Palestine Journal of Mathematics

Vol. 10(2)(2021) , 834–844

Closure prime spectrum of $MS-$Almost Distributive Lattices

N. Rafi[1], Ravi Kumar Bandaru[2] and M. Srujana[3]

Communicated by Ayman Badawi

MSC 2010 Classifications: 06D99,06D15.

Keywords and phrases: Almost Distributive Lattice(ADL), $MS-$ADL, closure ideal, Hausdorff space, compact space.

Abstract In this paper, the concept of closure ideal is introduced in an $MS-$ADL and their properties are studied. It is observed that the set of all closure ideals forms a De Morgan ADL and topological properties of prime closure ideals are studied in an $MS-$ADL. Finally, equivalent conditions are provided for prime closure ideal to become maximal.

1 Introduction

In 1981, the idea of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao as a common abstraction of almost all the current ring theoretical generalizations of Boolean algebra on the one side and distributive lattices on the other. An ADL is an algebraic structure $(L, \vee, \wedge, 0)$ that satisfies most of the distributive lattice conditions with the smallest element 0, except, if possible, the commutativity of two binary operations $\vee$ and $\wedge$ and the right distributivity of the binary operation "$\vee$" over "$\wedge$." It has also been noted that each of these three properties transforms an ADL into a lattice distributive. Subsequently, several researchers have extended concepts like the class of pseudo-complemented lattices, stone lattices and normal lattices to the class of almost distributive lattices. In [2], authors introduced the concept of closure ideal in $MS-$algebras and studied its properties. In [8], as a popular abstraction of De Morgan ADLs and Stone ADLs, G. M. Addis recently identified a new equational class of algebras called $MS-$ADLs. The $MS-$ADL class properly includes the $MS-$algebras class, and most of the $MS-$algebras properties are generalized to $MS-$ADL class. In this paper, we introduce the concepts of closure ideal in an $MS-$ADL and studied its properties. We discuss topological properties of prime closure ideals of an $MS-$ADL and give equivalent conditions for a prime closure ideal to become maximal.

2 Preliminaries

We recall certain definitions, properties of an ADL and an $MS-$ADL in this section. We can go through the references for further literature about ADL.

Definition 2.1. [4] An almost distributive lattice (ADL) is an algebraic structure $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying the following set of axioms:

1. $a \vee 0 = a$,
2. $0 \wedge a = 0$,
3. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,
4. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
5. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
6. $(a \vee b) \wedge b = b$, for all $a, b, c \in L$.

Note that an element m of an ADL L is called a maximal element if $m \wedge x = x$ for all $x \in L$.

Definition 2.2. [4] A nonempty subset I of L is called an ideal (respectively a filter) of L, if $a \vee b, a \wedge x \in I$ (respectively $a \wedge b, x \vee a \in I$) for all $a, b \in I$ and $x \in L$. The set of all ideals (respectively filters) of L is denoted by $\mathfrak{I}(L)$ (respectively $\mathfrak{F}(L)$).

Lemma 2.3. *[4] Let I be an ideal of an ADL L. Then, for any $a, b \in L$,*

1. *$a \wedge b \in I$ if and only if $b \wedge a \in I$*
2. *$a \leq b$ and $b \in I$ implies that $a \in I$.*

A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal.

Definition 2.4. [8] An $MS-$almost distributive lattice ($MS-$ADL) is an algebra $(L, \vee, \wedge, ^\circ, 0)$ of type $(2, 2, 1, 0)$ such that $(L, \vee, \wedge, 0)$ is an ADL with maximal elements and $x \mapsto x^\circ$ is a unary operation on L satisfying the following axioms:

1. $x^{\circ\circ} \wedge x = x$,
2. $(x \vee y)^\circ = x^\circ \wedge y^\circ$,
3. $(x \wedge y)^\circ = x^\circ \vee y^\circ$,
4. $m^\circ = 0$ for all maximal elements m of L,
   for all $x, y \in L$.
   In addition, if it satisfies the following condition:
5. $x^{\circ\circ} = x \wedge m$,

then L is called a De Morgan ADL.

Lemma 2.5. *[8] The following holds in an $MS-$ADL L:*

1. *0° is maximal,*
2. *$a \leq b \Rightarrow b^\circ \leq a^\circ$,*
3. *$a^{\circ\circ\circ} = a^\circ$,*
4. *$(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$,*
5. *$(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,*
6. *$(a \wedge m)^\circ = a^\circ$,*
7. *$(a \wedge b)^\circ = (b \wedge a)^\circ$ for all $a, b \in L$.*

Definition 2.6. [8] An element x of L is said to be dense if $x^\circ = 0$. The set of all dense elements of L is denoted by $D(L)$.

Throughout this paper, an ideal of an $MS-$ADL $(L, \vee, \wedge, ^\circ, 0)$ is an ideal of an ADL $(L, \vee, \wedge, 0)$.

3 Closure ideals of $MS-$ADLs

In this section, we introduce the concept of closure ideal in an $MS-$ADL and study their properties.

Definition 3.1. Let L be an $MS-$ADL and A be any nonempty subset of L. Define the dominator of S as $S_{\circ\circ} = \{a \in L \mid s^{\circ\circ} \wedge a = a, \text{ for some } s \in S\}$.

The following lemma can be proved easily.

Lemma 3.2. *Let L be an $MS-$ADL and S, T be any two nonempty subsets of L. Then we have the following:*

1. *$S \subseteq S_{\circ\circ}$*
2. *if $S \subseteq T$ then $S_{\circ\circ} \subseteq T_{\circ\circ}$*
3. *$(S_{\circ\circ})_{\circ\circ} = S_{\circ\circ}$.*

Lemma 3.3. *Let S, T be any two ideals of an $MS-$ADL L. Then we have the following:*

1. $S_{\circ\circ}$ *is an ideal of* L
2. $(S\cap T)_{\circ\circ} = S_{\circ\circ}\cap T_{\circ\circ}$
3. $(S\vee T)_{\circ\circ} = S_{\circ\circ}\vee T_{\circ\circ}$.

Proof. 1. Clearly, we have that $0\in S_{\circ\circ}$ and hence $S_{\circ\circ}\neq\emptyset$. Let $a,b\in S_{\circ\circ}$. Then there exist elements $s_1,s_2\in S$ such that $s_1^{\circ\circ}\wedge a = a$ and $s_2^{\circ\circ}\wedge b = b$. Since $s_1,s_2\in S$ and S is an ideal of L, we have that $s_1\vee s_2\in S$. Now, $(s_1\vee s_2)^{\circ\circ}\wedge(a\vee b) = (s_1^{\circ\circ}\vee s_2^{\circ\circ})\wedge(a\vee b)\wedge(a\vee b) = (((s_1^{\circ\circ}\vee s_2^{\circ\circ})\wedge a)\vee((s_1^{\circ\circ}\vee s_2^{\circ\circ})\wedge b))\wedge(a\vee b) = (((s_1^{\circ\circ}\wedge a)\vee(s_2^{\circ\circ}\wedge a))\vee((s_1^{\circ\circ}\wedge b)\vee(s_2^{\circ\circ}\wedge b)))\wedge(a\vee b) = ((a\vee(s_2^{\circ\circ}\wedge a))\vee((s_1^{\circ\circ}\wedge b)\vee b))\wedge(a\vee b) = (a\vee(s_2^{\circ\circ}\wedge a)\vee b)\wedge(a\vee b) = ((s_2^{\circ\circ}\wedge a)\vee a\vee b)\wedge(a\vee b) = (a\vee b)\wedge(a\vee b) = a\vee b$. Therefore $a\vee b\in S_{\circ\circ}$. Let $a\in S_{\circ\circ}$. Then there exists an element $s\in S$ such that $s^{\circ\circ}\wedge a = a$. Let r be any element of L. Clearly, we have that $s^{\circ\circ}\wedge a\wedge r = a\wedge r$ and hence $x\wedge r\in S_{\circ\circ}$. Therefore $S_{\circ\circ}$ is an ideal of L.
2. Since $S\cap T\subseteq S$ and $S\cap T\subseteq T$, we have that $(S\cap T)_{\circ\circ}\subseteq S_{\circ\circ}$ and $(S\cap T)_{\circ\circ}\subseteq T_{\circ\circ}$. Therefore $(S\cap T)_{\circ\circ}\subseteq S_{\circ\circ}\cap T_{\circ\circ}$. Let $a\in S_{\circ\circ}\cap T_{\circ\circ}$. Then $a\in S_{\circ\circ}$ and $b\in T_{\circ\circ}$. Since $a\in S_{\circ\circ}$, there exists an element $s\in S$ such that $s^{\circ\circ}\wedge a = a$. Since $a\in T_{\circ\circ}$, there exists an element $t\in T$ such that $t^{\circ\circ}\wedge a = a$. Since $s\in S,\ t\in T$ and $S,\ T$ are ideals of L, we have that $s\wedge t\in S\cap T$. Now $(s\wedge t)^{\circ\circ}\wedge a = s^{\circ\circ}\wedge t^{\circ\circ}\wedge a = a$. That implies $a\in(S\cap T)_{\circ\circ}$. Therefore $S_{\circ\circ}\cap T_{\circ\circ}\subseteq(S\cap T)_{\circ\circ}$. Thus $(S\cap T)_{\circ\circ} = S_{\circ\circ}\cap T_{\circ\circ}$.
3. Clearly, we have that $S_{\circ\circ}\vee T_{\circ\circ}\subseteq(S\vee T)_{\circ\circ}$. Let $a\in(S\vee T)_{\circ\circ}$. Then there exists an element $b\in S\vee T$ such that $b^{\circ\circ}\wedge a = a$. Since $b\in S\vee T$, there exist $s\in S$ and $t\in T$ such that $b = s\vee t$. Now, $a = b^{\circ\circ}\wedge a = (s\vee t)^{\circ\circ}\wedge a = (s^{\circ\circ}\vee t^{\circ\circ})\wedge a = (s^{\circ\circ}\wedge a)\vee(t^{\circ\circ}\wedge a)\in S_{\circ\circ}\vee T_{\circ\circ}$, (since $s^{\circ\circ}\wedge(s^{\circ\circ}\wedge b) = s^{\circ\circ}\wedge b\Rightarrow s^{\circ\circ}\wedge b\in S_{\circ\circ}$). Therefore $(S\vee T)_{\circ\circ}\subseteq S_{\circ\circ}\vee T_{\circ\circ}$. Hence $(S\vee T)_{\circ\circ} = S_{\circ\circ}\vee T_{\circ\circ}$. □

Corollary 3.4. *If $\{S_\alpha\}_{\alpha\in\Delta}$ is a family of ideals of L, then we have the following:*

1. $(\bigcap\limits_{\alpha\in\Delta} S_\alpha)_{\circ\circ} = \bigcap\limits_{\alpha\in\Delta}(S_\alpha)_{\circ\circ}$
2. $(\bigvee\limits_{\alpha\in\Delta} S_\alpha)_{\circ\circ} = \bigvee\limits_{\alpha\in\Delta}(S_\alpha)_{\circ\circ}$

Now we have the following definition

Definition 3.5. An ideal I of an $MS-$ADL L is said to be a closure ideal if $I = I_{\circ\circ}$.

By lemma-3.3, it is easy to get that the set $\Im_C(L)$ of all closure ideals of L forms a bounded distributive lattice. For any element a of an $MS-$ADL L, the dominator $\{a\}_{\circ\circ}$ is called a principal closure ideal of L. For any $MS-$ADL L we can define the set of closed elements $L^{\circ\circ} = \{x\in L \mid x = x^{\circ\circ}\}$.

Lemma 3.6. *Let L be an $MS-$ADL with maximal elements. Then for any $x,y\in L$, we have the following:*

1. $\{x\}_{\circ\circ} = (x]_{\circ\circ} = (x^{\circ\circ}]$
2. $\{0\}_{\circ\circ} = \{0\}$
3. *If m is any maximal element of L then* $\{m\}_{\circ\circ} = L$
4. $(x]_{\circ\circ} = (x^{\circ\circ}]_{\circ\circ}$
5. $x\in(y]_{\circ\circ}$ *if and only if* $(x]_{\circ\circ}\subseteq(y]_{\circ\circ}$
6. *if $x\leq y$ then* $(x]_{\circ\circ}\subseteq(y]_{\circ\circ}$
7. $(x]_{\circ\circ} = L$ *if and only if x is a dense element of L*
8. $(x]_{\circ\circ} = \{0\}$ *if and only if* $x = 0$.

Proof. 1. Clearly, we have that $\{x\}_{\circ\circ} \subseteq (x]_{\circ\circ}$. Let $a \in (x]_{\circ\circ}$. Then there exists an element $b \in (x]$ such that $b^{\circ\circ} \wedge a = a$. Since $b \in (x]$, we have $x \wedge b = b$. Now $x^{\circ\circ} \wedge a = (x \vee b)^{\circ\circ} \wedge a = (x^{\circ\circ} \vee b^{\circ\circ}) \wedge a = (x^{\circ\circ} \wedge a) \vee (b^{\circ\circ} \wedge a) = (x^{\circ\circ} \wedge a) \vee a = a$. That implies $a \in \{x\}_{\circ\circ}$ and hence $(x]_{\circ\circ} \subseteq \{x\}_{\circ\circ}$. Therefore $(x]_{\circ\circ} = \{x\}_{\circ\circ}$. Now $a \in \{x\}_{\circ\circ}$ iff $x^{\circ\circ} \wedge a = a$ iff $a \in (x^{\circ\circ}]$. Therefore $\{x\}_{\circ\circ} = (x]_{\circ\circ} = (x^{\circ\circ}]$.
2. Let $a \in \{0\}_{\circ\circ}$. Then $0^{\circ\circ} \wedge a = a$. That implies $0 \wedge a = a$ and hence $a = 0$. Therefore $\{0\}_{\circ\circ} = \{0\}$.
3. Let m be any maximal element of L. For any $a \in L$, we have that $m^{\circ\circ} \wedge a = m \wedge a = a$. Therefore $a \in \{m\}_{\circ\circ}$, for all $a \in L$. Hence $L = \{m\}_{\circ\circ}$.
4. Clearly, we have that $(x] \subseteq (x^{\circ\circ}]$ and hence $(x]_{\circ\circ} \subseteq (x^{\circ\circ}]_{\circ\circ}$. Let $a \in (x^{\circ\circ}]_{\circ\circ}$. Then there exists an element $b \in (x^{\circ\circ}]$ such that $b^{\circ\circ} \wedge a = a$. Since $b \in (x^{\circ\circ}]$, we have $x^{\circ\circ} \wedge b = b$ and hence $x^{\circ\circ} \wedge b^{\circ\circ} = b^{\circ\circ}$. Now $x^{\circ\circ} \wedge a = (x^{\circ\circ} \vee b^{\circ\circ}) \wedge a = (x^{\circ\circ} \wedge a) \vee (b^{\circ\circ} \wedge a) = (x^{\circ\circ} \wedge a) \vee a = a$. That implies $a \in (x^{\circ\circ}] = (x]_{\circ\circ}$. That implies $(x^{\circ\circ}]_{\circ\circ} \subseteq (x]_{\circ\circ}$. Therefore $(x^{\circ\circ}]_{\circ\circ} = (x]_{\circ\circ}$.
5. Assume that $x \in (y]_{\circ\circ}$. Then $y^{\circ\circ} \wedge x = x$. Let $a \in (x]_{\circ\circ}$. Then $x^{\circ\circ} \wedge a = a$. Now $a = x^{\circ\circ} \wedge a = (y^{\circ\circ} \wedge x)^{\circ\circ} \wedge a = y^{\circ\circ} \wedge x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$. That implies $a \in (y]_{\circ\circ}$. Therefore $(x]_{\circ\circ} \subseteq (y]_{\circ\circ}$. Assume that $(x]_{\circ\circ} \subseteq (y]_{\circ\circ}$. Clearly, we have that $x \in (x]_{\circ\circ} \subseteq (y]_{\circ\circ}$. Therefore $x \in (y]_{\circ\circ}$.
6. Assume that $x \leq y$. Then $x \wedge y = x$. Let $a \in (x]_{\circ\circ}$. Then $x^{\circ\circ} \wedge a = a$. Now $a = x^{\circ\circ} \wedge a = (x \wedge y)^{\circ\circ} \wedge a = x^{\circ\circ} \wedge y^{\circ\circ} \wedge a = y^{\circ\circ} \wedge x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$. That implies $a \in (y]_{\circ\circ}$. Therefore $(x]_{\circ\circ} \subseteq (y]_{\circ\circ}$.
7. Assume that $(x]_{\circ\circ} = L$. Then choose a maximal element m of L such that $m \in (x]_{\circ\circ}$. That implies $x^{\circ\circ} \wedge m = m$. Now $x^{\circ} = x^{\circ} \vee 0 = x^{\circ} \vee m^{\circ} = x^{\circ\circ\circ} \vee m^{\circ} = (x^{\circ\circ} \wedge m)^{\circ} = m^{\circ} = 0$. That implies x is a dense element of L. Conversely, assume that x is a dense element of L. Then $x^{\circ} = 0$. Let a be any element of L. Now $a = 0^{\circ} \wedge a = x^{\circ\circ} \wedge a$. That implies $a \in (x]_{\circ\circ}$, for all $a \in L$. Therefore $(x]_{\circ\circ} = L$.
8. Assume that $(x]_{\circ\circ} = \{0\}$. Clearly we have that $x \in (x]_{\circ\circ}$ and hence $x = 0$. Conversely assume that $x = 0$. Let $a \in (x]_{\circ\circ}$. Then $x^{\circ\circ} \wedge a = a$. That implies $0^{\circ\circ} \wedge a = a$ and hence $0 \wedge a = a$. Therefore $a = 0$. Thus $(x]_{\circ\circ} = \{0\}$. □

Theorem 3.7. *Let L be an $MS-$ADL with maximal elements. Then we have the following conditions:*

1. *The set $\mathfrak{M}_{\circ\circ}(L)$ of all principal closure ideals of L is a bounded sublattice of the lattice $\mathfrak{I}_C(L)$*
2. *L is homomorphic to $\mathfrak{M}_{\circ\circ}(L)$*
3. *$\mathfrak{M}_{\circ\circ}(L)$ is a De Morgan algebra*
4. *$L^{\circ\circ}$ is isomorphic to $\mathfrak{M}_{\circ\circ}(L)$.*

Proof. 1. Clearly, we have $\{0\}, L \in \mathfrak{M}_{\circ\circ}(L)$. Let $(x]_{\circ\circ}, (y]_{\circ\circ} \in \mathfrak{M}_{\circ\circ}(L)$. Now we have $(x]_{\circ\circ} \vee (y]_{\circ\circ} = ((x] \vee (y]]_{\circ\circ} = (x \vee y]_{\circ\circ}$ and $(x]_{\circ\circ} \cap (y]_{\circ\circ} = ((x] \cap (y]]_{\circ\circ} = (x \wedge y]_{\circ\circ}$. Therefore $(\mathfrak{M}_{\circ\circ}(L), \vee, \cap, \{0\}, L)$ is a bounded sublattices of $\mathfrak{I}_{\circ\circ}(L)$.
2. Define $f : L \longrightarrow \mathfrak{M}_{\circ\circ}(L)$ by $f(x) = (x]_{\circ\circ}$. Clearly, we have that $f(0) = \{0\}$ and $f(m) = L$, where m is any maximal element of L. Let $x, y \in L$. Now $f(x \vee y) = (x \vee y]_{\circ\circ} = (x]_{\circ\circ} \vee (y]_{\circ\circ} = f(x) \vee f(y)$ and $f(x \wedge y) = (x]_{\circ\circ} \cap (y]_{\circ\circ} = f(x) \cap f(y)$. Therefore f is homomorphism.
3. Define the unary operation $^{-}$ on $\mathfrak{M}_{\circ\circ}(L)$ by $\overline{(x]}_{\circ\circ} = (x^{\circ}]_{\circ\circ}$. Let $x, y \in L$. (i). Now $\overline{\overline{(x]}}_{\circ\circ} = \overline{(x^{\circ}]}_{\circ\circ} = (x^{\circ\circ}]_{\circ\circ} = (x]_{\circ\circ}$. (ii). Now $\overline{(x]_{\circ\circ} \vee (y]}_{\circ\circ} = \overline{(x \vee y]}_{\circ\circ} = ((x \vee y)^{\circ}]_{\circ\circ} = (x^{\circ} \wedge y^{\circ}]_{\circ\circ} = (x^{\circ}]_{\circ\circ} \cap (y^{\circ}]_{\circ\circ} = \overline{(x]}_{\circ\circ} \cap \overline{(y]}_{\circ\circ}$. (iii). Now $\overline{((x]_{\circ\circ} \cap (y]_{\circ\circ})} = \overline{(x \wedge y]}_{\circ\circ} = ((x \wedge y)^{\circ}]_{\circ\circ} = (x^{\circ} \vee y^{\circ}]_{\circ\circ} = (x^{\circ}]_{\circ\circ} \vee (y^{\circ}]_{\circ\circ} = \overline{(x]}_{\circ\circ} \vee \overline{(y]}_{\circ\circ}$. (iv). We have that $\overline{(0]}_{\circ\circ} = (0^{\circ}]_{\circ\circ} = L$. Therefore $(\mathfrak{M}_{\circ\circ}(L), \vee, \cap, ^{-}, \{0\}, L)$ is a De Morgan algebra.
4. Define $g : L^{\circ\circ} \longrightarrow \mathfrak{M}_{\circ\circ}(L)$ by $g(x) = (x^{\circ\circ}]_{\circ\circ}$, for all $x \in L^{\circ\circ}$. Let $x, y \in L^{\circ\circ}$. Then $x = x^{\circ\circ}$ and $y = y^{\circ\circ}$. Suppose $x = y$. Then $(x^{\circ\circ}]_{\circ\circ} = (y^{\circ\circ}]_{\circ\circ}$. That implies $g(x) = g(y)$ and hence g is well defined. Let $x, y \in L^{\circ\circ}$. Then $x = x^{\circ\circ}$ and $y = y^{\circ\circ}$. Suppose $g(x) = g(y)$. Then $(x^{\circ\circ}]_{\circ\circ} = (y^{\circ\circ}]_{\circ\circ}$. That implies $(x^{\circ\circ}] = (y^{\circ\circ}]$. That implies $x^{\circ\circ} \wedge y = y$ and $y^{\circ\circ} \wedge x = x$. That implies $(x^{\circ\circ} \wedge y)^{\circ\circ} = y^{\circ\circ}$ and $(y^{\circ\circ} \wedge x)^{\circ\circ} = x^{\circ\circ}$. That implies $x^{\circ\circ} \wedge y^{\circ\circ} = y^{\circ\circ}$ and $x^{\circ\circ} \wedge y^{\circ\circ} = x^{\circ\circ}$. Therefore $x^{\circ\circ} = y^{\circ\circ}$ and hence $x = y$. Thus g is one-one. Let $(x^{\circ\circ}]_{\circ\circ} \in \mathfrak{M}_{\circ\circ}(L)$. Clearly, we have that $(x^{\circ\circ}]_{\circ\circ} = g(x)$. Therefore g is onto. Let $x, y \in L^{\circ\circ}$. Then $x = x^{\circ\circ}$ and $y = y^{\circ\circ}$. Now $g(x \vee y) = ((x \vee y)]_{\circ\circ} = (x^{\circ\circ}]_{\circ\circ} \vee (y^{\circ\circ}]_{\circ\circ} = g(x) \vee g(y)$. Now

$g(x \wedge y) = ((x \wedge y)^{\circ\circ}]_{\circ\circ} = (x^{\circ\circ}]_{\circ\circ} \cap (y^{\circ\circ}]_{\circ\circ} = g(x) \wedge g(y)$. Therefore g is homomorphism and hence g is isomorphism. □

Theorem 3.8. *Let I be an ideal of an $MS-ADL$ L. Then $I_{\circ\circ} = \bigcup_{x \in I} (x]_{\circ\circ}$.*

Proof. Let $a \in I_{\circ\circ}$. Then there exists an element $x \in I$ such that $x^{\circ\circ} \wedge a = a$. That implies $a \in (x]_{\circ\circ}$ and hence $a \in \bigcup_{x \in I} (x]_{\circ\circ}$. Therefore $I_{\circ\circ} \subseteq \bigcup_{x \in I} (x]_{\circ\circ}$. Let $a \in \bigcup_{x \in I} (x]_{\circ\circ}$. Then there exists an element $y \in I$ such that $a \in (y]_{\circ\circ}$. That implies $y^{\circ\circ} \wedge a = a$. Since $y \in I$, we get that $a \in I_{\circ\circ}$. Therefore $\bigcup_{x \in I} (x]_{\circ\circ} \subseteq I_{\circ\circ}$ and hence $\bigcup_{x \in I} (x]_{\circ\circ} = I_{\circ\circ}$. □

Definition 3.9. Let L be an $MS-$ADL.

For any ideal I of L, define an operator $\sigma : \mathfrak{I}(L) \longrightarrow \mathfrak{I}(\mathfrak{M}_{\circ\circ}(L))$ as $\sigma(I) = \{(x]_{\circ\circ} \mid x \in I\}$.

For any ideal $\widetilde{I}$ of $\mathfrak{M}_{\circ\circ}(L)$, define an operator $\overleftarrow{\sigma} : \mathfrak{M}_{\circ\circ}(L) \longrightarrow \mathfrak{I}(L)$ as $\overleftarrow{\sigma}(\widetilde{I}) = \{a \in L \mid (a]_{\circ\circ} \in \widetilde{I}\}$.

Lemma 3.10. *Let L be an $MS-ADL$. Then we have*

1. *For any ideal I of L, $\sigma(I)$ is an ideal of $\mathfrak{M}_{\circ\circ}(L)$*
2. *For any ideal $\widetilde{I}$ of $\mathfrak{M}_{\circ\circ}(L)$, $\overleftarrow{\sigma}(\widetilde{I})$ is an ideal of L*
3. *$\overleftarrow{\sigma}$ and σ are isotones*
4. *$\sigma\overleftarrow{\sigma}(\widetilde{I}) = \widetilde{I}$, for all ideal $\widetilde{I}$ of $\mathfrak{M}_{\circ\circ}(L)$*
5. *σ is homomorphism.*

Proof. 1. Let I be any ideal of L. Clearly, we have that $(0]_{\circ\circ} \subseteq \sigma(I)$ and hence $\sigma(I) \neq \emptyset$. Let $(a]_{\circ\circ}, (b]_{\circ\circ} \in \sigma(I)$. Then $a, b \in I$. That implies $a \vee b \in I$. that implies $(a \vee b]_{\circ\circ} \in \sigma(I)$. Since $(a]_{\circ\circ} \vee (b]_{\circ\circ} = (a \vee b]_{\circ\circ}$, we have that $(a]_{\circ\circ} \vee (b]_{\circ\circ} \in \sigma(I)$. Let $(a]_{\circ\circ} \in \sigma(I)$ and $(r]_{\circ\circ} \in \mathfrak{M}_{\circ\circ}(L)$. Then $a \in I$ and hence $a \wedge r \in I$. That implies $(a \wedge r]_{\circ\circ} \in \sigma(I)$. Therefore $(a]_{\circ\circ} \cap (r]_{\circ\circ} \in \sigma(I)$ and hence $\sigma(I)$ is an ideal of $\mathfrak{M}_{\circ\circ}(L)$.
2. Let $\widetilde{I}$ be any ideal of $\mathfrak{M}_{\circ\circ}(L)$. Since $(0]_{\circ\circ} \in \widetilde{I}$, we have that $0 \in \overleftarrow{\sigma}(\widetilde{I})$. Therefore $\overleftarrow{\sigma}(\widetilde{I}) \neq \emptyset$. Let $a, b \in \overleftarrow{\sigma}(\widetilde{I})$. Then $(a]_{\circ\circ}, (b]_{\circ\circ} \in \widetilde{I}$. Since $\widetilde{I}$ is an ideal of $\mathfrak{M}_{\circ\circ}(L)$, we have that $(a]_{\circ\circ} \vee (b]_{\circ\circ} \in \widetilde{I}$ and hence $(a \vee b]_{\circ\circ} \in \widetilde{I}$. Therefore $a \vee b \in \overleftarrow{\sigma}(\widetilde{I})$. Let $a \in \overleftarrow{\sigma}(\widetilde{I})$ and $r \in L$. Then $(a]_{\circ\circ} \in \widetilde{I}$ and $(r]_{\circ\circ} \in \mathfrak{M}_{\circ\circ}(L)$. Since $\widetilde{I}$ is an ideal of $\mathfrak{M}_{\circ\circ}(L)$, we have that $(a]_{\circ\circ} \cap (r]_{\circ\circ} \in \widetilde{I}$ and hence $(a \wedge r]_{\circ\circ} \in \widetilde{I}$. Therefore $a \wedge r \in \overleftarrow{\sigma}(\widetilde{I})$. Thus $\overleftarrow{\sigma}(\widetilde{I})$ is an ideal of L.
3. Let $\widetilde{I}$ and $\widetilde{J}$ be two ideals of $\mathfrak{M}_{\circ\circ}(L)$ with $\widetilde{I} \subseteq \widetilde{J}$. Now we prove that $\overleftarrow{\sigma}(\widetilde{I}) \subseteq \overleftarrow{\sigma}(\widetilde{J})$. Let $a \in \overleftarrow{\sigma}(\widetilde{I})$. Then $(a]_{\circ\circ} \in \widetilde{I} \subseteq \widetilde{J}$. That implies $(a]_{\circ\circ} \in \widetilde{J}$. That implies $a \in \overleftarrow{\sigma}(\widetilde{J})$. Therefore $\overleftarrow{\sigma}(\widetilde{I}) \subseteq \overleftarrow{\sigma}(\widetilde{J})$ and hence $\overleftarrow{\sigma}$ is an isotone operator. Let I and J be two ideals of L with $I \subseteq J$. Let $(a]_{\circ\circ} \in \sigma(I)$. Then $a \in I \subseteq J$. That implies $(a]_{\circ\circ} \in \sigma(J)$ and hence $\sigma(I) \subseteq \sigma(J)$. Therefore σ is an isotone operator.
4. Let $\widetilde{I}$ be an ideal of $\mathfrak{M}_{\circ\circ}(L)$. Then $\overleftarrow{\sigma}(\widetilde{I})$ is an ideal of L. Let a be any element of L. Now $(a]_{\circ\circ} \in \widetilde{I}$ iff $a \in \overleftarrow{\sigma}(\widetilde{I})$ iff $(a]_{\circ\circ} \in \sigma(\overleftarrow{\sigma}(\widetilde{I}))$. Therefore $\widetilde{I} = \sigma(\overleftarrow{\sigma}(\widetilde{I}))$.
5. Let $I,\ J \in \mathfrak{I}(L)$. Clearly we have $\sigma(I \cap J) \subseteq \sigma(I) \cap \sigma(J)$. Let $(a]_{\circ\circ} \in \sigma(I) \cap \sigma(J)$. Then $(a]_{\circ\circ} \in \sigma(I)$ and $(a]_{\circ\circ} \in \sigma(J)$. Then there exist $i \in I$ and $j \in J$ such that $(a]_{\circ\circ} = (i]_{\circ\circ}$ and $(a]_{\circ\circ} = (j]_{\circ\circ}$. Now $(a]_{\circ\circ} = (i]_{\circ\circ} \cap (j]_{\circ\circ} = (i \wedge j]_{\circ\circ} \in \sigma(I \cap J)$. Therefore $\sigma(I) \cap \sigma(J) \subseteq \sigma(I \cap J)$. Hence $\sigma(I) \cap \sigma(J) = \sigma(I \cap J)$. Clearly, we have $\sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J)$. Let $(a]_{\circ\circ} \in \sigma(I \vee J)$. Then $a \in I \vee J$. Then there exist $x \in I$ and $y \in J$ such that $a = x \vee y$. Now $(a]_{\circ\circ} = (x \vee y]_{\circ\circ} = (x]_{\circ\circ} \vee (y]_{\circ\circ} \in \sigma(I) \vee \sigma(J)$. Therefore $\sigma(I \vee J) \subseteq \sigma(I) \vee \sigma(J)$. Hence $\sigma(I \vee J) = \sigma(I) \vee \sigma(J)$. Thus σ is homomorphism. □

Theorem 3.11. *The map $\overleftarrow{\sigma}\sigma : \mathfrak{I}(L) \longrightarrow \mathfrak{I}(L)$ is a closure operator.*

Proof. 1. Let I be any ideal of L and $a \in I$. Then $(a]_{\circ\circ} \in \sigma(I)$. Since $\sigma(I)$ is an ideal of $\mathfrak{M}_{\circ\circ}(L)$, we get that $a \in \overleftarrow{\sigma}(\sigma(I))$. Therefore $I \subseteq \overleftarrow{\sigma}(\sigma(I))$.
2. Let $I,\ J$ be any two ideals of L with $I \subseteq J$. Let $x \in \overleftarrow{\sigma}(\sigma(I))$. Then $(a]_{\circ\circ} \in \sigma(I)$. That implies $a \in I \subseteq J$. That implies $a \in J$ and hence $(a]_{\circ\circ} \in \sigma(J)$. Therefore $a \in \overleftarrow{\sigma}(\sigma(J))$. Thus

$\overleftarrow{\sigma}(\sigma(I)) \subseteq \overleftarrow{\sigma}(\sigma(J))$.
3. From 1 and 2, we have that $\overleftarrow{\sigma}(\sigma(I)) \subseteq \overleftarrow{\sigma}\sigma(\overleftarrow{\sigma}\sigma(I))$. Let $a \in \overleftarrow{\sigma}\sigma(\overleftarrow{\sigma}\sigma(I))$. Then $(a]_{\circ\circ} \in \sigma(\overleftarrow{\sigma}\sigma(I))$ and hence $a \in \overleftarrow{\sigma}(\sigma(I))$. Therefore $\overleftarrow{\sigma}\sigma(\overleftarrow{\sigma}\sigma(I)) \subseteq \overleftarrow{\sigma}(\sigma(I))$. Thus $\overleftarrow{\sigma}\sigma(\overleftarrow{\sigma}\sigma(I)) = \overleftarrow{\sigma}(\sigma(I))$. □

Corollary 3.12. *For any two ideals $I,\ J$ of an $MS-$ADL L, we have $\overleftarrow{\sigma}\sigma(I \cap J) = \overleftarrow{\sigma}(I) \cap \overleftarrow{\sigma}\sigma(J)$.*

Proof. Clearly, we have that $\overleftarrow{\sigma}\sigma(I \cap J) \subseteq \overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(J)$. Let $a \in \overleftarrow{\sigma}(I) \cap \overleftarrow{\sigma}\sigma(J)$. Then $a \in \overleftarrow{\sigma}(I)$ and $a \in \overleftarrow{\sigma}\sigma(J)$. That implies $(a]_{\circ\circ} \in \sigma(I)$ and $(a]_{\circ\circ} \in \sigma(J)$. That implies $(a]_{\circ\circ} \in \sigma(I) \cap \sigma(J) = \sigma(I \cap J)$. That implies $a \in \overleftarrow{\sigma}\sigma(I \cap J)$ and hence $\overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(J) \subseteq \overleftarrow{\sigma}\sigma(I \cap J)$. Therefore $\overleftarrow{\sigma}\sigma(I \cap J) = \overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(J)$. □

Theorem 3.13. *Let I be an ideal of an $MS-$ADL L. Then the following conditions are equivalent:*

1. *$\overleftarrow{\sigma}\sigma(I) = I$.*
2. *for any $a, b \in L$, $(a]_{\circ\circ} = (b]_{\circ\circ}$ and $a \in I$ imply $b \in I$*
3. *I is a closure ideal*
4. *$I = \bigcup\limits_{i \in I} (i]_{\circ\circ}$*
5. *if $a \in I$ then $(a]_{\circ\circ} \subseteq I$.*

Proof. $1 \Rightarrow 2$: Assume that $\overleftarrow{\sigma}\sigma(I) = I$. Let $a, b \in L$ with $(a]_{\circ\circ} = (b]_{\circ\circ}$ and $a \in I$. Then $a \in \overleftarrow{\sigma}\sigma(I)$. Then $(a]_{\circ\circ} \in \sigma(I)$. That implies $(b]_{\circ\circ} \in \sigma(I)$ and hence $b \in \overleftarrow{\sigma}\sigma(I) = I$. Therefore $b \in I$.
$2 \Rightarrow 3$: Assume 2. Clearly, $I \subseteq I_{\circ\circ}$. Let $a \in I_{\circ\circ}$. Then there exists an element $x \in I$ $x^{\circ\circ} \wedge a = a$. That implies $(x^{\circ\circ} \wedge]_{\circ\circ} = (a]_{\circ\circ}$. That implies $(a]_{\circ\circ} = (x^{\circ\circ}]_{\circ\circ} \cap (a]_{\circ\circ}$. That implies $(a]_{\circ\circ} = (x]_{\circ\circ} \cap (a]_{\circ\circ} = (x \wedge a]_{\circ\circ}$. Since I is an ideal of L, we have that $x \wedge a \in I$. By our assumption, we get that $a \in I$ and hence $I_{\circ\circ} \subseteq I$. Therefore $I = I_{\circ\circ}$.
$3 \Rightarrow 4$: Clear.
$4 \Rightarrow 5$: Assume that 4. Let $a \in I$. By our assumption we get that $(a]_{\circ\circ} \subseteq I$.
$5 \Rightarrow 1$: Assume that 5. Clearly, we have $I \subseteq \overleftarrow{\sigma}\sigma(I)$. Let $a \in \overleftarrow{\sigma}\sigma(I)$. Then $(a]_{\circ\circ} \in \sigma(I)$. Then there exists element $b \in I$ such that $(a]_{\circ\circ} = (b]_{\circ\circ}$. By our assumption, we get that $(b]_{\circ\circ} \subseteq I$ and hence $(a]_{\circ\circ} \subseteq I$. That implies $a \in I$. Therefore $\overleftarrow{\sigma}\sigma(I) \subseteq I$. Hence $\overleftarrow{\sigma}\sigma(I) = I$. □

Lemma 3.14. *Let L be an $MS-$ADL. Then we have the following conditions:*

1. *if $x \in L^{\circ\circ}$ then $(x]$ is a closure ideal of L*
2. *for any ideal I of L, $\overleftarrow{\sigma}\sigma(I) = I_{\circ\circ}$*
3. *for any ideal I of L, $I_{\circ\circ}$ is a closure ideal*
4. *the map $\overleftarrow{\sigma}\sigma : \mathfrak{I}(L) \longrightarrow \mathfrak{I}(L)$ is homomorphism.*

Proof. 1. Let $x \in L^{\circ\circ}$. Clearly, we have that $(x] \subseteq \overleftarrow{\sigma}\sigma((x])$. Let $a \in \overleftarrow{\sigma}\sigma((x])$. Then $(a]_{\circ\circ} \in \sigma((x])$. That implies there exists an element $b \in (x]$ such that $(a]_{\circ\circ} = (b]_{\circ\circ}$. That implies $(a]_{\circ\circ} = (b]_{\circ\circ} \subseteq (x]_{\circ\circ} = (x^{\circ\circ}] = (x]$. That implies $a \in (x]$ and hence $\overleftarrow{\sigma}\sigma((x]) \subseteq (x]$. Therefore $\overleftarrow{\sigma}\sigma((x]) = (x]$. Thus $(x]$ is a closure ideal of L.
2. Let I be any ideal of L. Now we prove that $I_{\circ\circ} = \overleftarrow{\sigma}\sigma(I)$. Let $a \in I_{\circ\circ}$. Then there exists an element $x \in I$ such that $x^{\circ\circ} \wedge a = a$. That implies $(a]_{\circ\circ} \subseteq (x]_{\circ\circ} \in \sigma(I)$. Since $\sigma(I)$ is an ideal of $\mathfrak{M}_{\circ\circ}(L)$ and by lemma-2.3, we get that $(a]_{\circ\circ} \in \sigma(I)$. Therefore $a \in \overleftarrow{\sigma}\sigma(I)$. Thus $I_{\circ\circ} = \overleftarrow{\sigma}\sigma(I)$. Let $a \in \overleftarrow{\sigma}\sigma(I)$. Then $(a]_{\circ\circ} \in \sigma(I)$. Then there exists an element $b \in I$ such that $(a]_{\circ\circ} = (b]_{\circ\circ}$. That implies $a \in (b]_{\circ\circ}$ and hence $b^{\circ\circ} \wedge a = a$. Since I is an ideal of L, we get that $a \in I_{\circ\circ}$. Therefore $\overleftarrow{\sigma}\sigma(I) \subseteq I_{\circ\circ}$. Hence $I_{\circ\circ} = \overleftarrow{\sigma}\sigma(I)$.
3. Clear
4. Let $I,\ J \in \mathfrak{I}(L)$. Now $\overleftarrow{\sigma}\sigma(I \cap J) = (I \cap J)_{\circ\circ} = I_{\circ\circ} \cap J_{\circ\circ} = \overleftarrow{\sigma}\sigma(I) \cap \overleftarrow{\sigma}\sigma(J)$. Now $\overleftarrow{\sigma}\sigma(I \vee J) = (I \vee J)_{\circ\circ} = I_{\circ\circ} \vee J_{\circ\circ} = \overleftarrow{\sigma}\sigma(I) \vee \overleftarrow{\sigma}\sigma(J)$. Clearly, we have that $\overleftarrow{\sigma}\sigma(\{0\}) = \overleftarrow{\sigma}(\{0\}) = \{0\}$ and $\overleftarrow{\sigma}\sigma(L) = \overleftarrow{\sigma}((M)_{\circ\circ}(L)) = L$. Therefore $\overleftarrow{\sigma}\sigma$ is homomorphism. □

Definition 3.15. A closure ideal I of an $MS-$ADL L is said to be prime if I is a prime ideal of L.

Theorem 3.16. *Let L be an $MS-$ADL. Then there is an isomorphism of the lattice of closure ideals of L onto the ideal lattice of $\mathfrak{M}_{\circ\circ}(L)$. Under this isomorphism the prime closure ideals corresponding to prime ideals of $\mathfrak{M}_{\circ\circ}(L)$.*

Proof. Define $g : \mathfrak{I}_C(L) \longrightarrow \mathfrak{I}(\mathfrak{M}_{\circ\circ}(L))$ by $g(I) = \sigma(I)$. Clearly g is well defined. Let $I,\ J \in \mathfrak{I}_C(L)$ with $g(I) = g(J)$. Then $\sigma(I) = \sigma(J)$ and hence $\overleftarrow{\sigma}\sigma(I) = \overleftarrow{\sigma}\sigma(J)$. Therefore $I = J$. Thus g is one-one. Let $\widetilde{I}$ be an ideal of $\mathfrak{M}_{\circ\circ}(L)$. Then $\overleftarrow{\sigma}(\widetilde{I})$ is an ideal of L. That implies $\sigma(\overleftarrow{(}\widetilde{I})) = \widetilde{I}$. That implies $\overleftarrow{\sigma}(\sigma(\overleftarrow{\sigma}(\widetilde{I}))) = \overleftarrow{\sigma}(\widetilde{I})$. That implies $\overleftarrow{\sigma}(\widetilde{I})$ is a closure ideal of L. Now $g(\overleftarrow{\sigma}(\widetilde{I})) = \sigma\overleftarrow{\sigma}(\widetilde{I}) = \widetilde{I}$. Therefore g is onto. Since σ is homomorphism, we get g is homomorphism. Hence g is an isomorphism. Let I be a prime closure ideal of L. Now we prove that $g(I)$ is a prime ideal of $\mathfrak{M}_{\circ\circ}(L)$. Let $(a]_{\circ\circ}, (b]_{\circ\circ} \in \mathfrak{M}_{\circ\circ}(L)$ with $(a]_{\circ\circ} \cap (b]_{\circ\circ} \in g(I) = \sigma(I)$. Then $(a \wedge b]_{\circ\circ} \in \sigma(I)$. Then there exists an element $c \in I$ such that $(a \wedge b]_{\circ\circ} = (c]_{\circ\circ}$. Since I is a closure ideal of L and $c \in I$, we get that $a \wedge b \in I$. Since I is a prime ideal of L, we have that either $a \in I$ or $b \in I$. That implies $(a]_{\circ\circ} \in \sigma(I)$ or $(b]_{\circ\circ} \in \sigma(I)$. Therefore $\sigma(I) = g(I)$ is a prime ideal of $\mathfrak{M}_{\circ\circ}(L)$. Let $\widetilde{I}$ be a prime ideal of $\mathfrak{M}_{\circ\circ}(L)$. Since g is onto, there exists an closure ideal I of L such that $g(I) = \widetilde{I}$. Since $g(I) = \sigma(I)$, we have that $\sigma(I) = \widetilde{I}$. Let $a, b \in L$ with $a \wedge b \in I$. Then $(a \wedge b]_{\circ\circ} \in \sigma(I)$. That implies $(a]_{\circ\circ} \cap (b]_{\circ\circ} \in \sigma(I) = \widetilde{I}$. Since $\widetilde{I}$ is a prime ideal of $\mathfrak{M}_{\circ\circ}(L)$, we have that $(a]_{\circ\circ} \subseteq \widetilde{I} = \sigma(I)$ or $(b]_{\circ\circ} \subseteq \widetilde{I} = \sigma(I)$. That implies $a \in \overleftarrow{\sigma}\sigma(I)$ or $b \in \overleftarrow{\sigma}\sigma(I)$. That implies $a \in I$ or $b \in I$. Therefore I is a prime ideal of L. □

Lemma 3.17. *Let L and L' be two $MS-$ADLs and $h : L \longrightarrow L'$, a homomorphism. Then we have the following:*

1. *for any nonempty subset S of L, $h(S_{\circ\circ}) \subseteq (h(S))_{\circ\circ}$*
2. *for any nonempty subset T of L', $(h^{-1}(T))_{\circ\circ} \subseteq h^{-1}(T_{\circ\circ})$.*

Proof. 1. Let S be any nonempty subset of L. Let $a \in h(S_{\circ\circ})$. Then there exists an element $b \in S_{\circ\circ}$ such that $a = h(b)$. Since $b \in S_{\circ\circ}$, there exists an element $s \in A$ such that $s^{\circ\circ} \wedge b = b$. Now $a = h(b) = h(s^{\circ\circ} \wedge b) = h(s^{\circ\circ}) \wedge h(b) = (h(s))^{\circ\circ} \wedge h(b) = (h(s))^{\circ\circ} \wedge h(b)$. Since $h(s) \in h(S)$, we get that $a = h(b) \in (h(S))_{\circ\circ}$ and hence $h(s_{\circ\circ}) \subseteq (h(S))_{\circ\circ}$.
2. Let T be any nonempty subset of L'. Let $a \in (h^{-1}(T))_{\circ\circ}$. Then there exists an element $b \in h^{-1}(T)$ such that $b^{\circ\circ} \wedge a = a$. Since $b \in h^{-1}(T)$, we get that $h(b) \in T$. Now $h(a) = h(b^{\circ\circ} \wedge a) = h(b^{\circ\circ}) \wedge h(a) = (h(b))^{\circ\circ} \wedge h(a)$. That implies $h(a) \in T_{\circ\circ}$ and hence $a \in h^{-1}(T_{\circ\circ})$. Therefore $(h^{-1}(T))_{\circ\circ} \subseteq h^{-1}(T_{\circ\circ})$. □

In general, $h(S_{\circ\circ}) \subseteq (h(S))_{\circ\circ}$ and $h^{-1}(T_{\circ\circ}) \subseteq (h^{-1}(T))_{\circ\circ}$ are not true.

Example 3.18. Let L be the five element chain $0 < a < b < c < d < 1$ and $a^\circ = b^\circ = b,\ d^\circ = 0$. Clearly L is an $MS-$algebra. Define $h : L \longrightarrow L$ by $h(0) = 0,\ h(a) = h(b),\ h(d) = d,\ h(1) = 1$. Clearly h is a homomorphism. Take $S = T = \{0, a\}$. Then $S_{\circ\circ} = \{0, a, b\}$ and $h(S) = \{0, b\}$. That implies $h(S_{\circ\circ}) = \{0, b\}$ and $(h(S))_{\circ\circ} = \{0, a, b\}$. Therefore $(h(S))_{\circ\circ} \nsubseteq h(A_{\circ\circ})$. We have that $h^{-1}(T) = \{0\}$ and $T_{\circ\circ} = \{0, a, b\}$. That implies $(h^{-1}(T))_{\circ\circ} = \{0\}$ and $h^{-1}(T_{\circ\circ}) = \{0, a, b\}$. Therefore $h^{-1}(T_{\circ\circ}) \nsubseteq (h^{-1}(T))_{\circ\circ}$.

Definition 3.19. Let L and L' be two $MS-$ADLs and $h : L \longrightarrow L'$, a homomorphism. h is called closure ideal preserving if $h(I_{\circ\circ}) = (h(I))_{\circ\circ}$, for any ideal I of L.

Theorem 3.20. *Let L and L' be two $MS-$ADLs and $h : L \longrightarrow L'$, onto homomorphism. Then h is a closure ideal preserving.*

Theorem 3.21. *Let L and L' be two $MS-$ADLs and $h : L \longrightarrow L'$, onto homomorphism. Then we have the following:*

1. *for any $x \in L,\ h((x]_{\circ\circ}) = (h(x)]_{\circ\circ}$*
2. *for any closure ideal I of $L,\ h(I)$ is a closure ideal of L'*

3. *for any closure ideal I of L, $h(I) = \bigcup\limits_{i\in I}(h(i)]_{\circ\circ}$.*

Proof. 1. Let $x \in L$. Now $a \in h((x]_{\circ\circ})$ iff $a = h(y)$, for some $y \in (x]_{\circ\circ}$ iff $a = h(x^{\circ\circ} \wedge y)$, (since $y \in (x]_{\circ\circ},\ x^{\circ\circ} \wedge y = y$) iff $a = h(x^{\circ\circ}) \wedge h(y)$ iff $a = (h(x))^{\circ\circ} \wedge h(y) = h(y)$ iff $h(y) \in ((h(x))^{\circ\circ}]_{\circ\circ}$ iff $a \in (h(x)]_{\circ\circ}$. Therefore $h((x]_{\circ\circ}) = (h(x)]_{\circ\circ}$.
2. Let I be a closure ideal of L. Let $a, b \in h(I)$. Then there exist elements $x, y \in I$ such that $a = h(x)$ and $b = h(y)$. Since $x, y \in I$, we get that $x \vee y \in I$. Now $a \vee b = h(x) \vee h(y) = h(x \vee y) \in h(I)$. Therefore $a \vee b \in h(I)$. Let $a \in h(I)$. Then there exists an element $x \in I$ such that $a = h(x)$. Let r be any element ofL'. Since h is onto, there exists an element $y \in L$ such that $h(y) = r$. Since $x \in I,\ y \in L$ and I is an ideal of L, we have that $x \wedge y \in I$. Now $a \wedge r = h(x) \wedge h(y) = h(x \wedge y) \in h(I)$. That implies $x \wedge r \in h(I)$. Therefore $h(I)$ is an ideal of L'. clearly, we have that $h(I) \subseteq \overleftarrow{\sigma}\sigma(h(I))$. Let $a \in \overleftarrow{\sigma}\sigma(h(I))$. Then $(a]_{\circ\circ} \in \sigma(h(I))$. Then there exists an element $b \in h(I)$ such that $(a]_{\circ\circ} = (y]_{\circ\circ}$. That implies $a \in (b]_{\circ\circ} \subseteq h(I_{\circ\circ}) = h(I)$, since $I_{\circ\circ} = I$. That implies $a \in h(I)$ and hence $\overleftarrow{\sigma}\sigma(h(I)) \subseteq h(I)$. Therefore $\overleftarrow{\sigma}\sigma(h(I)) = h(I)$. Thus $h(I)$ is a closure ideal of L'.
3. Let I be a closure ideal of L. Then $I = I_{\circ\circ} = \bigcup\limits_{i\in I}(i]_{\circ\circ}$. That implies $(i]_{\circ\circ} \subseteq I$, for all $i \in I$. That implies $h((i]_{\circ\circ}) \subseteq h(I)$ and hence $(h(i)]_{\circ\circ} \subseteq h(I)$. Therefore $\bigcup\limits_{i\in I}(h(i)]_{\circ\circ} \subseteq h(I)$. Let $a \in h(I)$. Then there exists an element $b \in I$ such that $a = h(b)$. Now $a = h(b) \in (h(b)]_{\circ\circ} \subseteq \bigcup\limits_{b\in I}(h(b)]_{\circ\circ}$ and hence $a \in \bigcup\limits_{i\in I}(h(i)]_{\circ\circ}$. Therefore $h(I) \subseteq \bigcup\limits_{i\in I}(h(i)]_{\circ\circ}$. Thus $h(I) = \bigcup\limits_{i\in I}(h(i)]_{\circ\circ}$. □

Theorem 3.22. *Let L and L' be two $MS-$ADLs and $h : L \longrightarrow L'$, a homomorphism. Then we have the following:*

1. *for any closure ideal I of L', $h^{-1}(I)$ is a closure ideal of L*
2. *$Kerh$ is a closure ideal of L*
3. *for a closure ideal I of L', $h^{-1}(I_{\circ\circ}) = (h^{-1}(I))_{\circ\circ}$.*

Proof. 1. Let I be any closure ideal of L'. Clearly, $h^{-1}(I)$ is an ideal of L. Since $h^{-1}(I) \subseteq \overleftarrow{\sigma}\sigma(h^{-1}(I))$, we have to prove that $\overleftarrow{\sigma}\sigma(h^{-1}(I)) \subseteq h^{-1}(I)$. Let $a \in \overleftarrow{\sigma}\sigma(h^{-1}(I))$. Then $(a]_{\circ\circ} \in \sigma(h^{-1}(I))$. Then there exists an element $b \in h^{-1}(I)$ such that $(a]_{\circ\circ} = (b]_{\circ\circ}$. That implies $(a]_{\circ\circ} = (b]_{\circ\circ}$. That implies $h((a]_{\circ\circ}) = h((b]_{\circ\circ})$ and hence $(h(a)]_{\circ\circ} = (h(b)]_{\circ\circ}$. Therefore $h(a) \in I$, since $h(b) \in I$ and I is a closure ideal of L'. That implies $a \in h^{-1}(I)$ and hence $\overleftarrow{\sigma}\sigma(h^{-1}(I)) \subseteq h^{-1}(I)$. Therefore $h^{-1}(I) = \overleftarrow{\sigma}\sigma(h^{-1}(I))$. Thus $h^{-1}(I)$ is a closure ideal of L.
2. Clearly, we have that $Kerh$ is an ideal of L and $kerh \subseteq \overleftarrow{\sigma}\sigma(kerh)$. Let $a \in \overleftarrow{\sigma}\sigma(kerh)$. Then $(a]_{\circ\circ} \in \sigma(kerh)$. Then there exists an element $b \in kerh$ such that $(a]_{\circ\circ} = (b]_{\circ\circ}$. That implies $(a]_{\circ\circ} = (b]_{\circ\circ}$ and $h(b) = 0'$. That implies $a \in (b]_{\circ\circ}$ and $h(b) = 0'$. That implies $b^{\circ\circ} \wedge a = a$ and $h(b) = 0'$. That implies $h(b^{\circ\circ} \wedge a) = h(a)$ and $h(b) = 0'$. That implies $h(b^{\circ\circ}) \wedge h(a) = h(a)$ and $h(b) = 0'$. That implies $(h(b))^{\circ\circ} \wedge h(a) = h(a)$ and $(h(b))^{\circ\circ} = (0)^{\circ\circ} = 0'$. That implies $h(a) = 0'$ and hence $a \in kerh$. Therefore $\overleftarrow{\sigma}\sigma(kerh) \subseteq kerh$. Thus $kerh$ is a closure ideal of L.
3. Let I be any closure ideal of L'. Then $I = I_{\circ\circ}$. That implies $h^{-1}(I) = h^{-1}(I_{\circ\circ})$ is a closure ideal of L. That implies $h^{-1}(I_{\circ\circ}) = (h^{-1}(I))_{\circ\circ}$. Therefore h^{-1} is a closure ideal preserving. □

Theorem 3.23. *Let L and L' be two $MS-$ADLs and $h : L \longrightarrow L'$, onto homomorphism. Then we have the following:*

1. *$\mathfrak{M}_{\circ\circ}(L)$ is De Morgan homomorphic of $\mathfrak{M}_{\circ\circ}(L')$*
2. *$\mathfrak{I}_C(L)$ is homomorphic of $\mathfrak{I}_C(L')$.*

Proof. 1. Define $f : \mathfrak{M}_{\circ\circ}(L) \longrightarrow \mathfrak{M}_{\circ\circ}(L')$ by $f((x]_{\circ\circ})h((x]_{\circ\circ})$. Let $(x]_{\circ\circ},\ (y]_{\circ\circ} \in \mathfrak{M}_{\circ\circ}(L)$. Now, $f((x]_{\circ\circ} \vee (y]_{\circ\circ}) = f((x \vee y]_{\circ\circ}) = h((x \vee y]_{\circ\circ}) = (h(x \vee y)]_{\circ\circ} = (h(x) \vee h(y)]_{\circ\circ} = (h(x)]_{\circ\circ} \vee (h(y)]_{\circ\circ} = h((x]_{\circ\circ}) \vee h((y]_{\circ\circ}) = f((x]_{\circ\circ}) \vee f((y]_{\circ\circ})$. Now, $f((x]_{\circ\circ} \cap (y]_{\circ\circ}) = f((x \wedge y]_{\circ\circ}) = h((x \wedge y]_{\circ\circ}) = (h(x \wedge y)]_{\circ\circ} = (h(x) \wedge h(y)]_{\circ\circ} = (h(x)]_{\circ\circ} \cap (h(y)]_{\circ\circ} = h((x]_{\circ\circ}) \cap h((y]_{\circ\circ}) = f((x]_{\circ\circ}) \cap f((y]_{\circ\circ})$. Clearly, we have that $f((0]_{\circ\circ}) = (0']_{\circ\circ}$ and $f((m]_{\circ\circ}) = (m']_{\circ\circ}$, where 0 and $0'$ are the zero elements of L and L' respectively, and m and m' are maximal

elements of L and L' respectively.
2. Define $g : \mathfrak{I}_C(L) \longrightarrow \mathfrak{I}_C(L')$ by $g(I) = h(I)$. Clearly we have that $g(L) = L'$, and $g(\{0\}) = \{0'\}$, where 0 and $0'$ are the zero elements of L and L' respectively. Let $I,\ J \in \mathfrak{I}_C(L)$. Now $g(I \vee J) = h(I \vee J) = h(I) \vee h(J) = g(I) \vee g(J)$ and now $g(I \cap J) = h(I \cap J) = h(I) \cap h(J) = g(I) \cap g(J)$. Therefore g is homomorphism. □

Theorem 3.24. *Let I be a closure ideal and F, a filter of an $MS-$ADL L with $F \cap I = \emptyset$. There exists a prime closure ideal P of L such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof. Consider $\mathfrak{F} = \{G \mid G$ is a closure ideal and $G \cap F = \emptyset\}$. Clearly, $I \in \mathfrak{F}$ and $\mathfrak{F}$ satisfies the Zorn's lemma hypothesis. Then $\mathfrak{F}$ has a maximal element say N. Let $a, b \in L$ with $a \wedge b \in N$. We prove that either $a \in N$ or $b \in N$. Suppose that $a \notin N$ and $b \notin N$. Then $N \subset N \vee (a] \subseteq \overleftarrow{\sigma}\sigma(N \vee (a])$ and $N \subset N \vee (b] \subseteq \overleftarrow{\sigma}\sigma(N \vee (b])$. That implies $N \subset \overleftarrow{\sigma}\sigma(N \vee (a])$ and $N \subset \overleftarrow{\sigma}\sigma(N \vee (b])$. Since $\overleftarrow{\sigma}\sigma(N \vee (a])$ and $\overleftarrow{\sigma}\sigma(N \vee (b])$ are closure ideals of L, we get that $\overleftarrow{\sigma}\sigma(N \vee (a]) \cap F \neq \emptyset$ and $\overleftarrow{\sigma}\sigma(N \vee (b]) \cap F \neq \emptyset$. Then choose $x \in \overleftarrow{\sigma}\sigma(N \vee (a]) \cap F$ and $y \in \overleftarrow{\sigma}\sigma(N \vee (b]) \cap F$. Therefore $x \wedge y \in F$ and $x \wedge y \in \overleftarrow{\sigma}\sigma(N \vee (a]) \cap \overleftarrow{\sigma}\sigma(N \vee (b]) = \overleftarrow{\sigma}\sigma((N \vee (a]) \cap (N \vee (b])) = \overleftarrow{\sigma}\sigma(N \vee (a \wedge b]) = \overleftarrow{\sigma}\sigma(N) = N$. Therefore $N \cap F \neq \emptyset$, which is a contradiction. Hence $a \in N$ or $b \in N$. Thus N is a prime closure ideal of L. □

Corollary 3.25. *Let I be a closure ideal of an $MS-$ADL L and $x \notin I$. Then there exists a prime closure ideal P of L such that $I \subseteq P$ and $x \notin P$.*

Corollary 3.26. *For any closure ideal I of an $MS-$ADL L, we have*
$I = \cap\{P/P$ is a closure ideal of L and $I \subseteq P\}$

Corollary 3.27. *The intersection of all prime closure ideals of an $MS-$ADL L is $\{0\}$.*

We discuss some topological properties of prime closure ideals. For this, we first need the following.

Theorem 3.28. *Let L be an $MS-$ADL. Then every proper closure ideal of L is the intersection of all prime closure ideals containing it.*

Proof. Let I be a proper closure ideal of L. Consider the following set
$\mathfrak{F}_0 = \cap\{P \mid P$ is a prime closure ideal and $I \subseteq P\}$. Clearly, $I \subseteq \mathfrak{F}_0$. Conversely, let $x \notin I$. Take $\mathfrak{F} = \{G \mid G$ is a closure ideal, $I \subseteq G, x \notin G\}$. Then clearly $I \in \mathfrak{F}$. Clearly $\mathfrak{F}$ satisfies the hypothesis of Zorn's lemma. Let N be a maximal element of $\mathfrak{F}$. Let $a, b \in L$ be such that $a \notin N$ and $b \notin N$. Then $N \subset N \vee (a] \subseteq \overleftarrow{\sigma}\sigma\{N \vee (a]\}$ and $N \subset N \vee (b] \subseteq \overleftarrow{\sigma}\sigma\{N \vee (b]\}$. By maximality of N, we get $x \in \overleftarrow{\sigma}\sigma\{N \vee (a]\}$ and $x \in \overleftarrow{\sigma}\sigma\{N \vee (b]\}$. Hence we get that $x \in \overleftarrow{\sigma}\sigma\{N \vee (a]\} \cap \overleftarrow{\sigma}\sigma\{N \vee (b]\} = \overleftarrow{\sigma}\sigma\{[N \vee (a]] \cap [N \vee (b]]\} = \overleftarrow{\sigma}\sigma\{N \vee (a \wedge b]\}$. If $a \wedge b \in N$, then $x \in \overleftarrow{\sigma}\sigma(N) = N$, which is a contradiction. Thus N is a prime closure ideal such that $x \notin N$. Therefore $x \notin \mathfrak{F}_0$ and hence $\mathfrak{F} = \mathfrak{F}_0$. Thus every proper closure ideal of L is the intersection of all prime closure ideals containing it. □

4 The Space of Prime closure ideals

In this section, we discuss some topological concepts on the collection of prime closure ideals of an $MS-$ADL. Let $Spec_C(L)$ be the set of all prime closure ideals of an $MS-$ADL L. For any $A \subseteq L$, let $h(A) = \{P \in Spec_C(L) \mid A \nsubseteq P\}$ and for any $x \in L$; $h(x) = h(\{x\})$. For any two subsets A and B of L, it is obvious that $A \subseteq B$ implies $h(A) \subseteq h(B)$. The following observations can be verified directly.

Lemma 4.1. *For any $x, y \in L$, the following conditions holds.*

1. $\bigcup_{x \in L} h(x) = Spec_C(L)$
2. $h(x) \cup h(y) = h(x \vee y)$
3. $h(x) \cap h(y) = h(x \wedge y)$

4. $h(x) = \emptyset \Leftrightarrow x = 0$
5. $h(x) = Spec_C(L) \Leftrightarrow x$ *is a maximal element of* L.

From the above Lemma, it can be easily observed that the collection $\{h(x) \mid x \in L\}$ forms a base for a topology on $Spec_C(L)$ which is called a hull-kernel topology.

Theorem 4.2. *For any ideal* I *of* L, $h(I) = h(\overleftarrow{\sigma}\sigma(I))$.

Proof. Clearly we get that $h(I) \subseteq h(\overleftarrow{\sigma}\sigma(I))$. Let $P \in h(\overleftarrow{\sigma}\sigma(I))$. Then $\overleftarrow{\sigma}\sigma(I) \nsubseteq P$. Therefore we can choose an element $x \in \overleftarrow{\sigma}\sigma(I)$ such that $x \notin P$. Since $x \in \overleftarrow{\sigma}\sigma(I)$, we have $(x]_{\circ\circ} \in \sigma(I)$ and hence $(x]_{\circ\circ} = (y]_{\circ\circ}$, for some $y \in I$. Suppose $I \subseteq P$. Then $y \in P$. Since P is a closure ideal of L, we get that $x \in P$, which is a contradiction. Therefore $I \nsubseteq P$ and hence $P \in h(I)$. Thus $h(\overleftarrow{\sigma}\sigma(I)) \subseteq h(I)$. □

In the following theorem, the compact open set of $Spec_C(L)$ are characterized.

Theorem 4.3. *For any* $MS-$*ADL, the set of all compact open sets of* $Spec_C(L)$ *is the base* $\{h(x) \mid x \in L\}$.

Proof. Let $x \in L$ with $h(x) \subseteq \bigcup_{i\in\Delta} h(x_i)$. Let I be a ideal generated by $\{x_i \mid i \in \Delta\}$. Suppose $x \notin \overleftarrow{\sigma}\sigma(I)$. Since $\overleftarrow{\sigma}\sigma(I)$ is a closure ideal of L, there exists a prime closure ideal P of L such that $x \notin P$ and $\overleftarrow{\sigma}\sigma(I) \subseteq P$. Since $x \notin P$, we get that $P \in h(x) \subseteq \bigcup_{i\in\Delta} h(x_i)$. That implies $x_i \notin P$, for some $i \in \Delta$, which is a contradiction to that $I \subseteq \overleftarrow{\sigma}\sigma(I) \subseteq P$. Therefore $x \in \overleftarrow{\sigma}\sigma(I)$. That implies $(x]_{\circ\circ} \in \sigma(I)$ and hence $(x]_{\circ\circ} = (y]_{\circ\circ}$, for some $y \in I$. Since I is an ideal generated by $\{x_i \mid i \in \Delta\}$, we get that $y = x_1 \vee x_2 \vee \cdots \vee x_n$, for some $x_1, x_2, ..., x_n \in \{x_i \mid i \in \Delta\}$. That implies $(y]_{\circ\circ} = (x_1 \vee x_2 \vee \cdots \vee x_n]_{\circ\circ}$. Let $P \in h(x)$. Then $x \notin P$. Suppose $P \notin \bigcup_{i\in\Delta} h(x_i)$. Then $x_i \in P$, for all $i = 1, 2, ..., n$ and hence $x_1 \vee x_2 \vee \cdots \vee x_n \in P$. That implies $y \in P$, which is a contradiction. Therefore $P \in \bigcup_{i\in\Delta} h(x_i)$ and hence $h(x) \subseteq \bigcup_{i=1}^{n} h(x_i)$. Thus $h(x)$ is a compact space. It is enough to show that every compact open subset of $Spec_C(L)$ is of the form $h(x)$, for some $x \in L$. Let C be a compact open subset of $Spec_C(L)$. Since C is open, we get that $C = \bigcup_{a\in A} h(a)$, for some $A \subseteq L$. Since C is compact, there exist $a_1, a_2, ..., a_n \in A$ such that $C = \bigcup_{i=1}^{n} h(a_i) = h(\bigvee_{i=1}^{n} a_i)$. Therefore $C = h(x)$, for some $x \in L$. □

Corollary 4.4. *Let* L *be an* $MS-$*ADL. Then* $Spec_C(L)$ *is a compact space.*

Theorem 4.5. *Let* L *be an* $MS-$*ADL. Then the following are equivalent:*

1. $Spec_C(L)$ *is* T_1-*space*
2. *every prime closure ideal is maximal*
3. *every prime closure ideal is minimal*
4. $Spec_C(L)$ *is Hausdorff space.*

Proof. $1 \Rightarrow 2$: Assume that $Spec_C(L)$ is T_1-space. Let P be a prime closure ideal of L. Suppose Q is any prime closure ideal of L with $P \subsetneqq Q$. Since $Spec_C(L)$ is T_1-space, there exist basic open sets $h(x)$ and $h(y)$ such that $P \in h(x) \setminus h(y)$ and $Q \in h(y) \setminus h(x)$. Since $P \notin h(y)$, we get that $y \in P \subsetneqq Q$. Therefore $Q \notin h(y)$, which is a contradiction. Hence P is maximal.
$2 \Rightarrow 3$: it is obvious
$3 \Rightarrow 4$: Assume that every prime closure ideal is minimal. Let $P, Q \in Spec_C(L)$ with $P \neq Q$. Choose an element $a \in P$ such that $a \notin Q$. By our assumption, P is minimal prime ideal of L. Since $a \in P$, then there $c \notin P$ such that $a \wedge c = 0$. So that $Q \in h(a)$ and $P \in h(c)$. Now $h(a) \cap h(c) = h(a \vee c) = \emptyset$, since $a \vee c = 0$.
$4 \Rightarrow 1$: Clear. □

References

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. XXV, Providence, U.S.A (1967).

[2] A.E. Badawy and M. Sambasiva Rao, *Closure ideals of MS-algebras,* Chamchuri Journal of Mathematics,**6**, 31–46, (2014).

[3] G. Gratzer, *General Lattice Theory*, Academic Press, New York, Sanfransisco, (1978).

[4] G.C. Rao, *Almost Distributive Lattices*, Doctoral Thesis, Dept. of Mathematics, Andhra University, Visakhapatnam, (1980).

[5] G.C. Rao, N. Rafi and B. Ravikumar, *On the prime filters of normal almost distributive lattices*, Southeast Asian Bulletin of Mathematics, **35**, 653–663, (2011).

[6] G.C. Rao, N. Rafi and B. Ravikumar, *Topological characterization of dually normal almost distributive lattices*, Asian European Journal of Mathematics, **5**, 1250043, DOI: 10-1142/S17355711250043X, (2012).

[7] G.C. Rao and S. Ravi Kumar, *Minimal prime ideals in an ADL*, Int. J. Contemp. Sciences, **4**, 475–484, (2009).

[8] G.M. Addis, $MS-$*almost distributive lattices*, Asian-European Journal of Mathematics, **13(1)**, 2050135 (14 pages), (2020).

[9] U.M. Swamy and G.C. Rao, *Almost Distributive Lattices*, J. Aust. Math. Soc. (Series A), **31**, 77–91, (1981).

[10] U.M. Swamy, G.C. Rao and G. Nanaji Rao, *Pseudo-complementation on Almost Distributive Lattices*, Southeast Asian Bullettin of Mathematics, **24**, 95–104, (2000).

Author information

N. Rafi[1], Ravi Kumar Bandaru[2] and M. Srujana[3],
[1,3]Department of Mathematics, Bapatla Engineering College, Bapatla, Andhra Pradesh, India-522 101
[2]Department of Mathematics, GITAM(Deemed to be University), Hyderabad Campus, Telangana, India-502 329,.
E-mail: rafimaths@gmail.com, ravimaths83@gmail.com, srujana.maths@gmail.com

Received: August 17, 2020
Accepted: October 27, 2020