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HARMONICITY OF (σ, σ')-HOLOMORPHIC SECTIONS OF A (SEMI-RIEMANNIAN) ALMOST PARA-QUATERNIONIC FIBER BUNDLE

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I'm very grateful to my father St Pierre for his tremendous help.

Abstract. In this paper, we study (σ, σ')-para-holomorphic maps between almost para-quaternionic Hermitian manifolds and obtain a criterion for the harmonicity of such (σ, σ')-para-holomorphic maps. We also study (σ, σ')-para-holomorphic sections of $(semi - Riemannian)$ almost para-quaternionic fiber bundles, and obtain a criterion for the harmonicity of such (σ, σ')-para-holomorphic sections.

1 Introduction

A semi-Riemannian manifold M is a C^∞-manifold endowed with a metric tensor g , that is, a symmetric non-degenerate $(0, 2)$ tensor field on M with constant indices of positivity and negativity ind_+M and ind_-M , respectively. The non-degeneracy means that $ind_+M + ind_-M = dimM$ (the dimension of M).

As in the Riemannian's case, a C^2-map $u : (M, g) \to (N, h)$ between semi-Riemannian manifolds is called harmonic, when the tension field $\tau(u)$ of u vanishes, where $\tau(u)$ is defined by $\tau(u) = Trace\nabla^{*'} du$.

Let $(N \overset{\pi}{\to} M)$ be a fiber bundle. A section of $(N \overset{\pi}{\to} M)$ is a continuous map $u : M \to N$ such that $\pi o u = Id$.

Let (N, h) and (M, g) be two semi-Riemannian manifolds of index $s'(0 \leq s' \leq dimN)$ and $s(0 \leq s \leq dimM)$ respectively with $s \leq s'$ and $(N \overset{\pi}{\to} M)$ a fiber bundle.
We say that $[(N, h) \overset{\pi}{\to} (M, g)]$ is a semi-Riemannian fiber bundle, when $\forall y \in N, du(y)|_{H_y}$: $H_y \to T_{\pi(y)}M$ is an isometry for the metric $h(y)$ on H_y and $g(\pi(y))$ on $T_{\pi(y)}M$, where $H_y = (T_y\pi^{-1}(\pi(y)))^\perp$ is the orthogonal complement of $T_y\pi^{-1}(\pi(y))$ and the fibers, $\pi^{-1}(x), \forall x \in M$ are semi-Riemannian submanifolds of N.

In other words a semi-Riemannian fiber bundle $[(N, h) \overset{\pi}{\to} (M, g)]$ is a fiber bundle, where (N, h) and (M, g) are two semi-Riemannian manifolds and π a semi-Riemannian submersion. Submersions of semi-Riemannian manifolds have been studied by many authors. (See e.g. [6] and [7]). As in the Riemannian's case (see [4], p.160), a C^2-section $u : M \to N$ of a semi-Riemannian fiber bundle is called harmonic, when the vertical tension field $\tau^v(u)$ of u vanishes, where $\tau^v(u) = Trace\nabla^{*^v} d^v u$.

An almost para-complex manifold (M, J) is a differentiable manifold M with a tensor J satisfying: $\forall x \in M, J_x : T_xM \to T_xM$ such that : $J_x^2 = Id_{T_xM}$.

An almost para-Hermitian manifold (M, g, J) is an almost para-complex manifold (M, J) with a semi-Riemannian metric g satisfying: $\forall X, Y \in \chi(M)$,
$g(JX, JY) = -g(X, Y)$, where $\chi(M)$ is the set of vector fields on M.

An almost para-Hermitian manifold (M, J, g) is said to be:
$\star$ para-$K\ddot{a}hler$ if $\nabla J = 0$;
$\star$ nearly para-$K\ddot{a}hler$ if $(\nabla_X J)X = 0, \forall X \in \chi(M)$;

⋆ almost para-$Kähler$ if $d\Phi = 0$, where Φ is the para-$Kähler$ 2-form defined by : $\forall X, Y \in \chi(M), \Phi(X,Y) = g(X, JY)$;
⋆ quasi-para-$Kähler$ if $(\nabla_X J)Y - (\nabla_{JX} J)JY = 0, \forall X, Y \in \chi(M)$;
⋆ almost semi-para-$Kähler$ if $\delta J = 0$, where δ denotes the codifferential in (M, g).

A C^2-map $u : (M, g, J) \to (N, h, \varphi)$ between almost para-Hermiian manifolds is said to be (J, φ)-holomorphic (or para-holomorphic or almost para-complex), when $duoJ = \varphi odu$

In [2], we study harmonic maps between almost para-Hermiian manifolds and give an expression of the tension field of a para-holomorphic map between almost para-Hermitian manifolds that we used to deduce a characterisation of its hamonicity. Namely we show that :

proposition 1

Let (M, J, g) and (N, φ, h) be two almost para-Hermitian manifolds and $u : M \to N$ be a (J, φ)- holomorphic map.
Then, we have:

$$\tau(u) = -\varphi[trac_g u^*(\nabla'\varphi) + du(\delta J)],$$

where $trac_g u^*(\nabla'\varphi) = \sum_{k=1}^{m}(\nabla'_{du(e_k)}\varphi)du(e_k) - \sum_{k=m+1}^{2m}(\nabla'_{du(e_k)}\varphi)du(e_k)$, with $\{e_1, ..., e_{2m}\}$ an orthonormal basis of TM.

A $(semi-Riemannian)$ almost para-Hermitian fiber bundle $[(N, \varphi, h) \xrightarrow{\pi} (M, J, g)]$ is a semi-Riemannian fiber bundle, where (N, φ, h) and (M, J, g) are almost para-Hermitian manifolds and π is (φ, J)-holomorphic map. We recall that an almost para-Hermitian submersion is a semi-Riemannian submersion which is additionaly almost para-complex. Almost para-Hermitian submersions have been studied by Yilmaz Gunduzalp. See e.g. [8].

In [2], we give an expression of the vertical tension field of a para-holomorphic section u of a $(semi-Riemannian)$ almost para-Hermitian fiber bundle that we used to deduce a characterisation of its hamonicity. Namely we show that :

proposition 2

Let $[(N, \varphi, h) \xrightarrow{\pi} (M, J, g)]$ be a $(semi-Riemannian)$ almost para-Hermitian fiber bundle and $u : M \to N$ be a (J, φ)-holomorphic section of $[(N, \varphi, h) \xrightarrow{\pi} (M, J, g)]$.

Then, we have:

$$\begin{aligned}\tau^v(u) &= -\varphi[trac_g(d^v u)^*(\nabla^v\varphi) + trac_g(\nabla^v_{d^h u}\varphi)(d^v u) + trac_g(A_{d^h u}d^h uoJ) \\ &+ trac_g(T_{d^v u}d^h uoJ) + d^v u(\delta J)],\end{aligned}$$

where

$$trac_g(d^v u)^*(\nabla^v J') = \sum_{k=1}^{m}(\nabla^v_{d^v u(X_k)}J')(d^v \widetilde{u(X_k)}) + \sum_{k=1}^{m}(\nabla^v_{d^v u(JX_k)}J')(d^v \widetilde{u(JX_k)}),$$

$$trac_g(A_{d^h u}d^h uoJ) = \sum_{k=1}^{m}[A_{d^h u(X_k)}(d^h uo\widetilde{J(X_k)}) + A_{d^h u(JX_k)}(d^h uo\widetilde{J(JX_k)})],$$

$$trac_g(T_{d^v u}d^h uoJ) = \sum_{k=1}^{m}[T_{d^v u(X_k)}(d^h uo\widetilde{J(X_k)}) + T_{d^v u(JX_k)}(d^h uo\widetilde{J(JX_k)})],$$

$$trac_g(\nabla^v_{d^h u}J')(d^v u = \sum_{k=1}^{m}[(\nabla^v_{d^h u(X_k)}J')d^v \widetilde{u(X_k)} + (\nabla^v_{d^h u(JX_k)}J')d^v \widetilde{u(JX_k)},$$

with $\{X_1, ..., X_m, JX_1, ..., JX_m\}$ an orthonormal local J-basis of TM.

An almost hyper para-complex manifold (M, J_1, J_2, J_3) is a differentiable manifold M with 3 tensors J_1, J_2 and J_3 such that : $J_1x, J_2x, J_3x \in End(T_xM), \forall x \in M$ and satisfying: $J_1^2x = -Id_{T_xM}, J_2^2x = J_3^2x = Id_{T_xM}$,
$J_1oJ_2 = J_3, J_3oJ_1 = J_2, J_2oJ_3 = -J_1$,
$J_1oJ_2 = -J_2oJ_1, J_3oJ_1 = -J_1oJ_3, J_3oJ_2 = -J_2oJ_3$.

An almost hyper para-Hermitian manifold (M, g, J_1, J_2, J_3) is an almost hyper para- complex manifold (M, g, J_1, J_2, J_3) with a semi-Riemannian metric g satisfying: $\forall \alpha \in \{1; 2; 3\}, \forall X, Y \in \chi(M)$,
$g(J_\alpha X, J_\alpha Y) = \epsilon_\alpha g(X, Y)$, where $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$ and $\chi(M)$ is the set of vector fields on M.

An almost hyper para-Hermitian manifold (M, g, J_1, J_2, J_3) is said to be:
$\star$ hyper para-$K\ddot{a}hler$ if $\nabla J_\alpha = 0, \forall \alpha \in \{1; 2; 3\}$;
$\star$ hyper nearly para-$K\ddot{a}hler$ if $(\nabla_X J_\alpha)X = 0, \forall \alpha \in \{1; 2; 3\}, \forall X \in \chi(M)$;
$\star$ almost hyper para-$K\ddot{a}hler$ if $d\Phi_\alpha = 0, \forall \alpha \in \{1; 2; 3\}$, where Φ_α is the para-$K\ddot{a}hler$ 2-form defined by : $\forall X, Y \in \chi(M), \Phi_\alpha(X, Y) = g(X, J_\alpha Y)$;
$\star$ hyper quasi para-$K\ddot{a}hler$ if $(\nabla_X J_\alpha)Y + \epsilon_\alpha(\nabla_{J_\alpha X} J_\alpha)J_\alpha Y = 0, \forall \alpha \in \{1; 2; 3\}, \forall X, Y \in \chi(M)$ and $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$;
$\star$ almost hyper semi-para-$K\ddot{a}hler$ if $\delta J_\alpha = 0, \forall \alpha \in \{1; 2; 3\}$, where δ denotes the codifferential in (M, g).

The concept of almost quaterionic Hermitian submersions is defined in [5]. According to [5] and [1], we introduce the following.

Let M be a differentiable manifold of dimension m and assume that there is a rank 3-subbundle σ of End(TM) such that a local basis $\{J_1, J_2, J_3\}$ exists of sections of σ such that $J_1, J_2, J_3 \in End(TM)$ and satisfying the hyper para-complex multiplication table above. Then the bundle σ is called almost para-quaternionic structure on M and $\{J_1, J_2, J_3\}$ is called canonical local basis of σ. Moerever, (M, σ) is said to be an almost para-quaternionic manifold and is of dimension $m = 4k$.

A $semi - Riemannian$ metric g on M is said to be adapted to σ if it satisfies :
$g(J_\alpha X, J_\alpha Y) = \epsilon_\alpha g(X, Y), \forall \alpha \in \{1, 2, 3\}$ for all vector fields X, Y on M and any canonical local basis $\{J_1, J_2, J_3\}$ of σ, $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$. Moerever, (M, σ, g) is said to be almost para-quaternionic Hermitian manifold.

Let (M, σ, g) and (N, σ', h) be two almost para-quaternionic Hermitian manifolds.

A map $u : (M, \sigma, g) \to (N, \sigma', h)$ between almost para-quaternionic manifolds is said to be (σ, σ')-para-holomorphic at a point $x \in M$, when for any $J \in \sigma_x$ exists $J' \in \sigma'_{u(x)}$ such that $duoJ = J'odu$. Moerever, we say that u is a (σ, σ')-para-holomorphic map, when u is a (σ, σ')-para-holomorphic map at each point $x \in M$.

A $(semi - Riemannian)$ almost para-quaternionic Hermitian fiber bundle $[(N, \sigma', h) \overset{\pi}{\to} (M, \sigma, g)]$ is a semi-Riemannian fiber bundle, where (N, σ', h) and (M, σ, g) are almost para-quaternionic Hermitian manifolds and π is a (σ, σ')-para-holomorphic map.

In other words a $(semi - Riemannian)$ almost para-quaternionic Hermitian fiber bundle $[(N, \sigma', h) \overset{\pi}{\to} (M, \sigma, g)]$ is a fiber bundle, where (N, σ', h) and (M, σ, g) are two almost para-quaternionic Hermitian manifolds and π is an almost para-quaternionic Hermitian submersion.

We recall that an almost para-quaternionic Hermitian submersion is a semi-Riemannian submersion which is (σ, σ')-para-holomorphic.

Almost para-quaternionic Hermitian submersions have been studied by Angelo V. Caldarella in [1].

A section $u : M \to N$ of a $(semi - Riemannian)$ almost para-quaternionic Hermitian fiber bundle $[(N, \sigma', h) \overset{\pi}{\to} (M, \sigma, g)]$ is said to be (σ, σ')-para-holomorphic at a point $x \in M$, when for any $J \in \sigma_x$ exists $J' \in \sigma'_{u(x)}$ such that $d^v uoJ = J'od^v u$. Moerever, we say that u is a (σ, σ')-holomorphic section, when u is a (σ, σ')-para-holomorphic section at each point $x \in M$.

The aim of this work is to give a version of ours above results in the para-quaternionic hermitian case.

We reach our aim by giving an expression of :

First the tension field $\tau(u)$ of a (σ, σ')-para-holomorphic map $u : (M, \sigma, g) \to (N, \sigma', h)$ between almost para-quaternionic Hermiian manifolds.

And secondly the vertical tension field $\tau^v(u)$ of a (σ, σ')-para-holomorphic section u of a $(semi{-}Riemannian)$ almost para-quaternionic Hermitian fiber bundle, $[(N, \sigma', h) \overset{\pi}{\to} (M, \sigma, g)]$.

The paper is organized as follows: In section II, we recall definitions and properties which are useful for the other sections. In section III, we give the results of the maps case. Section IV is devoted to proofs of the results of the maps case. In section V, we give the results of the sections case. Section VI is devoted to proofs of the results of sections case.

2 Definitions

2.1 Vertical and horizontal fiber bundles

Let $[(N,h) \overset{\pi}{\to} (M,g)]$ be a semi-Riemannian fiber bundle.
We define the vertical fiber bundle W to be the vector bundle over N which fibers are the vertical tangent spaces $T_y\pi^{-1}(\pi(y)), \forall y \in N$.
Also we define the horizontal fiber bundle H to be the vector bundle over N which fibers are the horizontal tangent spaces $H_y, \forall y \in N$.

2.2 Pull back bundle

Let $(B \overset{\pi}{\to} N)$ be a vector bundle, and $u : M \to N$ a smooth map, then there exists a unique vector bundle $(u^{-1}(B) \overset{\pi'}{\to} M)$ called pull-back bundle such that $\forall x \in M, {\pi'}^{-1}(\{x\}) = \pi^{-1}(\{u(x)\})$

Let's assume that $(B \overset{\pi}{\to} N)$ is equipped with a semi-Riemannian metric h and a compatible covariant derivative ∇. Then $(u^{-1}(B) \overset{\pi'}{\to} M)$ is equipped with a semi-Riemannian metric a and a compatible covariant derivative ∇^* defined respectively by:

$$a : M \to (u^{-1}(B))^* \otimes (u^{-1}(B))^*,$$

$$a_x(Y,Z) = h_{u(x)}(Y,Z), \forall x \in M, \forall Y,Z \in (u^{-1}(B))_x.$$

And

$$\nabla^*_X(wou) = \nabla_{du(X)}w, \forall X \in \chi(M), \forall w \in \Gamma(B).$$

2.3 Harmonic section

Let $u : M \to N$ be a C^2 section of a semi-Riemannian fiber bundle $[(N,h) \overset{\pi}{\to} (M,g)]$.
The vertical tension field of u is given by $\tau^v(u) = Trace\nabla^{*^v} d^v u$.
If $\{e_1, ..., e_{2m}\}$ is a local orthonormal frame of TM such that
$g(e_k, e_k) = 1, \forall 1 \le k \le m,$
$g(e_k, e_k) = -1, \forall m+1 \le k \le 2m,$
$g(e_i, e_j) = 0, \forall i \ne j.$
Then

$$\tau^v(u) = \sum_{k=1}^{m}[\nabla^{*^v}_{e_k}(d^v u(e_k)) - d^v u(\nabla_{e_k}e_k)] - \sum_{k=m+1}^{2m}[\nabla^{*^v}_{e_k}(d^v u(e_k)) - d^v u(\nabla_{e_k}e_k)],$$

where ∇ denotes the Levi-Civita covariant derivative on M, ∇^v denotes the Levi-Civita covariant derivative on W induced by the metric h and ∇^{*^v} the pull-back of the Levi-Civita covariant derivative ∇^v on W to the pull-back bundle $(u^{-1}(W) \to M)$.
We recall that u is called harmonic, when $\tau^v(u) = 0$.

2.4 Fundamental tensors of a semi Riemannian fiber bundle.(See [7])

Let $[(N,h) \overset{\pi}{\to} (M,g)]$ be a semi-Riemannian fiber bundle. The two fundamental tensors of π are defined by :

$$\begin{aligned} T_E F &= \nabla^h_{E^v} F^v + \nabla^v_{E^v} F^h \\ A_E F &= \nabla^h_{E^h} F^v + \nabla^v_{E^h} F^h \end{aligned}$$

for all vectors fields E and F on N, where the superscripts h and v mean respectively the projection onto the horizontal, respectively the vertical bundle.
Remark:
The tensor $A = 0$ if the horizontal distribution is integrable.
The tensor $T = 0$ if the fibers are totally geodesic.

2.5 Link of the covariant derivative of $Z \in \Gamma(u^{-1}(TN))$ to the one of its extension

Let $(N \stackrel{\pi}{\to} M)$ be a fiber bundle. $\Gamma(N)$ denotes the set of the sections of $(N \stackrel{\pi}{\to} M)$.

Let $Z \in \Gamma(u^{-1}(TN))$ and $(E_j)_{1\leq\alpha\leq m}$ be an orthonormal local frame in TN defined on a open subset V of N, then $\forall x_0 \in u^{-1}(V), Z(x_0) = Z^j(x_0)E_j(u(x_0))$.

To Z we associate the vector field $\widetilde{Z}$ of N defined on V by $\forall y \in V$, $\widetilde{Z}(y) = Z^j(\pi(y))E_j(y)$.

On the one hand :

$$\begin{aligned}\forall X \in \chi(M), \forall x \in u^{-1}(V), \nabla^{*'}_X Z(x) &= \nabla^{*'}_X[Z^j(x)E_j(u(x))] \\ &= D_X(Z^j)(x)E_j ou(x) + Z^j(x)\nabla^{*'}_X E_j(u(x)) \\ &= D_X(Z^j)(x)E_j ou(x) + Z^j(x)\nabla'_{du(X)(x)}E_j.\end{aligned}$$

On the other hand, with the fact that u is a section i.e. $\pi ou = Id_M$, one has :

$$\begin{aligned}\forall X \in \chi(M), \forall x \in u^{-1}(V), (\nabla^{*'}_X \widetilde{Z} ou)(x) &= \nabla'_X[(Z^j o\pi ou)E_j ou](x) \\ &= D_X(Z^j o\pi ou)(x)E_j ou(x) \\ &+ Z^j o\pi ou(x)(\nabla^{*'}_X E_j ou)(x) \\ &= D_X(Z^j)(x)E_j ou(x) \\ &+ Z^j(x)\nabla'_{du(X)(x)}E_j.\end{aligned}$$

It follows that :

$$\forall X \in \chi(M), \nabla^{*'}_X Z = \nabla^{*'}_X(\widetilde{Z} ou).$$

Since :

$\forall X \in \chi(M), \nabla^{*'}_X(\widetilde{Z} ou) = \nabla'_{du(X)}\widetilde{Z}$ we get:

$$\forall X \in \chi(M), \nabla^{*'}_X Z = \nabla'_{du(X)}\widetilde{Z}, \tag{2.1}$$

where ∇' denotes the Levi-Civita covariant derivative on N and $\nabla^{*'}$ the pull-back of ∇' to the pull-back bundle $(u^{-1}(TN) \to M)$.

2.6 Fundamental properties of a (semi-Riemannian) almost para-quaternionic Hermitian fiber bundle. (See prop. 4 and prop. 5 in [1])

Let $[(N, \sigma', h) \stackrel{\pi}{\to} (M, \sigma, g)]$ be a $(semi - Riemannian)$ almost para-quaternionic Hermitian fiber bundle, then we have the following properties:

i) The vertical and the horizontal distributions are φ-invariant (ie $\forall U \in \Gamma(W), J'U \in \Gamma(W)$ and $\forall Y \in \Gamma(H), J'Y \in \Gamma(H)$).

ii) The fibers are almost para-quaternionic Hermitian submanifolds of N.

2.7 Trace of some 2-covariant tensors

Let $u : M \to N$ be a (J, φ)-holomorphic section of a $(semi - Riemannian)$ almost para-quaternionic Hermitian fiber bundle, $[(N, \varphi, h) \stackrel{\pi}{\to} (M, J, g)]$

Let θ be a $u^{-1}(TN)$-valued 2-covariant tensor on M.

As g is non-degenerate, there exists an ordered basis $(e_1, e_2, ..., e_{4m})$ of TM such that:

$g(e_i, e_i) = 1, \forall 1 \leq i \leq 2m$,

$g(e_i, e_i) = -1, \forall 2m + 1 \leq i \leq 4m$,

$g(e_i, e_j) = 0, \forall i \neq j$.

Let $\{(e_1)^*, ..., (e_{4m})^*\}$ be its dual basis.

We call trace of θ, the section $trac_g(\theta)$ of $u^{-1}(TN)$ defined by:

$$trac_g(\theta) = \sum_{i,j=1}^{4m} g^{ij}\theta(e_i, e_j).$$

If $\{e_1, ..., e_{4m}\}$ is orthonormal, then $trac_g(\theta) = \sum_{i=1}^{2m} \theta(e_i, e_i) - \sum_{i=2m+1}^{4m} \theta(e_i, e_i)$

Like above let us assume that
$\{X_1; ...; X_m; J_1X_1; ...; J_1X_m; J_2X_1; ...; J_2X_m; J_3X_1; ...; J_3X_m\}$ is an orthonormal local frame of TM.

The trace of the tensor $(d^vu)^*(\nabla^v J'_\alpha)$ is given by:

$$\begin{aligned} trac_g(d^vu)^*(\nabla^v J'_\alpha) &= \sum_{k=1}^{m} (\nabla^v_{d^vu(X_k)} J'_\alpha)(d^v\widetilde{u(X_k)}) \\ &+ \sum_{k=1}^{m}\sum_{\beta=1}^{3} \epsilon_\beta(\nabla^v_{d^vu(J_\beta X_k)} J'_\alpha)(d^vu(\widetilde{J_\beta X_k})) \end{aligned}$$

The trace of the tensor $A_{d^hu}d^huoJ$ is given by:

$$\begin{aligned} trac_g(A_{d^hu}d^huoJ_\alpha) &= \sum_{k=1}^{m} A_{d^hu(X_k)}(d^huo\widetilde{J_\alpha}(X_k)) \\ &+ \sum_{k=1}^{m}\sum_{\beta=1}^{3} \epsilon_\beta A_{d^hu(J_\beta X_k)}(d^huo\widetilde{J_\alpha(J_\beta X_k)}) \end{aligned}$$

The trace of the tensor $T_{d^vu}d^huoJ_\alpha$ is given by:

$$\begin{aligned} trac_g(T_{d^vu}d^huoJ_\alpha) &= \sum_{k=1}^{m} T_{d^vu(X_k)}(d^huo\widetilde{J_\alpha}(X_k)) \\ &+ \sum_{k=1}^{m}\sum_{\beta=1}^{3} \epsilon_\beta T_{d^vu(J_\beta X_k)}(d^huo\widetilde{J_\alpha(J_\beta X_k)}) \end{aligned}$$

The trace of the tensor field $(\nabla^v_{d^hu} J'_\alpha)(d^vu)$ is given by:

$$\begin{aligned} trac_g(\nabla^v_{d^hu} J'_\alpha)(d^vu) &= \sum_{k=1}^{m} [(\nabla^v_{d^hu(X_k)} J'_\alpha)d^v\widetilde{u(X_k)} \\ &+ \sum_{k=1}^{m}\sum_{\beta=1}^{3} \epsilon_\beta(\nabla^v_{d^hu(J_\beta X_k)} J'_\alpha)d^vu(\widetilde{J_\beta X_k})], \end{aligned}$$

for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ', where $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$.

3 Results for maps.

3.1 Proposition

Proposition 3.1. *Let (M, σ, g) and (N, σ', h) be two almost para-quaternionic Hermitian manifolds and $u : M \to N$ be a (σ, σ')-para-holomorphic map.*
Then, we have:

$$\tau(u) = \epsilon_\alpha J'_\alpha[trac_g(du)^*(\nabla' J'_\alpha) + du(\delta J_\alpha)],$$

for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ', where $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$.

3.2 Corollary

Corollary 3.2. *Let $u : M \to N$ be a (σ, σ')-para-holomorphic map from a locally almost hyper semi-para-$K\ddot{a}hler$ almost para-quaternionic Hermitian manifold M to a locally hyper quasi para-$K\ddot{a}hler$ almost para-quaternionic Hermitian manifold N, then u is a harmonic map.*

3.3 Theorem

Theorem 3.3. *Let $u : M \to N$ be a (σ, σ')-para-holomorphic map between almost para-quaternionic Hermitian manifolds.*
Then, u is a harmonic map if only if

$$trac_g u^*(\nabla' J'_\alpha) = -du(\delta J_\alpha),$$

for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ'.

4 Proof of the results

4.1 Proof of Proposition 3.1

Proof. Let $\{X_1, ..., X_m, J_1X_1, ..., J_1X_m, J_2X_1, ..., J_2X_m, J_3X_1, ..., J_3X_m\}$ be an orthonormal local frame of TM

such that $g(X_k, X_k) = 1 = g(J_1X_k, J_1X_k)$, and $g(J_\beta X_k, J_\beta X_k) = -1, \forall\beta \in \{2.3\}$.

$$\begin{aligned}
\tau(u) &= \sum_{k=1}^{m}(\nabla^{*'} du)(X_k, X_k) + \sum_{k=1}^{m}(\nabla^{*'} du)(J_1X_k, J_1X_k) \\
&- \sum_{k=1}^{m}(\nabla^{*'} du)(J_2X_k, J_2X_k) - \sum_{k=1}^{m}(\nabla^{*'} du)(J_3X_k, J_3X_k) \\
&= \sum_{k=1}^{m}(\nabla^{*'} du)(X_k, X_k) + \epsilon_\alpha \sum_{k=1}^{m}(\nabla^{*'} du)(J_\alpha X_k, J_\alpha X_k) \\
&+ \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla^{*'} du)(J_\beta X_k, J_\beta X_k) \\
&= \sum_{k=1}^{m}[\nabla^{*'}_{X_k}(du(X_k)) - du(\nabla_{X_k}X_k)] + \epsilon_\alpha \sum_{k=1}^{m}[\nabla^{*'}_{J_\alpha X_k}(du(J_\alpha X_k)) - du(\nabla_{J_\alpha X_k}J_\alpha X_k)] \\
&+ \sum_{k=1}^{m}[\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla^{*'} du)(J_\beta X_k, J_\beta X_k)] \\
&= \sum_{k=1}^{m}[\nabla^{*'}_{X_k}(du(X_k)) + \epsilon_\alpha \nabla^{*'}_{J_\alpha X_k}(du(J_\alpha X_k))] - du[\sum_{k=1}^{m}(\nabla_{X_k}X_k + \epsilon_\alpha \nabla_{J_\alpha X_k}J_\alpha X_k)] \\
&+ R,
\end{aligned}$$

where we set $R = \sum_{k=1}^{m}[\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla^{*'} du)(J_\beta X_k, J_\beta X_k)]$

$$\begin{aligned}
-\delta J_\alpha &= \sum_{k=1}^{m}(\nabla J_\alpha)(X_k, X_k) + \epsilon_\alpha \sum_{k=1}^{m}(\nabla J_\alpha)(J_\alpha X_k, J_\alpha X_k) \\
&+ \sum_{k=1}^{m}\Big[\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla J_\alpha)(J_\beta X_k, J_\beta X_k)\Big] \\
&= \sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - J_\alpha(\nabla_{X_k} X_k)] + \epsilon_\alpha \sum_{k=1}^{m}[\nabla_{J_\alpha X_k} J_\alpha(J_\alpha X_k) - J_\alpha(\nabla_{J_\alpha X_k} J_\alpha X_k)] \\
&+ \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla J_\alpha)(J_\beta X_k, J_\beta X_k) \\
&= \sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - J_\alpha(\nabla_{X_k} X_k)] + \epsilon_\alpha \sum_{k=1}^{m}[\nabla_{J_\alpha X_k}(-\epsilon_\alpha X_k) - J_\alpha(\nabla_{J_\alpha X_k} J_\alpha X_k)] \\
&+ \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla J_\alpha)(J_\beta X_k, J_\beta X_k) \\
&= \sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k] - \sum_{k=1}^{m} J_\alpha(\nabla_{X_k} X_k + \epsilon_\alpha \nabla_{J_\alpha X_k} J_\alpha X_k) \\
&+ \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla J_\alpha)(J_\beta X_k, J_\beta X_k) \\
&= \sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k] - \sum_{k=1}^{m} J(\nabla_{X_k} X_k + \epsilon_\alpha \nabla_{J_\alpha X_k} J_\alpha X_k) \\
&+ S
\end{aligned}$$

since : $J_\alpha^2(Z) = -\epsilon_\alpha Z$. and we set $S = \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla J_\alpha)(J_\beta X_k, J_\beta X_k)$.

By applying J_α, we get :

$$J_\alpha(-\delta J_\alpha) = J_\alpha(\sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k]) - J_\alpha^2(\sum_{k=1}^{m}[\nabla_{X_k} X_k + \epsilon_\alpha \nabla_{J_\alpha X_k} J_\alpha X_k]) + J_\alpha(S)$$

As $J_\alpha^2 = -\epsilon_\alpha I$, we have:

$$\begin{aligned}
J_\alpha(-\delta J_\alpha) &= J_\alpha(\sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k]) \\
&\quad + \epsilon_\alpha \sum_{k=1}^{m}[\nabla_{X_k} X_k + \epsilon_\alpha \nabla_{J_\alpha X_k} J_\alpha X_k] + J_\alpha(S).
\end{aligned}$$

We deduce that

$$\begin{aligned}
\epsilon_\alpha \sum_{k=1}^{m}[\nabla_{X_k} X_k + \epsilon_\alpha \nabla_{J_\alpha X_k} J_\alpha X_k] &= -J_\alpha(\delta J_\alpha) - J_\alpha(\sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k]) \\
&\quad - J_\alpha(S).
\end{aligned}$$

By applying du, we get

$$\begin{aligned}
\epsilon_\alpha du(\sum_{k=1}^{m}[\nabla_{X_k} X_k + \epsilon_\alpha \nabla_{J_\alpha X_k} J_\alpha X_k]) &= -duoJ_\alpha(\delta J_\alpha) \\
&\quad - duoJ_\alpha(\sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k]) \\
&\quad - duoJ_\alpha(S).
\end{aligned}$$

Multiplying by $-\epsilon_\alpha$,

$$-du(\sum_{k=1}^{m}[\nabla_{X_k}X_k+\epsilon_\alpha\nabla_{J_\alpha X_k}J_\alpha X_k]) = \epsilon_\alpha\{duoJ_\alpha(\delta J_\alpha) + duoJ_\alpha(\sum_{k=1}^{m}[\nabla_{X_k}J_\alpha X_k-\nabla_{J_\alpha X_k}X_k]) + duoJ_\alpha(S)\}.$$

$\tau(u)$ becomes including the fact that $duoJ_\alpha = J'_\alpha odu$,

$$\begin{aligned}\tau(u) &= \sum_{k=1}^{m}[\nabla^{*'}_{X_k}(du(X_k))+\epsilon_\alpha\nabla^{*'}_{J_\alpha X_k}(du(J_\alpha X_k))] \\ &+ \epsilon_\alpha\{duoJ_\alpha(\delta J_\alpha)+duoJ_\alpha(\sum_{k=1}^{m}[\nabla_{X_k}J_\alpha X_k-\nabla_{J_\alpha X_k}X_k])+J'_\alpha odu(S)\}+R \\ &= \sum_{k=1}^{m}[\nabla^{*'}_{X_k}(du(X_k))+\epsilon_\alpha\nabla^{*'}_{J_\alpha X_k}(du(J_\alpha X_k))] \\ &+ \epsilon_\alpha duoJ_\alpha(\delta J_\alpha)+\epsilon_\alpha J'_\alpha odu(\sum_{k=1}^{m}[\nabla_{X_k}J_\alpha X_k-\nabla_{J_\alpha X_k}X_k])+\epsilon_\alpha J'_\alpha odu(S)+R.\end{aligned}$$

One knows that, for any C^2-map $f : M \to N$ and any vectors fields X and Y, we have

$$\begin{aligned}df(\nabla_X Y-\nabla_Y X) &= df[X,Y] \\ &= \nabla^{*'}_X(df(Y))-\nabla^{*'}_Y(df(X)).\end{aligned}$$

Then

$$du(\nabla_{X_k}J_\alpha X_k-\nabla_{J_\alpha X_k}X_k)=\nabla^{*'}_{X_k}(du(J_\alpha X_k))-\nabla^{*'}_{J_\alpha X_k}(du(X_k)),$$

we have

$$\begin{aligned}\tau(u) &= \sum_{k=1}^{m}[\nabla^{*'}_{X_k}(du(X_k))+\epsilon_\alpha\nabla^{*'}_{J_\alpha X_k}(du(J_\alpha X_k))]+\epsilon_\alpha duoJ_\alpha(\delta J_\alpha) \\ &+ \epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[\nabla^{*'}_{X_k}(J'_\alpha odu(X_k))-\nabla^{*'}_{J_\alpha X_k}(du(X_k))]\}+\epsilon_\alpha J'_\alpha odu(S)+R\end{aligned}$$

Let us set Z, the sum of the first two elements of $\tau(u)$ ie
$Z = \sum_{k=1}^{m}[\nabla^{*'}_{X_k}(du(X_k))+\epsilon_\alpha\nabla^{*'}_{J_\alpha X_k}(du(J_\alpha X_k))]$.

With $J'^2_\alpha(Z) = -\epsilon_\alpha Z$, we have :

$$Z=-\epsilon_\alpha J'_\alpha\{J'_\alpha\{\sum_{k=1}^{m}[\nabla^{*'}_{X_k}(du(X_k))+\epsilon_\alpha\nabla^{*'}_{J_\alpha X_k}(du(J_\alpha X_k))]\}\}.$$

So $\tau(u)$ turns to :

$$\begin{aligned}\tau(u) \;&=\; -\epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[J_\alpha(\nabla^{*'}_{X_k}(du(X_k)))+\epsilon_\alpha J'_\alpha(\nabla^{*'}_{J_\alpha X_k}(du(J_\alpha X_k)))]\} \\ &+\; \epsilon_\alpha duoJ_\alpha(\delta J_\alpha)+\epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[\nabla^{*'}_{X_k}(J'_\alpha odu(X_k))-\nabla^{*'}_{J_\alpha X_k}(du(X_k))]\} \\ &+\; \epsilon_\alpha J'_\alpha odu(S)+R.\end{aligned}$$

By gathering, we have :

$$\tau(u) = \epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[\nabla^{*'}_{X_k}(J'_\alpha odu(X_k)) - \nabla^{*'}_{J_\alpha X_k}(du(X_k))$$
$$- J'_\alpha(\nabla^{*'}_{X_k}(du(X_k))) - \epsilon_\alpha J'_\alpha(\nabla^{*'}_{J_\alpha X_k}(du(J_\alpha X_k)))]\}$$
$$+ \epsilon_\alpha duoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha J'_\alpha odu(S) + R.$$

Since $\nabla^{*'}_X V = \nabla'_{du(X)}V$, where V is a C^∞-section of the induced bundle $u^{-1}(TN)$,

$$\tau(u) = \epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[\nabla'_{du(X_k)}(J'_\alpha odu(X_k)) - \nabla'_{du(J_\alpha X_k)}(du(X_k))$$
$$- J'_\alpha(\nabla'_{du(X_k)}(du(X_k))) - \epsilon_\alpha J'_\alpha(\nabla'_{du(J_\alpha X_k)}(du(J_\alpha X_k)))]\}$$
$$+ \epsilon_\alpha duoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha J'_\alpha odu(S) + R.$$

Let us call the elements in the brakets $E_i (i \in \{1, ..., 4\})$.

$E_1 + E_3 = (\nabla'_{du(X_k)} J'_\alpha)(du(X_k))$.
Since $J^2_\alpha(X_k) = -\epsilon_\alpha X_k$, we have

$$-\nabla'_{du(J_\alpha X_k)}(du(X_k)) = \epsilon_\alpha \nabla'_{du(J_\alpha X_k)}(du(J^2_\alpha X_k)) = \epsilon_\alpha \nabla'_{du(J_\alpha X_k)}(J'_\alpha(du(J_\alpha X_k))).$$

Then
$E_2 + E_4 = \epsilon_\alpha \nabla'_{du(J_\alpha X_k)}(J'_\alpha(du(J_\alpha X_k))) - \epsilon_\alpha J'_\alpha(\nabla'_{du(J_\alpha X_k)}(du(J_\alpha X_k)))$
$= \epsilon_\alpha(\nabla'_{du(J_\alpha X_k)} J'_\alpha)(du(J_\alpha X_k))$.

Putting all this together,

$$\tau(u) = \epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[(\nabla'_{du(X_k)} J'_\alpha)(du(X_k)) + \epsilon_\alpha(\nabla'_{du(J_\alpha X_k)} J'_\alpha)(du(J_\alpha X_k))]\}$$
$$+ \epsilon_\alpha duoJ_\alpha(\delta J_\alpha) + \{\epsilon_\alpha J'_\alpha odu(S) + R\}.$$

Let's evaluate $\epsilon_\alpha J'_\alpha odu(S) + R$

$$\epsilon_\alpha J'_\alpha odu(S) + R = \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta\{\epsilon_\alpha J'_\alpha odu[(\nabla J_\alpha)(J_\beta X_k, J_\beta X_k)]$$
$$+ (\nabla^{*'} du)(J_\beta X_k, J_\beta X_k)\}$$
$$= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta\{\epsilon_\alpha J'_\alpha odu[\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k) - J_\alpha(\nabla_{J_\beta X_k} J_\beta X_k)]$$
$$+ \nabla^{*'}_{J_\beta X_k}(du(J_\beta X_k)) - du(\nabla_{J_\beta X_k} J_\beta X_k)\}$$
$$= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta\{\epsilon_\alpha J'_\alpha odu[\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k)]$$
$$- \epsilon_\alpha duoJ^2_\alpha(\nabla_{J_\beta X_k} J_\beta X_k) + \nabla^{*'}_{J_\beta X_k}(du(J_\beta X_k))$$
$$- du(\nabla_{J_\beta X_k} J_\beta X_k)\}$$
$$= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta\{\epsilon_\alpha J'_\alpha odu[\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k)] + du(\nabla_{J_\beta X_k} J_\beta X_k)$$
$$+ \nabla^{*'}_{J_\beta X_k}(du(J_\beta X_k)) - du(\nabla_{J_\beta X_k} J_\beta X_k)\},$$

which gives by simplification

$$\begin{aligned}
\epsilon_\alpha J'_\alpha odu(S) + R &= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta\{\epsilon_\alpha J'_\alpha odu[\nabla_{J_\beta X_k}J_\alpha(J_\beta X_k)] + \nabla^{*'}_{J_\beta X_k}(du(J_\beta X_k))\} \\
&= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq 2}^{3}\epsilon_\beta\{\epsilon_\alpha J'_\alpha odu[\nabla_{J_\beta X_k}J_\alpha(J_\beta X_k)] \\
&- \epsilon_\alpha J'_\alpha[J'_\alpha(\nabla^{*'}_{J_\beta X_k}(du(J_\beta X_k)))]\} \\
&= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} - \epsilon_\beta\epsilon_\alpha J'_\alpha\{-du(\nabla_{J_\beta X_k}J_\alpha(J_\beta X_k)) \\
&+ J'_\alpha(\nabla'_{du(J_\beta X_k)}(du(J_\beta X_k)))\}.
\end{aligned}$$

As $(\nabla^{*'}du)(J_\beta X_k, J_\alpha(J_\beta X_k)) = \nabla^{*'}_{J_\beta X_k}du(J_\alpha(J_\beta X_k)) - du(\nabla_{J_\beta X_k}J_\alpha(J_\beta X_k))$,

$$\begin{aligned}
-du(\nabla_{J_\beta X_k}J_\alpha(J_\beta X_k)) &= -\nabla'_{du(J_\beta X_k)}du(J_\alpha(J_\beta X_k)) + (\nabla^{*'}du)(J_\beta X_k, J_\alpha(J_\beta X_k)) \\
&= -\nabla'_{du(J_\beta X_k)}J'_\alpha du(J_\beta X_k) + (\nabla^{*'}du)(J_\beta X_k, J_\alpha(J_\beta X_k)) \\
&= (\nabla^{*'}du)(J_\beta X_k, J_\alpha(J_\beta X_k)) - \nabla'_{du(J_\beta X_k)}J'_\alpha du(J_\beta X_k).
\end{aligned}$$

Then, one gets

$$\begin{aligned}
\epsilon_\alpha J'_\alpha odu(S) + R &= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} - \epsilon_\beta\epsilon_\alpha J'_\alpha\{(\nabla^{*'}du)(J_\beta X_k, J_\alpha(J_\beta X_k)) \\
&- \nabla'_{du(J_\beta X_k)}J'_\alpha du(J_\beta X_k) + J'_\alpha(\nabla'_{du(J_\beta X_k)}(du(J_\beta X_k)))\} \\
&= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} - \epsilon_\beta\epsilon_\alpha J'_\alpha\{(\nabla^{*'}du)(J_\beta X_k, J_\alpha(J_\beta X_k)) \\
&- (\nabla'_{du(J_\beta X_k)}J'_\alpha)du(J_\beta X_k)\},
\end{aligned}$$

by gathering.

And $\tau(u)$ becomes

$$\begin{aligned}
\tau(u) &= \epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[(\nabla'_{du(X_k)}J'_\alpha)(du(X_k)) + \epsilon_\alpha(\nabla'_{du(J_\alpha X_k)}J'_\alpha)(du(J_\alpha X_k))]\} \\
&+ \epsilon_\alpha duoJ_\alpha(\delta J_\alpha) + \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} - \epsilon_\beta\epsilon_\alpha J'_\alpha\{(\nabla^{*'}du)(J_\beta X_k, J_\alpha(J_\beta X_k)) \\
&- (\nabla'_{du(J_\beta X_k)}J'_\alpha)du(J_\beta X_k)\} \\
&= \epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[(\nabla'_{du(X_k)}J'_\alpha)(du(X_k)) + \epsilon_\alpha(\nabla'_{du(J_\alpha X_k)}J'_\alpha)(du(J_\alpha X_k)) \\
&+ \sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla'_{du(J_\beta X_k)}J'_\alpha)du(J_\beta X_k) - \sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla^{*'}du)(J_\beta X_k, J_\alpha(J_\beta X_k))]\} \\
&+ \epsilon_\alpha duoJ_\alpha(\delta J_\alpha).
\end{aligned}$$

Let us evaluate
$\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla^{*'}du)(J_\beta X_k, J_\alpha(J_\beta X_k)) := B$.

Suppose for example that $\alpha = 2$, we have:
$B = (\nabla^{*'}du)(J_1 X_k, J_2(J_1 X_k)) - (\nabla^{*'}du)(J_3 X_k, J_2(J_3 X_k))$. As $J_2 J_1 = -J_3$ and $J_2 J_3 = -J_1$,

$$\begin{aligned}
B &= (\nabla^{*'}du)(J_1 X_k, -J_3 X_k) - (\nabla^{*'}du)(J_3 X_k, -J_1 X_k)) \\
&= -(\nabla^{*'}du)(J_1 X_k, J_3 X_k) + (\nabla^{*'}du)(J_3 X_k, J_1 X_k)) = 0,
\end{aligned}$$

because the second fundamental form is symmetric.

Then

$$\tau(u) = \epsilon_\alpha J'_\alpha \{ \sum_{k=1}^{m} [(\nabla'_{du(X_k)} J'_\alpha)(du(X_k)) + \epsilon_\alpha (\nabla'_{du(J_\alpha X_k)} J'_\alpha)(du(J_\alpha X_k)) + \sum_{\beta=1,\beta\neq\alpha}^{3} (\nabla'_{du(J_\beta X_k)} J'_\alpha) du(J_\beta X_k)]\} + \epsilon_\alpha du o J_\alpha(\delta J_\alpha).$$

We can conclude that

$$\tau(u) = \epsilon_\alpha J'_\alpha [trac_g(du)^*(\nabla' J'_\alpha) + du(\delta J_\alpha)],$$

for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ', where $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$.

□

4.2 Lemma

Lemma 4.1. *Let $u : M \to N$ be a (σ, σ')-para-holomorphic map between almost para-quaternionic Hermitian manifolds.*
If (N, σ', h) is locally hyper quasi para-$K\ddot{a}hler$, then

$$trac_g(du)^*(\nabla' J'_\alpha) = 0,$$

for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J'_1, J'_2, J'_3\}$ of σ'

Proof. Assume (N, σ', h) is locally hyper quasi para-$K\ddot{a}hler$ ie
$(\nabla'_X J'_\alpha)Y + \epsilon_\alpha(\nabla_{J'_\alpha X} J'_\alpha)J'_\alpha Y = 0, \forall X, Y \in \chi(N)$, for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J'_1, J'_2, J'_3\}$ of σ', where $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$.

Let's recall that

$$\begin{aligned} trac_g(du)^*(\nabla' J'_\alpha) &= \sum_{k=1}^{m} [(\nabla'_{du(X_k)} J'_\alpha)(du(X_k)) + \epsilon_\alpha (\nabla'_{du(J_\alpha X_k)} J'_\alpha)(du(J_\alpha X_k)) \\ &+ \sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta (\nabla'_{du(J_\beta X_k)} J'_\alpha) du(J_\beta X_k)] \\ &= \sum_{k=1}^{m} [(\nabla'_{du(X_k)} J'_\alpha)(du(X_k)) + \epsilon_\alpha (\nabla'_{J'_\alpha du(X_k)} J'_\alpha)(J'_\alpha du(X_k)) \\ &+ \sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta (\nabla'_{J'_\beta du(X_k)} J'_\alpha) J'_\beta du(X_k)] \\ &= \sum_{k=1}^{m} \sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta (\nabla'_{J'_\beta du(X_k)} J'_\alpha) J'_\beta du(X_k), \end{aligned}$$

since u is a (σ, σ')-para-holomorphic map and the first line of the second egality vanishes because $(\nabla'_Z J'_\alpha)Y + \epsilon_\alpha(\nabla_{J'_\alpha Z} J'_\alpha)J'_\alpha Y = 0, \forall Z, Y \in \chi(N)$.
Let' s look at $\sum_{k=1}^{m} \sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta (\nabla'_{J'_\beta du(X_k)} J'_\alpha) J'_\beta du(X_k)$.
Suppose $\alpha = 1$, then

$$trac_g(du)^*(\nabla' J'_1) = -\sum_{k=1}^{m} [(\nabla'_{J'_2 du(X_k)} J'_1) J'_2 du(X_k) + (\nabla'_{J'_3 du(X_k)} J'_1) J'_3 du(X_k)].$$

As $J_3 = J_1 J_2$,

$$\begin{aligned} trac_g(du)^*(\nabla' J'_1) &= -[(\nabla'_{J'_2 du(X_k)} J'_1) J'_2 du(X_k) + (\nabla'_{J'_1(J'_2 du(X_k))} J'_1) J'_1(J'_2 du(X_k))] \\ &= 0. \end{aligned}$$

□

4.3 Proof of Corollary 3.2

Proof. : $trac_g(du)^*(\nabla' J'_\beta) = 0$, if (N, σ', h) is a locally hyper quasi para-$K\ddot{a}hler$ and $du(\delta J_\beta) = 0$, if (M, σ, g) a locally almost hyper semi-para-$K\ddot{a}hler$ manifold, for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ' and, then by virtue of proposition 3.1, u is a harmonic map. □

4.4 Proof of theorem 3.3

Proof. : A C^2-map u is a harmonic map if only if $\tau(u) = 0$ which is equivalent by virtue of proposition 3.1 to the egality $trac_g u^*(\nabla' J'_\alpha) = -du(\delta J_\alpha)$, for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ'. □

5 Results for sections

5.1 Proposition

Proposition 5.1. *Let* $[(N, \sigma', h) \stackrel{\pi}{\to} (M, \sigma, g)]$ *be a* $(semi{-}Riemannian)$ *almost para-quaternionic Hermitian fiber bundle and* $u : M \to N$ *be a* (σ, σ')*-para-holomorphic section of* $[(N, \sigma', h) \stackrel{\pi}{\to} (M, \sigma, g)]$.
Then, we have:

$$\tau^v(u) = \epsilon_\alpha J'_\alpha[trac_g(d^v u)^*(\nabla^v J'_\alpha) + (d^v u)(\delta J_\alpha) + trac_g(\nabla^v_{d^h u} J'_\alpha)(d^v u) \\ + trac_g(A_{d^h u} d^h u o J_\alpha) + trac_g(T_{d^v u} d^h u o J_\alpha)],$$

for all $\alpha \in \{1.2.3\}$*, for any canonical local basis* $\{J_1, J_2, J_3\}$ *of* σ *and corresponding local basis* $\{J'_1, J'_2, J'_3\}$ *of* σ'*, where* $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$.

Corollary 5.2. *Let* $u : M \to N$ *be a* (σ, σ')*-para-holomorphic section of a* $(semi{-}Riemannian)$ *almost para-quaternionic Hermitian fiber bundle* $[(N, \sigma', h) \stackrel{\pi}{\to} (M, \sigma, g)]$.
If the fibers are locally hyper quasi para-$K\ddot{a}hler$ *almost para-quaternionic Hermitian manifolds,* M *is a locally almost hyper semi- para-*$K\ddot{a}hler$ *almost para-quaternionic Hermitian manifold and*

$$\begin{aligned} trac_g(\nabla^v_{d^h u} J'_\alpha)(d^v u) &= 0, \\ trac_g(A_{d^h u} d^h u o J_\alpha) &= 0, \\ trac_g(T_{d^v u} d^h u o J_\alpha) &= 0, \end{aligned}$$

for all $\alpha \in \{1.2.3\}$*, for any canonical local basis* $\{J_1, J_2, J_3\}$ *of* σ *and corresponding local basis* $\{J'_1, J'_2, J'_3\}$ *of* σ'*, then u is a harmonic section.*

Corollary 5.3. *:*
Let $u : M \to N$ *be a* (σ, σ')*-para-holomorphic section of a* $(semi - Riemannian)$ *almost para-quaternionic Hermitian fiber bundle* $[(N, \sigma', h) \stackrel{\pi}{\to} (M, \sigma, g)]$.
If the fibers are totally geodesic, locally hyper quasi para-$K\ddot{a}hler$ *almost para-quaternionic Hermitian manifolds,* M *is a locally almost hyper semi-para-*$K\ddot{a}hler$ *almost para-quaternionic Hermitian manifold, the horizontal distribution is integrable and* $(\nabla^v_Y J'_\alpha)U = 0$ *for all* Y *horizontal and* U *vertical, for all* $\alpha \in \{1.2.3\}$*, for any canonical local basis* $\{J_1, J_2, J_3\}$ *of* σ *and corresponding local basis* $\{J'_1, J'_2, J'_3\}$ *of* σ'*, then u is a harmonic section.*

6 Proof of the results

6.1 Proof of proposition

Proof. Let $\{X_1, ..., X_m, J_1X_1, ..., J_1X_m, J_2X_1, ..., J_2X_m, J_3X_1, ..., J_3X_m\}$ be an orthonormal local frame of TM
such that $g(X_k, X_k) = 1 = g(J_1X_k, J_1X_k)$, and $g(J_\beta X_k, J_\beta X_k) = -1, \forall \beta \in \{2.3\}$.

$$\begin{aligned}\tau^v(u) &= \sum_{k=1}^{m}(\nabla^{*^v} d^v u)(X_k, X_k) + \sum_{k=1}^{m}(\nabla^{*^v} d^v u)(J_1 X_k, J_1 X_k)\\
&- \sum_{k=1}^{m}(\nabla^{*^v} d^v u)(J_2 X_k, J_2 X_k) - \sum_{k=1}^{m}(\nabla^{*^v} d^v u)(J_3 X_k, J_3 X_k)\\
&= \sum_{k=1}^{m}(\nabla^{*^v} d^v u)(X_k, X_k) + \epsilon_\alpha \sum_{k=1}^{m}(\nabla^{*^v} d^v u)(J_\alpha X_k, J_\alpha X_k)\\
&+ \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta (\nabla^{*^v} d^v u)(J_\beta X_k, J_\beta X_k)\\
&= \sum_{k=1}^{m}[\nabla^{*^v}_{X_k}(d^v u(X_k)) - d^v u(\nabla_{X_k} X_k)]\\
&+ \epsilon_\alpha \sum_{k=1}^{m}[\nabla^{*^v}_{J_\alpha X_k}(d^v u(J_\alpha X_k)) - d^v u(\nabla_{J_\alpha X_k} J_\alpha X_k)]\\
&+ \sum_{k=1}^{m}[\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta (\nabla^{*^v} d^v u)(J_\beta X_k, J_\beta X_k)]\\
&= \sum_{k=1}^{m}[\nabla^{*^v}_{X_k}(d^v u(X_k)) + \epsilon_\alpha \nabla^{*^v}_{J_\alpha X_k}(d^v u(J_\alpha X_k))]\\
&- d^v u[\sum_{k=1}^{m}(\nabla_{X_k} X_k + \epsilon_\alpha \nabla_{J_\alpha X_k} J_\alpha X_k)] + R,\end{aligned}$$

where we set $R = \sum_{k=1}^{m}[\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta (\nabla^{*^v} du)(J_\beta X_k, J_\beta X_k)]$.

In the map's case, we shew

$$\begin{aligned}-du(\sum_{k=1}^{m}[\nabla_{X_k} X_k + \epsilon_\alpha \nabla_{J_\alpha X_k} J_\alpha X_k]) &= \epsilon_\alpha \{duoJ_\alpha(\delta J_\alpha)\\
&+ duoJ_\alpha(\sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k])\\
&+ duoJ_\alpha(S)\},\end{aligned}$$

where $S = \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta (\nabla J_\alpha)(J_\beta X_k, J_\beta X_k)$.

Taking the vertical part, we have

$$\begin{aligned}-d^v u(\sum_{k=1}^{m}[\nabla_{X_k} X_k + \epsilon_\alpha \nabla_{J_\alpha X_k} J_\alpha X_k]) &= \epsilon_\alpha \{d^v uoJ_\alpha(\delta J_\alpha)\\
&+ d^v uoJ_\alpha(\sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k])\\
&+ d^v uoJ_\alpha(S)\}.\end{aligned}$$

$\tau^v(u)$ becomes including the fact that $d^v uoJ_\alpha = J'_\alpha o d^v u$,

$$\begin{aligned}\tau^v(u) &= \sum_{k=1}^{m}[\nabla^{*^v}_{X_k}(d^v u(X_k)) + \epsilon_\alpha \nabla^{*^v}_{J_\alpha X_k}(d^v u(J_\alpha X_k))]\\
&+ \epsilon_\alpha \{d^v uoJ_\alpha(\delta J_\alpha) + J'_\alpha d^v u(\sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k]) + d^v uoJ_\alpha(S)\} + R\\
&= \sum_{k=1}^{m}[\nabla^{*^v}_{X_k}(d^v u(X_k)) + \epsilon_\alpha \nabla^{*^v}_{J_\alpha X_k}(d^v u(J_\alpha X_k))]\\
&+ \epsilon_\alpha d^v uoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha J'_\alpha d^v u(\sum_{k=1}^{m}[\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k]) + \epsilon_\alpha d^v uoJ_\alpha(S)\\
&+ R.\end{aligned}$$

We shew in the map's case that

$$du(\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k) = \nabla^{*'}_{X_k}(du(J_\alpha X_k)) - \nabla^{*'}_{J_\alpha X_k}(du(X_k)),$$

If we Take the vertical part,

$$\begin{aligned} d^v u(\nabla_{X_k} J_\alpha X_k - \nabla_{J_\alpha X_k} X_k) &= \nabla^{*^v}_{X_k}(du(J_\alpha X_k)) - \nabla^{*^v}_{J_\alpha X_k}(du(X_k)) \\ &= \nabla^{*^v}_{X_k}(d^v u(J_\alpha X_k) + d^h u(J_\alpha X_k)) \\ &- \nabla^{*^v}_{J_\alpha X_k}(d^v u(X_k) + d^h u(X_k)) \\ &= \nabla^{*^v}_{X_k}(d^v u(J_\alpha X_k)) + \nabla^{*^v}_{X_k}(d^h u(J_\alpha X_k)) \\ &- \nabla^{*^v}_{J_\alpha X_k}(d^v u(X_k)) - \nabla^{*^v}_{J_\alpha X_k}(d^h u(X_k)) \end{aligned}$$

This leads to

$$\begin{aligned} \tau^v(u) = &\sum_{k=1}^{m} [\nabla^{*^v}_{X_k}(d^v u(X_k)) + \epsilon_\alpha \nabla^{*^v}_{J_\alpha X_k}(d^v u(J_\alpha X_k))] \\ &+ \epsilon_\alpha d^v uoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha J'_\alpha\{\nabla^{*^v}_{X_k}(d^v u(J_\alpha X_k)) + \nabla^{*^v}_{X_k}(d^h u(J_\alpha X_k)) \\ &- \nabla^{*^v}_{J_\alpha X_k}(d^v u(X_k)) - \nabla^{*^v}_{J_\alpha X_k}(d^h u(X_k))\} + \varepsilon_\alpha d^v uoJ_\alpha(S) + R. \end{aligned}$$

Let us set Z, the sum of the first two elements of $\tau(u)$ ie
$Z = \sum_{k=1}^{m} [\nabla^{*^v}_{X_k}(d^v u(X_k)) + \epsilon_\alpha \nabla^{*^v}_{J_\alpha X_k}(d^v u(J_\alpha X_k))]$.
With $J'^2_\alpha(Z) = -\varepsilon_\alpha Z$, we have :

$$Z = -\epsilon_\alpha J'_\alpha\{J'_\alpha\{\sum_{k=1}^{m} [\nabla^{*^v}_{X_k}(d^v u(X_k)) + \epsilon_\alpha \nabla^{*^v}_{J_\alpha X_k}(d^v u(J_\alpha X_k))]\}\}.$$

So $\tau^v(u)$ turns to

$$\begin{aligned} \tau^v(u) = &-\epsilon_\alpha J'_\alpha\{J'_\alpha\{\sum_{k=1}^{m} [\nabla^{*^v}_{X_k}(d^v u(X_k)) + \epsilon_\alpha \nabla^{*^v}_{J_\alpha X_k}(d^v u(J_\alpha X_k))]\}\} \\ &+ \epsilon_\alpha d^v uoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha J'_\alpha\{\nabla^{*^v}_{X_k}(d^v u(J_\alpha X_k)) + \nabla^{*^v}_{X_k}(d^h u(J_\alpha X_k)) \\ &- \nabla^{*^v}_{J_\alpha X_k}(d^v u(X_k)) - \nabla^{*^v}_{J_\alpha X_k}(d^h u(X_k))\} + \varepsilon_\alpha d^v uoJ_\alpha(S) + R. \\ &= \epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m} [-J'_\alpha(\nabla^{*^v}_{X_k}(d^v u(X_k))) - \epsilon_\alpha J'_\alpha(\nabla^{*^v}_{J_\alpha X_k}(d^v u(J_\alpha X_k)))]\} \\ &+ \epsilon_\alpha d^v uoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha J'_\alpha\{\nabla^{*^v}_{X_k}(d^v u(J_\alpha X_k)) + \nabla^{*^v}_{X_k}(d^h u(J_\alpha X_k)) \\ &- \nabla^{*^v}_{J_\alpha X_k}(d^v u(X_k)) - \nabla^{*^v}_{J_\alpha X_k}(d^h u(X_k))\} + \varepsilon_\alpha d^v uoJ_\alpha(S) + R. \end{aligned}$$

By gatering, we have :

$$\begin{aligned} \tau^v(u) = \epsilon_\alpha J'_\alpha\{&\sum_{k=1}^{m} [\nabla^{*^v}_{X_k}(d^v u(J_\alpha X_k)) + \nabla^{*^v}_{X_k}(d^h u(J_\alpha X_k)) \\ &- \nabla^{*^v}_{J_\alpha X_k}(d^v u(X_k)) - \nabla^{*^v}_{J_\alpha X_k}(d^h u(X_k)) \\ &- J'_\alpha(\nabla^{*^v}_{X_k}(d^v u(X_k))) - \epsilon_\alpha J'_\alpha(\nabla^{*^v}_{J_\alpha X_k}(d^v u(J_\alpha X_k)))]\} \\ &+ \epsilon_\alpha d^v uoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha d^v uoJ_\alpha(S) + R. \end{aligned}$$

From relation (2.1) and $d^v uoJ_\alpha = J'_\alpha od^v u$, it comes that

$$\tau^v(u) = \epsilon_\alpha J'_\alpha\{\sum_{k=1}^m[\nabla^v_{du(X_k)}(J'_\alpha(d^v\widetilde{u(X_k)})) + \nabla^v_{du(X_k)}(d^h u(\widetilde{J_\alpha X_k}))$$
$$- \nabla^v_{du(J_\alpha X_k)}(d^v\widetilde{u(X_k)}) - \nabla^v_{du(J_\alpha X_k)}(d^h\widetilde{u(X_k)})$$
$$- J'_\alpha(\nabla^v_{du(X_k)}(d^v\widetilde{u(X_k)})) - \epsilon_\alpha J'_\alpha(\nabla^v_{du(J_\alpha X_k)}(d^v u(\widetilde{J_\alpha X_k})))]\}$$
$$+ \epsilon_\alpha d^v uoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha d^v uoJ_\alpha(S) + R.$$

Let us call the elements in the brakets $E_i (i \in \{1, ..., 6\})$.
$E_1 + E_5 = (\nabla^v_{du(X_k)}J'_\alpha)(d^v\widetilde{u(X_k)})$.

Since $J^2_\alpha X_k = -\epsilon_\alpha X_k$, we have
$-\nabla^v_{du(J_\alpha X_k)}(d^v\widetilde{u(X_k)}) = \epsilon_\alpha\nabla^v_{du(J_\alpha X_k)}(d^v u(\widetilde{J^2_\alpha X_k}))$
$= \epsilon_\alpha\nabla^v_{du(J_\alpha X_k)}J'_\alpha(d^v\widetilde{u(J_\alpha X_k)})$.
Then $E_3 + E_6 = \epsilon_\alpha\nabla^v_{du(J_\alpha X_k)}(J'_\alpha(d^v\widetilde{u(J_\alpha X_k)})) - \epsilon_\alpha J'_\alpha(\nabla^v_{du(JX_k)}(d^v u(\widetilde{JX_k})))$
$= \epsilon_\alpha(\nabla^v_{du(J_\alpha X_k)}J'_\alpha)(d^v u(\widetilde{J_\alpha X_k}))$. With this,

$$\tau^v(u) = \epsilon_\alpha J'_\alpha\sum_{k=1}^m[(\nabla^v_{du(X_k)}J'_\alpha)(d^v\widetilde{u(X_k)}) + \epsilon_\alpha(\nabla^v_{du(J_\alpha X_k)}J'_\alpha)(d^v u(\widetilde{J_\alpha X_k}))$$
$$+ \nabla^v_{du(X_k)}(d^h u(\widetilde{J_\alpha X_k})) - \nabla^v_{du(J_\alpha X_k)}(d^h\widetilde{u(X_k)})]\}$$
$$+ \epsilon_\alpha d^v uoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha d^v uoJ_\alpha(S) + R.$$

As $du(Y) = d^v u(Y) + d^h u(Y)$,

$$\tau^v(u) = \epsilon_\alpha J'_\alpha\{\sum_{k=1}^m[(\nabla^v_{d^v u(X_k)}J'_\alpha)(d^v\widetilde{u(X_k)}) + (\nabla^v_{d^h u(X_k)}J'_\alpha)(d^v\widetilde{u(X_k)})$$
$$+ \epsilon_\alpha(\nabla^v_{d^v u(J_\alpha X_k)}J'_\alpha)(d^v u(\widetilde{J_\alpha X_k})) + \epsilon_\alpha(\nabla^v_{d^h u(J_\alpha X_k)}J'_\alpha)(d^v u(\widetilde{J_\alpha X_k}))$$
$$+ \nabla^v_{d^v u(X_k)}(d^h u(\widetilde{J_\alpha X_k})) + \nabla^v_{d^h u(X_k)}(d^h u(\widetilde{J_\alpha X_k}))$$
$$- \nabla^v_{d^v u(J_\alpha X_k)}(d^h\widetilde{u(X_k)}) - \nabla^v_{d^h u(J_\alpha X_k)}(d^h\widetilde{u(X_k)})]\}$$
$$+ \epsilon_\alpha d^v uoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha d^v uoJ_\alpha(S) + R.$$

Using the fondamental tensors T and A,

$$\tau^v(u) = \epsilon_\alpha J'_\alpha\{\sum_{k=1}^m[(\nabla^v_{d^v u(X_k)}J'_\alpha)(d^v\widetilde{u(X_k)}) + (\nabla^v_{d^h u(X_k)}J'_\alpha)(d^v\widetilde{u(X_k)})$$
$$+ \epsilon_\alpha(\nabla^v_{d^v u(J_\alpha X_k)}J'_\alpha)(d^v u(\widetilde{J_\alpha X_k})) + \epsilon_\alpha(\nabla^v_{d^h u(J_\alpha X_k)}J'_\alpha)(d^v u(\widetilde{J_\alpha X_k}))$$
$$+ T_{d^v u(X_k)}(d^h u(\widetilde{J_\alpha X_k})) + A_{d^h u(X_k)}(d^h u(\widetilde{J_\alpha X_k}))$$
$$- T_{d^v u(J_\alpha X_k)}(d^h\widetilde{u(X_k)}) - A_{d^h u(J_\alpha X_k)}(d^h\widetilde{u(X_k)})]\}$$
$$+ \epsilon_\alpha d^v uoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha d^v uoJ_\alpha(S) + R.$$

Using one more time $J^2_\alpha X_k = -\epsilon_\alpha X_k$, in the fourth line,

$$\tau^v(u) = \epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[(\nabla^v_{d^v u(X_k)} J'_\alpha)(d^v \widetilde{u(X_k)}) + (\nabla^v_{d^h u(X_k)} J'_\alpha)(d^v \widetilde{u(X_k)})$$
$$+ \epsilon_\alpha(\nabla^v_{d^v u(J_\alpha X_k)} J'_\alpha)(d^v u(\widetilde{J_\alpha X_k})) + \epsilon_\alpha(\nabla^v_{d^h u(J_\alpha X_k)} J'_\alpha)(d^v u(\widetilde{J_\alpha X_k}))$$
$$+ T_{d^v u(X_k)}(d^h u(\widetilde{J_\alpha X_k})) + A_{d^h u(X_k)}(d^h u(\widetilde{J_\alpha X_k}))$$
$$+ \epsilon_\alpha T_{d^v u(J_\alpha X_k)}(d^h uo\widetilde{J_\alpha(J_\alpha X_k)}) + \epsilon_\alpha A_{d^h u(J_\alpha X_k)}(d^h uo\widetilde{J_\alpha(J_\alpha X_k)})]\}$$
$$+ \epsilon_\alpha d^v uoJ_\alpha(\delta J_\alpha) + \epsilon_\alpha d^v uoJ_\alpha(S) + R.$$

Let evaluate $\epsilon_\alpha J' od^v u(S) + R$,

$$\epsilon_\alpha J' od^v u(S) + R = \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta\{\epsilon_\alpha J'_\alpha od^v u[(\nabla J_\alpha)(J_\beta X_k, J_\beta X_k)]$$
$$+ (\nabla^{*^v} d^v u)(J_\beta X_k, J_\beta X_k)\}$$
$$= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta\{\epsilon_\alpha J'_\alpha od^v u[\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k) - J_\alpha(\nabla_{J_\beta X_k} J_\beta X_k)]$$
$$+ \nabla^{*^v}_{J_\beta X_k}(d^v u(J_\beta X_k)) - d^v u(\nabla_{J_\beta X_k} J_\beta X_k)\}$$
$$= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta\{\epsilon_\alpha J'_\alpha od^v u[\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k)]$$
$$- \epsilon_\alpha d^v uoJ^2_\alpha(\nabla_{J_\beta X_k} J_\beta X_k)$$
$$+ \nabla^{*^v}_{J_\beta X_k}(d^v u(J_\beta X_k)) - d^v u(\nabla_{J_\beta X_k} J_\beta X_k)\}$$
$$= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta\{\epsilon_\alpha J'_\alpha od^v u[\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k)] + d^v u(\nabla_{J_\beta X_k} J_\beta X_k)$$
$$+ \nabla^{*^v}_{J_\beta X_k}(d^v u(J_\beta X_k)) - d^v u(\nabla_{J_\beta X_k} J_\beta X_k)\},$$

which gives by simplification

$$\epsilon_\alpha J'_\alpha od^v u(S) + R = \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\beta}^{3} \epsilon_\beta\{\epsilon_\alpha J'_\alpha od^v u[\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k)] + \nabla^{*^v}_{J_\beta X_k}(d^v u(J_\beta X_k))\}$$
$$= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta\{\epsilon_\alpha J'_\alpha od^v u[\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k)]$$
$$- \epsilon_\alpha J'_\alpha[J'_\alpha(\nabla^{*^v}_{J_\beta X_k}(d^v u(J_\beta X_k)))]\}$$
$$= \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3} - \epsilon_\beta\epsilon_\alpha J'_\alpha\{-d^v u(\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k))$$
$$+ J'_\alpha(\nabla^{*^v}_{J_\beta X_k}(d^v u(J_\beta X_k)))\}.$$

As $(\nabla^{*'} du)(J_\beta X_k, J(J_\beta X_k)) = \nabla^{*'}_{J_\beta X_k} du(J_\alpha(J_\beta X_k)) - du(\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k))$,

$$-\sum_{\beta=1,\beta\neq\alpha}^{3} du(\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k)) = -\sum_{\beta=1,\beta\neq\alpha}^{3} \nabla^{*'}_{J_\beta X_k} du(J_\alpha(J_\beta X_k))$$
$$+ \sum_{\beta=1,\beta\neq\alpha}^{3} (\nabla^{*'} du)(J_\beta X_k, J_\alpha(J_\beta X_k)).$$

In the map's case, we shew that $\sum_{\beta=1,\beta\neq\alpha}^{3} (\nabla^{*'} du)(J_\beta X_k, J_\alpha(J_\beta X_k)) = 0$ because of the symmetry of the second fondamental form, then

$$-\sum_{\beta=1,\beta\neq\alpha}^{3} du(\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k)) = -\sum_{\beta=1,\beta\neq\alpha}^{3} \nabla^{*'}_{J_\beta X_k} du(J_\alpha(J_\beta X_k))$$

Taking the vertical part, we have

$$\begin{aligned}
-\sum_{\beta=1,\beta\neq\alpha}^{3} d^v u(\nabla_{J_\beta X_k} J_\alpha(J_\beta X_k)) &= -\sum_{\beta=1,\beta\neq\alpha}^{3} \nabla^{*^v}_{J_\beta X_k} du(J_\alpha(J_\beta X_k)) \\
&= \sum_{\beta=1,\beta\neq\alpha}^{3} [-\nabla^{*^v}_{J_\beta X_k} d^v u(J_\alpha(J_\beta X_k)) \\
&- \nabla^{*^v}_{J_\beta X_k} d^h u(J_\alpha(J_\beta X_k))].
\end{aligned}$$

And then, we have

$$\begin{aligned}
\epsilon_\alpha J'_\alpha o d^v u(S) + R = \sum_{k=1}^{m} \sum_{\beta=1,\beta\neq\alpha}^{3} &- \epsilon_\beta \epsilon_\alpha J'_\alpha \{-\nabla^{*^v}_{J_\beta X_k} d^v u(J_\alpha(J_\beta X_k)) \\
&- \nabla^{*^v}_{J_\beta X_k} d^h u(J_\alpha(J_\beta X_k)) + J'_\alpha(\nabla^{*^v}_{J_\beta X_k}(d^v u(J_\beta X_k)))\}.
\end{aligned}$$

And with relation (2.1)

$$\begin{aligned}
\epsilon_\alpha J'_\alpha o d^v u(S) + R = \sum_{k=1}^{m} \sum_{\beta=1,\beta\neq\alpha}^{3} &\epsilon_\beta \epsilon_\alpha J'_\alpha \{\nabla^v_{du(J_\beta X_k)}(d^v u(J_\alpha \widetilde{(J_\beta X_k)})) \\
&+ \nabla^v_{du(J_\beta X_k)}(d^h u(J_\alpha \widetilde{(J_\beta X_k)})) - J'_\alpha(\nabla^v_{du(J_\beta X_k)}(d^v u(\widetilde{J_\beta} X_k)))\}.
\end{aligned}$$

Since $d^v u o J_\alpha = J'_\alpha o d^v u$, the first element and the third can be gather, then

$$\begin{aligned}
\epsilon_\alpha J'_\alpha o d^v u(S) + R &= \sum_{k=1}^{m} \sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta \epsilon_\alpha J'_\alpha \{(\nabla^v_{du(J_\beta X_k)} J'_\alpha)(d^v u((\widetilde{J_\beta} X_k))) \\
&+ \nabla^v_{du(J_\beta X_k)}(d^h u(J\widetilde{(J_\beta X_k)}))\} \\
&= \sum_{k=1}^{m} \sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\alpha \epsilon_\alpha J'_\alpha \{(\nabla^v_{d^v u(J_\beta X_k)} J'_\alpha)(d^v u((\widetilde{J_\beta} X_k))) \\
&+ (\nabla^v_{d^h u(J_\beta X_k)} J'_\alpha)(d^v u((\widetilde{J_\beta} X_k))) + \nabla^v_{d^v u(J_\beta X_k)}(d^h u(J_\alpha \widetilde{(J_\beta X_k)})) \\
&+ \nabla^v_{d^h u(J_\beta X_k)}(d^h u(J_\alpha \widetilde{(J_\beta X_k)}))\},
\end{aligned}$$

since $du(Y) = d^v u(Y) + d^h u(Y)$.
And with the fondamentals tensors T and A

$$\begin{aligned}
\epsilon_\alpha J' o d^v u(S) + R &= \sum_{k=1}^{m} \sum_{\beta=1,\beta\neq\alpha}^{3} \epsilon_\beta \epsilon_\alpha J'_\alpha \{(\nabla^v_{d^v u(J_\beta X_k)} J'_\alpha)(d^v u((\widetilde{J_\beta} X_k))) \\
&+ (\nabla^v_{d^h u(J_\beta X_k)} J'_\alpha)(d^v u((\widetilde{J_\beta} X_k))) + T_{d^v u(J_\beta X_k)}(d^h u(J_\alpha \widetilde{(J_\beta X_k)})) \\
&+ A_{d^h u(J_\beta X_k)}(d^h u(J_\alpha \widetilde{(J_\beta X_k)}))\}.
\end{aligned}$$

So $\tau^v(u)$ becomes

$$\begin{aligned}\tau^v(u) = \epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[(\nabla^v_{d^v u(X_k)}J'_\alpha)(d^v u\widetilde{(X_k)}) + (\nabla^v_{d^h u(X_k)}J'_\alpha)(d^v u\widetilde{(X_k)})\\
+ \epsilon_\alpha(\nabla^v_{d^v u(J_\alpha X_k)}J'_\alpha)(d^v u(\widetilde{J_\alpha X_k})) + \epsilon_\alpha(\nabla^v_{d^h u(J_\alpha X_k)}J'_\alpha)(d^v u(\widetilde{J_\alpha X_k}))\\
+ T_{d^v u(X_k)}(d^h u(\widetilde{J_\alpha X_k})) + A_{d^h u(X_k)}(d^h u(\widetilde{J_\alpha X_k}))\\
+ \epsilon_\alpha T_{d^v u(J_\alpha X_k)}(d^h uoJ_\alpha\widetilde{(J_\alpha X_k)}) + \epsilon_\alpha A_{d^h u(J_\alpha X_k)}(d^h uoJ_\alpha\widetilde{(J_\alpha X_k)})]\}\\
+ \epsilon_\alpha J'_\alpha d^v u(\delta J_\alpha) + \sum_{k=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta\epsilon_\alpha J'_\alpha\{(\nabla^v_{d^v u(J_\beta X_k)}J'_\alpha)(d^v u((\widetilde{J_\beta X_k})))\\
+ (\nabla^v_{d^h u(J_\beta X_k)}J'_\alpha)(d^v u((\widetilde{J_\beta X_k}))) + T_{d^v u(J_\beta X_k)}(d^h u(J_\alpha\widetilde{(J_\beta X_k)}))\\
+ A_{d^h u(J_\beta X_k)}(d^h u(J_\alpha\widetilde{(J_\beta X_k)}))\},\end{aligned}$$

Finaly, one gets

$$\begin{aligned}\tau^v(u) = \epsilon_\alpha J'_\alpha\{\sum_{k=1}^{m}[(\nabla^v_{d^v u(X_k)}J'_\alpha)(d^v u\widetilde{(X_k)}) + \epsilon_\alpha(\nabla^v_{d^v u(J_\alpha X_k)}J'_\alpha)(d^v u(\widetilde{J_\alpha X_k}))\\
+ \sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla^v_{d^v u(J_\beta X_k)}J'_\alpha)(d^v u(\widetilde{J_\beta X_k}))\\
+ (\nabla^v_{d^h u(X_k)}J'_\alpha)(d^v u\widetilde{(X_k)}) + \epsilon_\alpha(\nabla^v_{d^h u(J_\alpha X_k)}J'_\alpha)(d^v u(\widetilde{J_\alpha X_k}))\\
+ \sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta(\nabla^v_{d^h u(J_\beta X_k)}J'_\alpha)(d^v u(\widetilde{J_\beta X_k}))\\
+ T_{d^v u(X_k)}(d^h u(\widetilde{J_\alpha X_k})) + \epsilon_\alpha T_{d^v u(J_\alpha X_k)}(d^h uoJ_\alpha\widetilde{(J_\alpha X_k)})\\
+ \sum_{\beta=1,\beta\neq\alpha}^{3}\epsilon_\beta T_{d^v u(J_\alpha X_k)}(d^h uoJ_\alpha\widetilde{(J_\alpha X_k)})\\
+ A_{d^h u(X_k)}(d^h u(\widetilde{J_\alpha X_k})) + \epsilon_\alpha A_{d^h u(J_\alpha X_k)}(d^h uoJ_\alpha\widetilde{(J_\alpha X_k)})\\
+ \sum_{\beta=1,\beta\neq\alpha}^{3} A_{d^h u(J_\beta X_k)}(d^h uoJ_\alpha\widetilde{(J_\beta X_k)})]\\
+ d^v u(\delta J_\alpha)\}.\end{aligned}$$

We can conclude that

$$\begin{aligned}\tau^v(u) = \epsilon_\alpha J'_\alpha[trac_g(d^v u)^*(\nabla^v J'_\alpha) + (d^v u)(\delta J_\alpha)\\
+ trac_g(\nabla^v_{d^h u}J'_\alpha)(d^v u) + trac_g(A_{d^h u}d^h uoJ_\alpha) + trac_g(T_{d^v u}d^h uoJ_\alpha)],\end{aligned}$$

for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ', where $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$.

□

6.2 Proof of Corollary 5.2.

Proof. :
Let $u : M \to N$ be a (σ, σ')-para-holomorphic section of a $(semi - Riemannian)$ almost para-quaternionic Hermitian fiber bundle, $[(N, \sigma', h) \xrightarrow{\pi} (M, \sigma, g)]$.
Assume $trac_g(\nabla^v_{d^h u}\varphi)(d^v u) = 0, trac_g(A_{d^h u}d^h uoJ) = 0$ and $trac_g(T_{d^v u}d^h uoJ) = 0$, then the formula in proposition 5.1 gives $\tau^v(u) = \epsilon_\alpha J'_\alpha[trac_g(d^v u)^*(\nabla^v J'_\alpha) + d^v u(\delta J_\alpha)]$. Morever if

M is a locally almost hyper semi-para-$Kähler$ almost para-quaternionic Hermitian manifold (ie $\delta J_\alpha = 0$) and the fibers are locally hyper quasi para-$Kähler$ almost para-quaternionic Hermitian manifolds, which implies $trac_g(d^v u)^*(\nabla^v J'_\alpha) = 0$, for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ', where $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$. Then one gets $\tau^v(u) = 0$, so u is a harmonic section. □

6.3 Proof of Corollary 5.3

Proof. :
Let $u : M \to N$ be a (σ, σ')-para-holomorphic section of a $(semi - Riemannian)$ almost para-quaternionic Hermitian fiber bundle, $[(N, \sigma', h) \xrightarrow{\pi} (M, \sigma, g)]$.
Assume the fibers are locally hyper quasi para-$Kähler$ almost para-quaternionic Hermitian manifolds, and M is a locally hyper almost semi-para-$Kähler$ almost para-quaternionic Hermitian manifold, then the formula in proposition 5.1 gives $\tau^v(u) = \epsilon_\alpha J'_\alpha[trac_g(\nabla^v_{d^h u} J'_\alpha)(d^v u) + trac_g(A_{d^h u} d^h uoJ_\alpha) + trac_g(T_{d^v u} d^h uoJ_\alpha)]$. Morever if the fibers are totally geodesic (ie $T = 0$), the horizontal distribution is integrable (ie $A = 0$) and $(\nabla^v_Y J'_\alpha)U = 0$ for all Y horizontal and U vertical, for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ', where $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$.

Then one gets $\tau^v(u) = 0$, so u is a harmonic section.

□

7 Examples

Before giving the first example let's give two theorems

First let's notice that
$\forall B \in \chi(N), J'^v B := (J'B)^v = (J'B^h + J'B^v)^v = (J'B^v)^v = J'^v B^v = J'B^v$,since the vertical and the horizontal distributions are φ-invariant.
In the same way, $J'^h B := (J'B)^h = (J'B^h + J'B^v)^h = (J'B^h)^h = J'^h B^h = J'B^h$.

7.1 Theorem

Theorem 7.1. *Let $[(N, \sigma', h) \xrightarrow{\pi} (M, \sigma, g)]$ be a $(semi-Riemannian)$ almost para-quaternionic Hermitian fiber bundle such that the total space is a locally hyper quasi para-$Kähler$ almost para-quaternionic Hermitian manifold.*

1- Let X, Y and Z be horizontal vector fields, then
a) $A_X J_\alpha X = 0$
b) $A_X J_\alpha Y = -A_Y J_\alpha X$
c) $A = 0$.

2- Let U, V and W be vertical vector fields, then
a) $T_U U = -\epsilon_\alpha T_{J'U} J'U$
b) $T_U V = -\epsilon_\alpha T_{J'U} J'V$
c) $T = 0$ *(ie The fibers are totally geodesic submanifolds).*

Proof

The hyper quasi para-$Kähler$ strcture gives

$$\begin{aligned}
(\nabla_X J'_\alpha)Y &= -\epsilon_\alpha(\nabla_{J'_\alpha X} J'_\alpha)J'_\alpha Y \\
\nabla_X J'_\alpha Y - J'_\alpha(\nabla_X Y) &= -\epsilon_\alpha\{\nabla_{J'_\alpha X} J'^2_\alpha Y - J'_\alpha(\nabla_{J'_\alpha X} J'_\alpha Y)\} \\
&= -\epsilon_\alpha\{-\epsilon_\alpha \nabla_{J'_\alpha X} Y - J'_\alpha(\nabla_{J'_\alpha X} J'_\alpha Y)\} \\
&= \nabla_{J'_\alpha X} Y + \epsilon_\alpha J'_\alpha(\nabla_{J'_\alpha X} J'_\alpha Y)
\end{aligned}$$

Taking the vertical part, we get

$$A_X J'_\alpha Y - J'_\alpha(A_X Y) \quad = \quad A_{J'_\alpha X} Y + \epsilon_\alpha J'_\alpha(A_{J'_\alpha X} J'_\alpha Y).$$

If we make $Y = X$

$$\begin{aligned} A_X J'_\alpha X - J'_\alpha(A_X X) &= A_{J'_\alpha X} X + \epsilon_\alpha J'_\alpha(A_{J'_\alpha X} J'_\alpha X) \\ A_X J'_\alpha X &= A_{J'_\alpha X} X \\ A_X J'_\alpha X &= -A_X J'_\alpha X \\ 2A_X J'_\alpha X &= 0 \\ A_X J'_\alpha X &= 0, \end{aligned}$$

since the antisymmetry of fondamental tensor A implies $A_Z Z = 0$.

Let show b) ie $A_X J_\alpha Y = -A_Y J_\alpha X$.
With a) we have

$$\begin{aligned} 0 &= A_{(X-Y)} J'_\alpha(X - Y) \\ &= A_X J'_\alpha X - A_X J'_\alpha Y - A_Y J'_\alpha X + A_Y J'_\alpha Y \\ &= -A_X J'_\alpha Y - A_Y J'_\alpha X, \end{aligned}$$

then $A_X J'_\alpha Y = -A_Y J'_\alpha X$.

Let show c) ie $A = 0$.

With b) we have $A_X J_\alpha Z = -A_Z J_\alpha X$. If we set $Z = J'_\alpha Y$,

$$\begin{aligned} A_X J'^2_\alpha Y &= -A_{J_\alpha Y} J_\alpha X \\ -\epsilon_\alpha A_X Y &= -A_{J_\alpha Y} J_\alpha X \\ A_X Y &= \epsilon_\alpha A_{J_\alpha Y} J_\alpha X, \end{aligned}$$

for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J'_1, J'_2, J'_3\}$ of σ', where $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$, which leads to

$$\begin{aligned} A_X Y &= -A_{J'_2 Y} J'_2 X \\ &= -A_{J'_1 J'_2 X} J'_1 J'_2 Y \\ &= -A_{J'_3 X} J'_3 Y \\ &= -(-A_Y X) \\ &= A_Y X \\ &= -A_X Y, \end{aligned}$$

since $J'_1 J'_2 = J'_3$.
Then $A_X Y = 0$, for all X and Y horizontal which means that $A = 0$ (see prop. 2.7 e) in [3]).

Let' s show 2-a)

The hyper quasi para-$K\ddot{a}hler$ strcture gives

$$\nabla_U J'_\alpha V - J'_\alpha(\nabla_U V) \quad = \quad \nabla_{J'_\alpha U} V + \epsilon_\alpha J'_\alpha(\nabla_{J'_\alpha U} J'_\alpha V)$$

Taking the horizontal part, we get

$$T_U J'_\alpha V - J'_\alpha(T_U V) = T_{J'_\alpha U} V + \epsilon_\alpha J'_\alpha(T_{J'_\alpha U} J'_\alpha V).$$

If we make $V = U$

$$T_U J'_\alpha U - J'_\alpha(T_U U) = T_{J'_\alpha U} U + \epsilon_\alpha J'_\alpha(T_{J'_\alpha U} J'_\alpha U)$$

since the fondamental tensor T is symmetric.

$$\begin{aligned} -J'_\alpha(T_U U) &= \epsilon_\alpha J'_\alpha(T_{J'_\alpha U} J'_\alpha U) \\ T_U U &= -\epsilon_\alpha T_{J'_\alpha U} J'_\alpha U \end{aligned}$$

Let' s show b) ie $T_U V = -\epsilon_\alpha T_{J'_\alpha U} J'_\alpha V$

$$\begin{aligned} T_{(U-V)}(U-V) &= -\epsilon_\alpha T_{J'_\alpha(U-V)} J'_\alpha(U-V) \\ T_U U - T_U V - T_V U + T_V V &= -\epsilon_\alpha\{T_{J'_\alpha U} J'_\alpha U - T_{J'_\alpha U} J'_\alpha V - T_{J'_\alpha V} J'_\alpha U \\ &+ T_{J'_\alpha V} J'_\alpha V\} \\ -T_U V - T_V U &= -\epsilon_\alpha\{-T_{J'_\alpha U} J'_\alpha V - T_{J'_\alpha V} J'_\alpha U\} \\ -2T_U V &= 2\epsilon_\alpha T_{J'_\alpha U} J'_\alpha V \\ T_U V &= -\epsilon_\alpha T_{J'_\alpha U} J'_\alpha V. \end{aligned}$$

for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J'_1, J'_2, J'_3\}$ of σ', where $\epsilon_1 = 1 = -\epsilon_2 = -\epsilon_3$.

Let' s show c) ie $T = 0$.

b) leads to

$$\begin{aligned} T_U V &= T_{J'_2 U} J'_2 V \\ &= -T_{J'_1 J'_2 U} J'_1 J'_2 V \\ &= -T_{J'_3 U} J'_3 V \\ &= -T_U V, \end{aligned}$$

since $J'_1 J'_2 = J'_3$.
Then $T_U V = 0$, for all U and V vertical, which means that $T = 0$ (see prop. 2.7 d) in [3]).

7.2 Theorem

Theorem 7.2. *Let* $[(N, \sigma', h) \xrightarrow{\pi} (M, \sigma, g)]$ *be a* H $(semi-Riemannian)$ *almost para-quaternionic ermitian fiber bundle. If the total space is locally hyper quasi para-*$K\ddot{a}hler$ *almost para-quaternionic Hermitian manifold, then*

1) the base space is locally hyper quasi para-$K\ddot{a}hler$ *almost para-quaternionic Hermitian manifold.*

2)The fibers are locally hyper quasi para-$K\ddot{a}hler$ *almost para-quaternionic Hermitian manifolds.*

Proof

Let us show 1).
For a semi Riemaniann submersion, If X' and Y' are basic vector field π-related respectively to

X and Y, then $\nabla^h_{X'}Y'$ is π-related to $\nabla_X Y$.
For a $(semi-Riemannian)$ almost para-quaternionic Hermitian fiber bundle if X' is π-related to X, then $J'_\alpha X'$ is the basic vector field π-related to $J_\alpha X$, for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ'.
In fact $d\pi J'_\alpha X' = J_\alpha d\pi X' = J_\alpha X$, since π is (σ', σ)-para-holomorphic.

In the following we denote by the symbol $\curvearrowright$ fact to be π-related to.

$\nabla^h_{X'}Y' \curvearrowright \nabla_X Y$ and $J'_\alpha X' \curvearrowright J_\alpha X$ lead to the following relations
$\nabla^h_{X'}J'_\alpha Y' \curvearrowright \nabla_X J_\alpha Y$ and $J'_\alpha(\nabla^h_{X'}Y') \curvearrowright J(\nabla_X Y)$, which the difference gives the relation $(\nabla^h_{X'}J'_\alpha)Y' \curvearrowright (\nabla_X J_\alpha)Y$ which itself leads to $(\nabla^h_{J'_\alpha X'}J'_\alpha)J'_\alpha Y' \curvearrowright (\nabla_{J_\alpha X}J_\alpha)J_\alpha Y$.
From both the last relations we derive

$(\nabla^h_{X'}J'_\alpha)Y' + \epsilon_\alpha(\nabla^h_{J'_\alpha X'}J'_\alpha)J'_\alpha Y' \curvearrowright (\nabla_X J_\alpha)Y + \epsilon_\alpha(\nabla_{J_\alpha X}J_\alpha)J_\alpha Y$.
So If the total space is locally quasi hyper para-$K\ddot{a}hler$ almost para-quaternionic Hermitian manifold, then the base space is locally hyper quasi para-$K\ddot{a}hler$ almost para-quaternionic Hermitian manifold,

Let's now show 2)

$$\begin{aligned}
(\nabla_U J'_\alpha)V &= -\epsilon_\alpha(\nabla_{J'_\alpha U}J'_\alpha)J'_\alpha V \\
\nabla_U J'_\alpha V - J'_\alpha(\nabla_U V) &= -\epsilon_\alpha\{\nabla_{J'_\alpha U}J'^2_\alpha V - J'_\alpha(\nabla_{J'_\alpha U}J'_\alpha V)\} \\
\nabla_U J^v_\alpha V - J'_\alpha(\nabla_U V) &= -\epsilon_\alpha\{\nabla_{J^v_\alpha U}J^{v^2}_\alpha V - J'_\alpha(\nabla_{J^v_\alpha U}J^v_\alpha V)\}
\end{aligned}$$

By taking the vertical part, we get

$$\begin{aligned}
\nabla^v_U J^v_\alpha V - J^v_\alpha(\nabla^v_U V) &= -\epsilon_\alpha\{\nabla^v_{J^v_\alpha U}J^{v^2}_\alpha V - J^v_\alpha(\nabla^v_{J^v_\alpha U}J^v_\alpha V)\} \\
(\nabla^v_U J^v_\alpha)V &= -\epsilon_\alpha(\nabla^v_{J^v_\alpha U}J^v)J^v_\alpha V.
\end{aligned}$$

Then fibers are locally hyper quasi para-$K\ddot{a}hler$ almost para-quaternionic Hermitian manifolds.

7.3 Example 1

An example of $(semi-Riemannian)$ almost para-Hermitian fiber bundle for which the corollary 5.3 applies, is a fiber bundle which projection is para-quaternionic Hermitian submersion, whose total space is a locally hyper quasi para-$K\ddot{a}hler$ and satisfy $(\nabla^v_Y J'_\alpha)U = 0$ for all Y horizontal and U vertical, for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J'_1, J'_2, J'_3\}$ of σ'.

In fact:
For almost para-quaternionic Hermitian submersion whose total space is locally hyper quasi para-$K\ddot{a}hler$ almost para-quaternionic Hermitian manifold, the horizontal distribution is integrable (ie $A = 0$), the fibers are locally quasi hyper para-$K\ddot{a}hler$ almost para-quaternionic Hermitian manifolds and totally geodesic(ie $T = 0$), the base space is locally almost hyper semi-para-$K\ddot{a}hler$ almost para-quaternionic Hermitian manifold since it is locally hyper quasi para-$K\ddot{a}hler$ almost para-quaternionic Hermitian manifold (see both the theorems above).

Before giving the second example let's give one theorem

7.4 Theorem

Theorem 7.3. *Let* $[(N, \sigma', h) \stackrel{\pi}{\rightarrow} (M, \sigma, g)]$ *be a* $(semi-Riemannian)$ *almost para-quaternionic Hermitian submersion.*
If the toal space is locally hyper almost para-$K\ddot{a}hler$ *almost para-quaternionic Hermitian manifold, then*

$(\nabla^v_X J'_\alpha)U = 0$, *for all* Y *horizontal and* U *vertical, for all* $\alpha \in \{1.2.3\}$, *for any canonical local basis* $\{J'_1, J'_2, J'_3\}$ *of* σ'.

Proof

Proof. Let W and U be vertical vector fields on N and X horizontal . Since N is locally hyper almost para-$Kähler$ almost para-quaternionic Hermitian manifold, it implies $d\Phi_\alpha = 0$.

$$\begin{aligned}
0 &= d\Phi_\alpha(W, J'_\alpha V, X)\\
0 &= W.\Phi_\alpha(J'_\alpha V, X) - J'_\alpha V.\Phi_\alpha(W, X) + X.\Phi(W, J'_\alpha V)\\
&- \Phi_\alpha([W, J'_\alpha V], X) + \Phi_\alpha([W, X], J'_\alpha V) - \Phi_\alpha([J'_\alpha V, X], W)\\
0 &= W.h(J'_\alpha V, J_{\alpha'} X) - J'_\alpha V.h(W, J'_\alpha X) + X.h(W, J'^2 V)\\
&- h([W, V], J'_\alpha X) + h([W, X], J'^2_\alpha V) - h([J'_\alpha V, X], J'_\alpha W)\\
0 &= W.h(J'_\alpha V, J_{\alpha'} X) - J''_\alpha V.h(W, J'_\alpha X) + X.h(W, -\epsilon_\alpha V)\\
&- h([W, V], J'_\alpha X) + h([W, X], -\varepsilon_\alpha V) - h([J'_\alpha V, X], J'_\alpha W).\\
0 &= W.h(J'_\alpha V, J_{\alpha'} X) - J'_\alpha V.h(W, J'_\alpha X) - \epsilon_\alpha X.h(W, V)\\
&- h([W, V], J'_\alpha X) - \epsilon_\alpha h([W, X], V) - h([J'_\alpha V, X], J'_\alpha W).
\end{aligned}$$

Since the dot product of a vertical and a horizontal is zero, the first two terms and the fourth vanish. Thus one get

$$\begin{aligned}
0 &= -\epsilon_\alpha X.h(W, V) - \epsilon_\alpha h([W, X], V) - h([J'_\alpha V, X], J'_\alpha W).\\
0 &= -\epsilon_\alpha h(\nabla_X W, V) - \epsilon_\alpha h(W, \nabla_X V) - \epsilon_\alpha h(\nabla_W X - \nabla_X W, V)\\
&- h(\nabla_{J'_\alpha V} X - \nabla_X J'_\alpha V, J'_\alpha W)\\
0 &= -\epsilon_\alpha h(\nabla_X W, V) - \epsilon_\alpha h(W, \nabla_X V) - \epsilon_\alpha h(\nabla_W X, V) + \epsilon_\alpha h(\nabla_X W, V)\\
&- h(\nabla_{J'_\alpha V} X, J'_\alpha W) + h(\nabla_X J'_\alpha V, J'_\alpha W)\\
0 &= -\epsilon_\alpha h(W, \nabla_X V) + h(\nabla_X J'_\alpha V, J'_\alpha W) - \epsilon_\alpha h(\nabla_W X, V) - h(\nabla_{J'_\alpha V} X, J' W)\\
0 &= -\epsilon^2_\alpha h(J'_\alpha W, J'_\alpha(\nabla_X V)) + h(\nabla_X J'_\alpha V, J'_\alpha W) - \epsilon_\alpha h(\nabla_W X, V) - h(\nabla_{J'_\alpha V} X, J'_\alpha W)\\
0 &= -h(J'_\alpha W, J'_\alpha(\nabla_X V)) + h(\nabla_X J'_\alpha V, J'_\alpha W) - \epsilon_\alpha h(\nabla_W X, V) - h(\nabla_{J'_\alpha V} X, J'_\alpha W)\\
0 &= h((\nabla_X J'_\alpha) V, J'_\alpha W) - \epsilon_\alpha h(T_W X, V) - h(T_{J'_\alpha V} X, J'_\alpha W).
\end{aligned}$$

Since the total space is locally almost hyper para-$Kähler$ almost para-quaternionic Hermitian manifold; it is locally hyper quasi para-$Kähler$ almost para-quaternionic Hermitian manifold then $T = 0$ so $h((\nabla_X J'_\alpha)V, \varphi W) = 0$, for all vertical vector field W, and $(\nabla^v_X J'_\alpha)V = 0$, for all Y horizontal and U vertical, for all $\alpha \in \{1.2.3\}$, for any canonical local basis $\{J'_1, J'_2, J'_3\}$ of σ'. □

7.5 Example 2

An example of $(semi - Riemannian)$ almost para-Hermitian fiber bundle for which the corollary 5.3 applies is a fiber bundle which projection is para-quaternionic Hermitian submersion, whose total space is a locally almost hyper almost para-$Kähler$ almost para-quaternionic manifold.

7.6 Example 3

Let $[(N, \sigma', h) \xrightarrow{\pi} (M, \sigma, g)]$ be a $(semi - Riemannian)$ almost para-quaternionic Hermitian fiber bundle such that (N, σ', h) is a locally almost hyper quasi para-$Kähler$ almost para-quaternionic manifold and let $u : M \to N$ be a section.
If u is a (σ, σ')-para-holomorphic map, then u is a harmonic section.
In fact :

On the one hand, a section that is a (σ,σ')-para-holomorphic map is obvious a (σ,σ')-para-holomorphic section.
In fact

$$\begin{aligned} duoJ_\alpha &= J'_\alpha odu \\ d^v uoJ &= J'^v_\alpha odu \\ d^v uoJ_\alpha &= J'_\alpha od^v u \end{aligned}$$

because $\forall E \in \chi(N), J'^v(E) = J'(E^v)$, since π is an almost para-quaternionic Hermitian submersion.
On the other hand, since u is a (σ,σ')-para-holomorphic map from a locally almost hyper semi para-$K\ddot{a}hler$ almost para-quaternionic manifold (since it is quasi) to a locally hyper quasi para-$K\ddot{a}hler$ almost para-quaternionic manifold, then by virtue of corollary 3.2, u is a harmonic map; morever one knows that a section that is harmonic map is a harmonic section.

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