal of R, then we have*

(iii) *the radical and the symbolic powers of $\mathfrak{a}$ are monomials.*

(iv) *$\mathfrak{a}$ is an intersection of monomial prime ideals, whenever $\mathfrak{a}$ is square-free.*

(v) *$\mathfrak{a}^{(k)} = \mathfrak{a}^k$ and $\mathfrak{a}$ is normal, whenever $\mathfrak{a}$ is square-free and $\mathrm{Ass}_R R/\mathfrak{a}^k \subseteq \mathrm{Ass}_R R/\mathfrak{a}$, for every integer $k \geq 1$.*

We say that a monomial $m = x_1^{e_1} \dots x_d^{e_d}$ with respect to a regular R-sequence $\mathbf{x}$ is *square-free* if the all e_i are 0 or 1. Also, a monomial ideal $\mathfrak{a}$ with respect to $\mathbf{x}$ is called a *square-free monomial ideal* if $\mathfrak{a}$ is generated by square-free monomials.

One of our tools for proving Theorem 1.1 is the following:

Proposition 1.2. *Let R denote a Noetherian ring and let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence contained in the Jacobson radical of R. Suppose that $\mathfrak{a}$ is a monomial ideal of R with respect to $\mathbf{x}$, and let $u_1, \dots, u_r$ be a monomial generating sequence for $\mathfrak{a}$. Suppose that $u_1 = vw$, where v and w are co-prime monomials with respect to $\mathbf{x}$ and $v \neq 1 \neq w$. Then*

$$\mathfrak{a} = (vR + u_2R + \cdots + u_rR) \cap (wR + u_2R + \cdots + Ru_r).$$

Pursuing this point of view further we compute explicitly the radical of a monomial ideal. In fact, we derive the following consequence of Theorem 1.1.

Proposition 1.3. *Let R denote a Noetherian ring and let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence contained in the Jacobson radical of R. Let $\mathfrak{a}$ be a monomial ideal of R with respect to $\mathbf{x}$, and let $u_1, \dots, u_r$ be a monomial generating sequence for $\mathfrak{a}$. Then the sequence $\omega_1, \dots, \omega_r$ is a monomial generating for $\mathrm{Rad}(\mathfrak{a})$, where $\omega_i = \mathrm{rad}(u_i)$, for all $i = 1, \dots, r$.*

If $m = x_1^{e_1} \dots x_d^{e_d}$ is a monomial with respect to a regular R-sequence $\mathbf{x}$, then the *support* of m, denoted by $\mathrm{supp}(m)$, is defined to be the set $\{j | j \in \{1, \dots, d\}$ and $e_j \neq 0\}$. Also the *radical* of m, denoted by $\mathrm{rad}(m)$, is defined as $\mathrm{rad}(m) := \Pi_{j \in \mathrm{supp}(m)} x_j$. It is clear that if $m \in \mathfrak{a}$, then $(\mathrm{rad}(m))^t \in \mathfrak{a}$, for some integer $t \geq 1$. Also, it is easy to see that $m = \mathrm{rad}(m)$ if and only if m is a square-free monomial.

Throughout this paper all rings are commutative and Noetherian, with identity, unless otherwise specified. We shall use R to denote such a ring and $\mathfrak{a}$ an ideal of R. The *radical* of $\mathfrak{a}$, denoted by $\mathrm{Rad}(\mathfrak{a})$, is defined to be the set $\{x \in \mathfrak{a} : x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}\}$. Further, we denote by $\mathrm{mAss}_R R/\mathfrak{a}$ the set of minimal prime ideals of $\mathrm{Ass}_R R/\mathfrak{a}$. We say that $x_1, \dots, x_d$ form an *R-sequence* (of elements of R) precisely when $x_1R + \cdots + x_dR \neq R$ and for each $i = 1, \dots, d$, the element x_i is a non-zerodivisor on the R-module $R/(x_1R + \cdots + x_{i-1}R)$.

For any unexplained notation and terminology we refer the reader to [1] or [10].

2 The Results

The aim of this paper is to add several new results concerning monomial ideals with respect to a regular R-sequence $\mathbf{x} = x_1, \dots, x_d$ which is contained in the Jacobson radical of R. Specifically, we first shall show that if $\mathfrak{a}$ is a monomial ideal of R with respect to $\mathbf{x}$, then $\mathfrak{a}$ has a unique decomposition of generalized-parametric ideals. Several corollaries of this result are included. The following proposition will be quite useful in the proof of that result. We begin with

Definition 2.1. Assume that R denotes a Noetherian ring, and let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence of elements of R.

(i) Let $u = x_1^{e_1} \dots x_d^{e_d}$ and $v = x_1^{t_1} \dots x_d^{t_d}$ be two monomials with respect to $\mathbf{x}$. For all $i \in \{1, \dots, d\}$, we set $k_i = \min\{e_i, t_i\}$ and $s_i = \max\{e_i, t_i\}$. Then, we define

$$\gcd(u, v) = x_1^{k_1} \dots x_d^{k_d}, \quad \operatorname{lcm}(u, v) = x_1^{s_1} \dots x_d^{s_d},$$

the *greatest common divisor* resp. the *least common multiple* of u and v. We say that u and v are co-prime if $\gcd(u, v) = 1$.

(ii) Suppose that s is an integer such that $1 \leq s \leq d$, let σ be a permutation of $\{1, \dots, d\}$, and let $e_1, \dots, e_d$ be positive integers. Then the ideal generated by the monomials $x_{\sigma(1)}^{e_1}, \dots, x_{\sigma(s)}^{e_s}$ is called a *generalized-parametric ideal*.

Proposition 2.2. *Suppose that R denotes a Noetherian ring and let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence contained in the Jacobson radical of R. Let $\mathfrak{a}$ be a non-zero monomial ideal of R with respect to $\mathbf{x}$, and assume that $u_1, \dots, u_r$ is a monomial generating sequence for $\mathfrak{a}$. Set $u_1 = v_1 w_1$, where $v_1 \neq 1 \neq w_1$ are co-prime monomials. Then*

$$\mathfrak{a} = (v_1 R + u_2 R + \dots + u_r R) \cap (w_1 R + u_2 R + \dots + u_r R).$$

Proof. Put

$$\mathfrak{b} := v_1 R + u_2 R + \dots + u_r R \quad \text{and} \quad \mathfrak{c} := w_1 R + u_2 R + \dots + u_r R.$$

Then, it is clear that $\mathfrak{a} \subseteq \mathfrak{b} \cap \mathfrak{c}$. On the other hand, in view of [7, Proposition 1], we have

$$\begin{aligned} \mathfrak{b} \cap \mathfrak{c} &= \operatorname{lcm}(v_1, w_1)R + \operatorname{lcm}(v_1, u_2)R + \dots + \operatorname{lcm}(v_1, u_r)R \\ &+ \operatorname{lcm}(u_2, w_1)R + \operatorname{lcm}(u_2, u_2)R + \dots + \operatorname{lcm}(u_2, u_r)R \\ &\vdots \\ &+ \operatorname{lcm}(u_r, w_1)R + \operatorname{lcm}(u_r, u_2)R + \dots + \operatorname{lcm}(u_r, u_r)R. \end{aligned}$$

Now, since $\gcd(w_1, v_1) = 1$, it follows that $\operatorname{lcm}(w_1, v_1) = v_1 w_1 = u_1$, and so

$$\mathfrak{b} \cap \mathfrak{c} \subseteq u_1 R + u_2 R + \dots + u_r R = \mathfrak{a},$$

as required. □

We are now ready to state and prove one of our main results, which provides a new and short proof of the main results of [3, Corollary 4.10] and [4, Theorems 4.1 and 4.10]).

Theorem 2.3. *Suppose that R denotes a Noetherian ring and let $\mathbf{x} := x_1, \dots, x_d$ be a regular R-sequence contained in the Jacobson radical of R. Let $\mathfrak{a}$ be a non-zero monomial ideal of R with respect to $\mathbf{x}$. Then $\mathfrak{a}$ has a finite irredundant intersection generalized-parametric ideals, say, $\mathfrak{a} = \cap_{i=1}^m \mathfrak{q}_i$, where each $\mathfrak{q}_i$ is of the form $x_{i_1}^{e_{i_1}} R + \dots + x_{i_k}^{e_{i_k}} R$. Moreover, such an irredundant presentation, up to the order of the factors, is unique.*

Proof. Let $\mathfrak{a}$ be a non-zero monomial ideal of R with respect to $\mathbf{x}$, and let the monomials $u_1, \dots, u_r$ generate $\mathfrak{a}$. If every u_i has pure power, then being nothing to prove. So suppose that some u_i is not a pure power, say u_1. Then we can write $u_1 = vw$, where v and w are monomials with respect to $\mathbf{x}$ with $\gcd(v, w) = 1$. Hence, in view of Proposition 2.2, we have $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$, where

$$\mathfrak{b} := vR + u_2 R + \dots + u_r R \quad \text{and} \quad \mathfrak{c} := wR + u_2 R + \dots + u_r R.$$

Now, if $\{v, u_2, \dots, u_r\}$ or $\{w, u_2, \dots, u_r\}$ contains an element which is not a pure power, we proceed as before and obtain a finite number of steps a presentation of $\mathfrak{a}$ as an intersection of monomial ideals generated by pure powers. That is $\mathfrak{a}$ is a finite intersection of generalized-parameter ideals. Now, by omitting those ideals which contains the intersection of the others we end up with an irredundant intersection of generalized-parameter ideals.

Therefore it remains to show that such a presentation for $\mathfrak{a}$ is unique. For this, suppose that $\mathfrak{a}$ has two irredundant decompositions, say

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r \quad \text{and} \quad \mathfrak{a} = \mathfrak{q}'_1 \cap \ldots \cap \mathfrak{q}'_s,$$

where $\mathfrak{q}_i$ and $\mathfrak{q}'_j$ are generated-parameter ideals, for all $1 \leq i \leq r$ and $1 \leq j \leq s$. We need to show that $r = s$ and that $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\} = \{\mathfrak{q}'_1 \ldots, \mathfrak{q}'_s\}$. It is enough for us to show that for each $i = 1, \ldots, r$, there exists $j = 1, \ldots, s$ such that $\mathfrak{q}'_j \subseteq \mathfrak{q}_i$, and by symmetry we then also have that for each $k = 1, \ldots, s$, there exists $l = 1, \ldots, r$ such that $\mathfrak{q}_l \subseteq \mathfrak{q}'_k$. In order to do so, let $i \in \{1, \ldots, r\}$, and we suppose that $\mathfrak{q}'_j \nsubseteq \mathfrak{q}_i$ for all $j = 1, \ldots, s$, and look for a contradiction. We may assume that

$$\mathfrak{q}_i = x_1^{e_1} R + \cdots + x_t^{e_t} R \quad \text{and} \quad \mathfrak{q}'_j = x_{1_j}^{b_1} R + \cdots + x_{l_j}^{b_l} R.$$

Then for every $j = 1, \ldots, s$, there exists $1 \leq \mu_j \leq l$ such that $x_{\mu_j}^{b_j} \in \mathfrak{q}'_j \setminus \mathfrak{q}_i$. Whence, it follows that either $\mu_j \notin \{1, \ldots, t\}$ or $b_j < e_{\mu_j}$. Now, let us set $u = \operatorname{lcm}(x_{\mu_1}^{b_1}, \ldots, x_{\mu_s}^{b_s})$. Then, we have $u \in \cap_{j=1}^{s} \mathfrak{q}'_j = \mathfrak{a}$, and so $u \in \cap_{j=1}^{r} \mathfrak{q}_j$. In particular, it follows that

$$u \in \mathfrak{q}_i = x_1^{e_1} R + \cdots + x_t^{e_t} R.$$

Consequently, in view of [7, Corollary 3], there exists $1 \leq i \leq t$ such that $x_i^{e_i} | u$ and so $u \in x_i^{e_i} R$. Hence, it follows from [7, Remark 1] that there exists $1 \leq j \leq s$ such that $b_j \geq e_{\mu_j}$, which is a contradiction. □

The first application of Theorem 2.3 shows that two important notions of monomial irreducible ideal and generalized-parameter monomial ideal are the same. Namely:

Corollary 2.4. *Let R denote a Noetherian ring and assume that $\mathbf{x} := x_1, \ldots, x_d$ is a regular R-sequence contained in the Jacobson radical of R. Then a monomial ideal with respect to $\mathbf{x}$ is irreducible if and only if it is a generalized-parameter ideal.*

Proof. ($\Longrightarrow$) Let $\mathfrak{a}$ be a monomial ideal of R with respect to $\mathbf{x}$. If $\mathfrak{a}$ is irreducible and $\{u_1, \ldots, u_r\}$ is a monomial system of generators of $\mathfrak{a}$ such that some u_i is not a pure power, say u_1, then we can write $u_1 = vw$, where u and w are coprime monomials and $v \neq 1 \neq w$. Then, in view of Proposition 2.2, we have

$$\mathfrak{a} = (vR + u_2 R + \cdots + u_r R) \cap (wR + u_2 R + \cdots + u_r R),$$

which is a contradiction.

($\Longleftarrow$) Conversely, let $\mathfrak{a}$ be a generalized-parameter ideal, and suppose that $\mathfrak{a}$ is not irreducible. Then, in view of the definition, there exist integers $e_i \geq 1$ such that

$$\mathfrak{a} = x_{i_1}^{e_1} R + \cdots + x_{i_k}^{e_k} R,$$

and that there are two monomial ideals $\mathfrak{b}$ and $\mathfrak{c}$ properly containing $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$. In view of the Theorem 2.3, we have $\mathfrak{b} = \cap_{i=1}^{r} \mathfrak{q}_i$ and $\mathfrak{c} = \cap_{j=1}^{s} \mathfrak{q}'_j$, where $\mathfrak{q}_i$ and $\mathfrak{q}'_i$ are generalized-parameter ideals. Hence $\mathfrak{a} = (\cap_{i=1}^{r} \mathfrak{q}_i) \cap (\cap_{i=1}^{s} \mathfrak{q}'_i)$. By omitting suitable ideals in the intersection on the right-hand side, we derive an irredundant decomposition for $\mathfrak{a}$. Now, the uniqueness statement in Theorem 2.3 implies that $\mathfrak{a} = \mathfrak{q}_i$ or $\mathfrak{a} = \mathfrak{q}'_j$, for some i or j, which is a contradiction. □

The second consequence of Theorem 2.3, which is an extension of a result of M. Hochster, shows that the symbolic powers and the ordinary powers, of a monomial prime ideal are equal (see [5, Theorem 1(A)]).

Corollary 2.5. *Let R denote a Noetherian ring and assume that $\mathbf{x} := x_1, \ldots, x_d$ is a regular R-sequence contained in the Jacobson radical of R. Let $\mathfrak{p}$ be a monomial prime ideal of R with respect to $\mathbf{x}$. Then for all integers $n \geq 1$, $\mathfrak{p}^{(n)} = \mathfrak{p}^n$.*

Proof. Since $\mathfrak{p}$ is a monomial prime ideal, it easily follows that $\mathfrak{p}$ is a generalized-parameter ideal. Hence we can assume that $\mathfrak{p} = x_{i_1}^{e_1} R + \cdots + x_{i_s}^{e_s} R$, where $1 \leq s \leq d$ and $e_i \in \mathbb{N}$. As $\mathfrak{p}$ is prime, it yields that $x_{i_1}, \ldots, x_{i_s} \in \mathfrak{p}$, and so $\mathfrak{p} = x_{i_1} R + \cdots + x_{i_s} R$. Therefore, since $\mathfrak{p}$ is generated by a regular R-sequence, it follows from [6, Theorem 125 and Exercise 13] that $\operatorname{Ass}_R R/\mathfrak{p}^n \subseteq \operatorname{Ass}_R R/\mathfrak{p}$ for any $n \in \mathbb{N}$. Consequently, $\operatorname{Ass}_R R/\mathfrak{p}^n = \{\mathfrak{p}\}$, for every $n \in \mathbb{N}$, and so $\mathfrak{p}^n$ is a $\mathfrak{p}$-primary ideal. Hence $\mathfrak{p}^{(n)} = \mathfrak{p}^n$, as required. □

As a third conclusion of Theorem 2.3, we derive the following result which shows that the radical and the symbolic powers of a monomial ideals are also monomial.

Corollary 2.6. *Let R denote a Noetherian ring and assume that $\mathbf{x} := x_1, \dots, x_d$ is a regular R-sequence contained in the Jacobson radical of R such that $\mathbf{x}R$ is a prime ideal. Then for any monomial ideal $\mathfrak{a}$ of R, the ideals $\mathrm{Rad}(\mathfrak{a})$ and $\mathfrak{a}^{(n)}$ are also monomials, for every integer $n \geq 1$.*

Proof. In view of Theorem 2.3, we have $\mathfrak{a} = \cap_{i=1}^{m} \mathfrak{q}_i$, with $\mathfrak{q}_i = x_{\sigma(i_1)}^{e_{i_1}} R + \cdots + x_{\sigma(i_k)}^{e_{i_k}} R$, where σ is a permutation on $\{1, \dots, d\}$ and $k \geq 1$ is an integer. As, by hypothesis the ideal $\mathbf{x}R$ is prime, it follows from [12, Theorem 3.4] that $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, for all $i = 1, \dots, m$, where $\mathfrak{p}_i = x_{\sigma(i_1)} R + \cdots + x_{\sigma(i_k)} R$. Whence

$$\mathrm{Rad}(\mathfrak{a}) = \cap_{i=1}^{m} \mathfrak{p}_i \quad \text{and} \quad \mathfrak{a}^{(n)} = \cap_{i \in T} \mathfrak{q}_i,$$

where $T \subseteq \{1, \dots, m\}$. Now, the assertion follows from [7, Proposition 1]. □

In the sequel of this paper, we will study some properties of the square-free monomial ideals. Specially, we show that if $\mathfrak{a}$ is a square-free monomial ideal of R such that $\mathrm{Ass}_R R/\mathfrak{a}^n \subseteq \mathrm{Ass}_R R/\mathfrak{a}$, for all integers $n \geq 1$, then all powers of $\mathfrak{a}$ and $\mathfrak{a}^{(n)}$ are integrally closed. The following proposition which explicitly describes the radical of a monomial ideal is needed in the proof of that theorem.

Recall that for a monomial $m = x_1^{e_1} \dots x_d^{e_d}$, the radical of m is is defined as $\mathrm{rad}(m) = \Pi_{j \in \mathrm{supp}(m)} x_j$, where the set $\mathrm{supp}(m) := \{j | \, 1 \leq j \leq d \text{ and } e_j \neq 0\}$ denotes the support of m.

Proposition 2.7. *Let R denote a Noetherian ring and assume that $\mathbf{x} := x_1, \dots, x_d$ is a regular R-sequence contained in the Jacobson radical of R such that $\mathbf{x}R$ is a prime ideal. Let $\mathfrak{a}$ be a monomial ideal of R with monomial generating sequence $u_1, \dots, u_r$. Then the monomials $\mathrm{rad}(u_1), \dots, \mathrm{rad}(u_r)$ is a monomial generating sequence for $\mathrm{Rad}(\mathfrak{a})$.*

Proof. Let us put $\mathfrak{a} = u_1 R + \cdots + u_r R$, and set $\omega_i = \mathrm{rad}(u_i)$, for all $i = 1, \dots, r$. We shall show that $\mathrm{Rad}(\mathfrak{a}) = \omega_1 R + \cdots + \omega_r R$. To do this, because of $\mathrm{rad}(u_j) \in \mathrm{Rad}(\mathfrak{a})$ for every $1 \leq j \leq r$, it follows that

$$\omega_1 R + \cdots + \omega_r R \subseteq \mathrm{Rad}(\mathfrak{a}).$$

Now in order to show the opposite inclusion, since in view of Corollary 2.6, $\mathrm{Rad}(\mathfrak{a})$ is a monomial ideal with respect to $\mathbf{x}$, it is enough for us to show that for each monomial $m \in \mathrm{Rad}(\mathfrak{a})$ there exists a monomial m' and $1 \leq i \leq r$ such that $m = m'\omega_i$. To this end, it follows from $m \in \mathrm{Rad}(\mathfrak{a})$ that $m^l \in I$ for some integer $l \geq 1$. Hence, in view of [7, Corollary 3], there exists a monomial m' such that $m^l = m' u_j$ for some $1 \leq j \leq r$. Now it is easy to see that this yields the desired conclusion. □

Corollary 2.8. *Let R denote a Noetherian ring and assume that $\mathbf{x} := x_1, \dots, x_d$ is a regular R-sequence contained in the Jacobson radical of R such that $\mathbf{x}R$ is a prime ideal. Suppose that $\mathfrak{a}$ is a monomial ideal with respect to $\mathbf{x}$. Then, $\mathrm{Rad}(\mathfrak{a}) = \mathfrak{a}$ if and only if $\mathfrak{a}$ is square-free. In particular, every monomial prime ideal is square-free.*

Proof. The assertion readily follows from Proposition 2.7. □

Corollary 2.9. *Let R denote a Noetherian ring and assume that $\mathbf{x} := x_1, \dots, x_d$ is a regular R-sequence contained in the Jacobson radical of R such that the ideal $\mathbf{x}R$ is prime. Then every square-free monomial ideal $\mathfrak{a}$ with respect to $\mathbf{x}$ is a finite intersection of monomial prime ideals. In fact, $\mathfrak{a} = \cap_{\mathfrak{p} \in \mathrm{mAss}_R R/\mathfrak{a}} \mathfrak{p}$.*

Proof. In view of Theorem 2.3, $\mathfrak{a}$ has a finite irredundant primary representation, say $\mathfrak{a} = \cap_{i=1}^{m} \mathfrak{q}_i$, where for all $i = 1, \dots, m$, we have

$$\mathfrak{q}_i = x_{i_1}^{e_{i_1}} R + \cdots + x_{i_k}^{e_{i_k}} R,$$

for some positive integers $e_{i_1}, \dots, e_{i_k}$. Now, as $\mathrm{Rad}(\mathfrak{q}_i) = x_{i_1} R + \cdots + x_{i_k} R$ is a monomial prime ideal (cf. [12, Theorem 3.4]), the desired conclusion follows from Corollary 2.8. □

Corollary 2.10. *Let R denote a Noetherian ring and assume that $\mathbf{x} := x_1, \ldots, x_d$ is a regular R-sequence contained in the Jacobson radical of R such that the ideal $\mathbf{x}R$ is prime. Let $\mathfrak{a}$ be a square-free monomial ideal with respect to $\mathbf{x}$. Then*

$$\mathfrak{a}^{(k)} = \bigcap\nolimits_{\mathfrak{p}\in \mathrm{mAss}_R\, R/\mathfrak{a}} \mathfrak{p}^k,$$

for any integer $k \geq 1$.

Proof. In view of Corollary 2.9, we have $\mathfrak{a} = \cap_{\mathfrak{p}\in \mathrm{mAss}_R\, R/\mathfrak{a}}\mathfrak{p}$. Now, it is easy to see that

$$\mathfrak{a}^{(k)} = \bigcap\nolimits_{\mathfrak{p}\in \mathrm{mAss}_R\, R/\mathfrak{a}} \mathfrak{p}^{(k)},$$

for any integer $k \geq 1$, and so the assertion follows from Corollary 2.5. □

We are now ready to state and prove the final main result of this paper.

Theorem 2.11. *Let R denote a Noetherian ring and assume that $\mathbf{x} := x_1, \ldots, x_d$ is a regular R-sequence contained in the Jacobson radical of R such that the ideal $\mathbf{x}R$ is prime. Let $\mathfrak{a}$ be a square-free monomial ideal with respect to $\mathbf{x}$ such that $\mathrm{Ass}_R\, R/\mathfrak{a}^k \subseteq \mathrm{mAss}_R\, R/\mathfrak{a}$ for all integers $k \geq 1$. Then $\mathfrak{a}^{(k)} = \mathfrak{a}^k$ and $\mathfrak{a}$ is a normal ideal.*

Proof. Since, $\mathfrak{a}$ is square-free, it follows from Corollary 2.9 that $\mathfrak{a} = \cap_{\mathfrak{p}\in \mathrm{Ass}_R\, R/\mathfrak{a}}\mathfrak{p}$, and so $\mathrm{Ass}_R\, R/\mathfrak{a} = \mathrm{mAss}_R\, R/\mathfrak{a}$. Therefore $\mathrm{Ass}_R\, R/\mathfrak{a} = \mathrm{mAss}_R\, R/\mathfrak{a}^k$, and hence it follows from $\mathrm{Ass}_R\, R/\mathfrak{a}^k \subseteq \mathrm{Ass}_R\, R/\mathfrak{a}$ that $\mathrm{Ass}_R\, R/\mathfrak{a}^k = \mathrm{mAss}_R\, R/\mathfrak{a}^k$. Now, let $\mathfrak{a}^k = \cap_{i=1}^m \mathfrak{q}_i$ be an irredundant primary decomposition of $\mathfrak{a}^k$ with $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Then, as

$$\mathfrak{a}^{(k)} = \bigcap\nolimits_{\mathfrak{p}_i\in \mathrm{mAss}_R\, R/\mathfrak{a}^k} \mathfrak{q}_i \quad \text{and} \quad \mathrm{Ass}_R\, R/\mathfrak{a}^k = \mathrm{mAss}_R\, R/\mathfrak{a}^k,$$

it follows that $\mathfrak{a}^{(k)} = \mathfrak{a}^k$.

Now, we show that for all integers $k \geq 1$, the ideal $\mathfrak{a}^k$ is integrally closed, i.e., $\overline{\mathfrak{a}^k} = \mathfrak{a}^k$. To do this, in view of [7, Proposition 4], it is enough for us to show that for a monomial m in which $m^l \in \mathfrak{a}^{kl}$ for some integer $l \geq 1$, we have $m \in \mathfrak{a}^k$; note that by virtue of [7, Theorem 1] the ideal $\overline{\mathfrak{a}^k}$ is monomial. Since $\mathfrak{a}^{(j)} = \mathfrak{a}^j$ for all integer $j \geq 1$, and according to Corollary 2.10,

$$\mathfrak{a}^{(j)} = \bigcap\nolimits_{\mathfrak{p}\in \mathrm{mAss}_R\, R/\mathfrak{a}} \mathfrak{p}^j,$$

it suffices to prove that whenever $m^l \in \cap_{\mathfrak{p}\in \mathrm{mAss}_R\, R/\mathfrak{a}}\mathfrak{p}^{lk}$, for some integer $l \geq 1$, we have

$$m \in \bigcap\nolimits_{\mathfrak{p}\in \mathrm{mAss}_R\, R/\mathfrak{a}} \mathfrak{p}^k.$$

To do this so, let $m = x_1^{e_1} \ldots x_d^{e_d}$. Then $m^l = x_1^{le_l} \ldots x_d^{le_d}$, and it easily follows from

$$m^l \in \bigcap\nolimits_{\mathfrak{p}\in \mathrm{mAss}_R\, R/\mathfrak{a}} \mathfrak{p}^{lk}$$

that $le_i \geq lk$, for all $i = 1, \ldots, d$, in which $x_i \in \mathfrak{p}$, for all $\mathfrak{p} \in \mathrm{mAss}_R\, R/\mathfrak{a}$. This then implies that $e_i \geq k$ for all $i = 1, \ldots, d$ for which $x_i \in \mathfrak{p}$ and for all $\mathfrak{p} \in \mathrm{mAss}_R\, R/\mathfrak{a}$, which yields the desired conclusion. □

References

[1] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics, Vol. 39, Cambridge Univ. Press, Cambridge, UK, 1998.

[2] J. A. Eagon and M. Hochster, *R-sequences and indeterminates,* Quart. J. Math. **25** (1974), 61-71.

[3] W. Heinzer, A. Mirbagheri, L. J. Ratliff, Jr. and K. Shah, *Parametric decomposition of monomial ideals* (II), J. Algebra **187** (1997), 120-149.

[4] W. Heinzer, L. J. Ratliff, Jr. and K. Shah, *Parametric decomposition of monomial ideals* (I), Houston J. Math. **21** (1995), 29-52.

[5] M. Hochster, *Criteria for equality of ordinary and symbolic powers of primes,* Math. Z. **133** (1973), 53-65.

[6] I. Kaplansky, *Commutative Rings*, Univ. of Chicago Press, Chicago, 1974.

[7] K. Kiyek and J. Stückrad, *Integral closure of monomial ideals on regular sequences,* Rev. Mat. Iberoamericana **19** (2003), 483-508.

[8] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1986.

[9] S. McAdam, *Asymptotic prime divisors*, Lecture Notes in Math. 1023, Springer-Verlag, Neow York, 1983.

[10] M. Nagata, *Local Rings*, Interscience, New York, 1961.

[11] D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145-158.

[12] M. Sedghi, *Asymptotic behaviour of monomial ideals on regular sequences,* Rev. Mat. Iberoamericana **22** (2006), 955-962.

[13] D. Taylor, *Ideals generated by monomials in an R-sequence,* Ph.D. dissertation, University of Chicago, 1966.

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