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ON *n*-ABSORBING PRIMARY SUBMODULES

Nilofer I. Shaikh and Rajendra P. Deore

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Abstract Let R be a commutative ring with $1 \neq 0$, n be a positive integer and M be an R-module. In this paper, we introduce the concept of n-absorbing primary submodules generalising n-absorbing primary ideals of rings. A proper submodule N of an R-module M is called an n-absorbing primary submodule if whenever $a_1 \dots a_n m \in N$ for $a_1, \dots, a_n \in R$ and $m \in M$, then either $a_1 \dots a_n \in \sqrt{(N : M)}$ or there are $n - 1$ of the $a_i's$ whose product with m is in N. We have tried to prove some results on n-absorbing primary submodules.

1 Introduction

In this paper, all rings are commutative with non-zero identity and all modules are unital. Let R be a ring, I be an ideal of R, M be an R-module and N be a submodule of M. The radical of I is denoted by $\sqrt{I}$ i.e. $\sqrt{I} = \{r \in R : r^k \in I \text{ for some } k \in \mathbb{N}\}$. We denote the residual of N over M by $(N : M)$ i.e. $(N : M) = \{r \in R : rM \subseteq N\}$.

The first generalisation of prime ideals in commutative rings was introduced by Ayman Badawi in [4], where he defined a non zero proper ideal I of R to be a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Expanding on this definition, Anderson and Badawi in [1] introduced the concept of n-absorbing ideals of R for a positive integer n. A proper ideal I of a commutative ring R is called as n-absorbing ideal if whenever $x_1 \dots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$, then there are n of the $x_i's$ whose product is in I.

In [5], Badawi introduced a generalisation of primary ideals, where he defined a proper ideal I of R to be a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Recentely, A. E. Becker generalised 2-absorbing primary ideals to n-absorbing primary ideals for positive integer n in [6]. A proper ideal I of a commutative ring R is said to be an n-absorbing primary ideal of R if whenever $x_1, \dots, x_{n+1} \in R$ and $x_1 x_2 \dots x_{n+1} \in I$, then either $x_1 x_2 \dots x_n \in I$ or a product of n of the $x_i's$ (other than $x_1 \dots x_n$) is in $\sqrt{I}$.

The concept of 2-absorbing and weakly 2-absorbing submodules, which are generalisations of prime and weak prime submodules, was introduced and investigated by Darani and Soheilnia in [7]. They defined a proper submodule N of an R-module M to be 2-absorbing (respectively weakly 2-absorbing) submodule of M if whenever $a, b \in R,\ m \in M$ and $abm \in N$ (resp. $0 \neq abm \in N$), then $ab \in (N : M)$ or $am \in N$ or $bm \in N$. Later in [8], Darani and Soheilnia introduced and studied the concept of n-absorbing submodules generalising n-absorbing ideals of rings. They defined a proper submodule N of an R-module M to be an n-absorbing submodule if whenever $a_1 \dots a_n m \in N$ for $a_1, \dots, a_n \in R$ and $m \in M$, then either $a_1 \dots a_n \in (N : M)$ or there are $n - 1$ of the $a_i's$ whose product with m is in N. In [9], M. K. Dubey and P. Aggarwal introduced the concept of 2-absorbing primary submodule which is a generalisation of primary submodule. They defined a proper submodule N of an R-module M to be 2-absorbing primary submodule if whenever $a, b \in R,\ m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N : M)}$.

In this paper, we generalise the concept of n-absorbing primary ideals of a ring R to that of n-absorbing primary submodules of an R-module M. Let n be a positive integer. A proper submodule N of an R-module M is said to be an n-absorbing primary submodule of M if whenever $a_1, \ldots, a_n \in R$, $m \in M$ and $a_1 \ldots a_n m \in N$, then either $a_1 \ldots a_n \in \sqrt{(N : M)}$ or there are $n-1$ of the $a_i's$ whose product with m is in N. We have proved several properties of n-absorbing primary submodules. Most of the results are related to the references [6], [8] and [9] which have been proved for n-absorbing primary submodules.

In Theorem 2.3, we have proved that if N is an n-absorbing primary submodule of a cyclic multiplication R-module M, then $(N : M)$ is an n-absorbing primary ideal of R. In Theorem 2.8, we have shown that any n-absorbing primary submodule is an m-absorbing primary submodule for $m \geq n$. In Theorem 2.15, we have given a characterisation of an n-absorbing primary submodule when it is irreducible.

2 n-absorbing primary submodule

In this section, we define n-absorbing primary submodule and prove several results related to the same.

Definition 2.1. Let n be a positive integer. Let M be an R-module and N be a proper submodule of M. N is said to be an n-absorbing primary submodule of M if for any $a_1, \ldots, a_n \in R$ and $m \in M$, $a_1 \ldots a_n m \in N$ implies either $a_1 \ldots a_n \in \sqrt{(N : M)}$ or there are $n-1$ of the $a_i's$ whose product with m is in N.

Let $\hat{a}_i$ denote the element of R obtained by eliminating a_i from the product $a_1 \ldots a_n$. Then the above condition can be written as $a_1 \ldots a_n m \in N$ implies either $a_1 \ldots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i m \in N$ for some $1 \leq i \leq n$.

It is easy to see that every n-absorbing submodule is an n-absorbing primary submodule but the converse need not be true which is illustrated as follows.

Example 2.2. Consider $R = \mathbb{Z}$ and an R-module $M = \mathbb{Z}_{162}$. Take a submodule $N = \{0, 81\}$ of M. Then $(N : M) = \{r \in R : rM \subseteq N\} = \{0, 81, 162, \ldots\} = 81\mathbb{Z}$ and $\sqrt{(N : M)} = \{r \in R : r^k M \subseteq N \text{ for some } k \in \mathbb{N}\} = \{0, 3, 6, 9, \ldots\} = 3\mathbb{Z}$. Now, $3 \cdot 3 \cdot 3 \cdot 3 \in N$ but $3 \cdot 3 \cdot 3 \notin N$ and $3 \cdot 3 \cdot 3 \notin (N : M)$. Therefore, N is not a 3-absorbing submodule of M but it is a 3-absorbing primary submodule of M since $3 \cdot 3 \cdot 3 \in \sqrt{(N : M)}$.

A natural question is that if N is an n-absorbing primary submodule of an R-module M, then is the ideal $(N : M)$ an n-absorbing primary ideal of R? This is true in the case when $n = 2$ and has been proved in [9, Theorem 2.6]. For the case where M is cyclic, we get the following result.

Theorem 2.3. *Let M be a cyclic multiplication R-module. Let N be an n-absorbing primary submodule of M. Then $(N : M)$ is an n-absorbing primary ideal of R.*

Proof. Let M be a cyclic R-module generated by m. Let $a_1 \ldots a_{n+1} \in (N : M)$ for some $a_1, \ldots, a_{n+1} \in R$. Assume all products of n of the $a_i's$ except $a_1 \ldots a_n$ are not in $\sqrt{(N : M)}$. Then $\hat{a}_i a_{n+1} \notin (N : M)$ for every $1 \leq i \leq n$, that is, $\hat{a}_i a_{n+1} m \notin N$ for every $1 \leq i \leq n$. Since $a_1 \ldots a_{n+1} \in (N : M)$, $a_1 \ldots a_{n+1} m \in N$, which we can write as $(a_2 \ldots a_{n+1})(a_1 m) \in N$. As N is an n-absorbing primary submodule of M, this implies either $(a_2 \ldots a_n)(a_1 m) \in N$ or $(a_3 \ldots a_{n+1})(a_1 m) \in N$ or $(a_2 a_4 \ldots a_{n+1})(a_1 m) \in N$ or . . . or $(a_2 \ldots a_{n-1} a_{n+1})(a_1 m) \in N$ or $a_2 \ldots a_{n+1} \in \sqrt{(N : M)}$ i.e. either $a_1 a_2 \ldots a_n m \in N$ or $\hat{a}_i a_{n+1} m \in N$ for some $2 \leq i \leq n$ or $\hat{a}_1 a_{n+1} \in \sqrt{(N : M)}$. Since by assumption, $\hat{a}_i a_{n+1} \notin \sqrt{(N : M)}$ for every $1 \leq i \leq n$, both the latter cases are not possible. Therefore $a_1 \ldots a_n m \in N$, which implies $a_1 \ldots a_n \in (N : M)$. Thus $(N : M)$ is an n-absorbing primary ideal of R. □

We state the following theorem which is used in this paper.

Theorem 2.4. *([6, Theorem 9]) If I is an n-absorbing primary ideal of R, then $\sqrt{I}$ is an n-absorbing ideal of R.*

Theorem 2.5. *Let N be an n-absorbing primary submodule of a cyclic multiplication R-module M. Then $\sqrt{(N : M)}$ is an n-absorbing ideal of R.*

Proof. By Theorem 2.3, we get that $(N : M)$ is an n-absorbing primary ideal of R. Then by Theorem 2.4, $\sqrt{(N : M)}$ is an n-absorbing ideal of R. □

We now give the following result using the ideal $(N : m)$, defined as $(N : m) = \{r \in R : rm \in N\}$, where R is a commutative ring, M is an R-module, N is a submodule of M and $m \in M$.

Theorem 2.6. *Let N be an n-absorbing primary submodule of an R-module M. If $m \in N$, then $(N : m) = R$. If $m \notin N$, then $(N : m)$ is an n-absorbing primary ideal of R containing $(N : M)$.*

Proof. If $m \in N$, then there is nothing to prove. Let $m \in M \setminus N$. Then $(N : m)$ is a proper ideal of R containing $(N : M)$. Let $a_1 \dots a_{n+1} \in (N : m)$ for some $a_1, \dots, a_{n+1} \in R$. Assume all products of n of the $a_i's$ except $a_1 \dots a_n$ are not in $\sqrt{(N : m)}$. Since $a_1 \dots a_{n+1} \in (N : m)$, $a_1 \dots a_{n+1}m \in N$, that is, $(a_2 \dots a_{n+1})(a_1m) \in N$ and N is an n-absorbing primary submodule of M. This implies either $(a_2 \dots a_n)(a_1m) \in N$ or $(a_3 \dots a_{n+1})(a_1m) \in N$ or $(a_2a_4 \dots a_{n+1})(a_1m) \in N$ or . . . or $(a_2 \dots a_{n-1}a_{n+1})(a_1m) \in N$ or $a_2 \dots a_{n+1} \in \sqrt{(N : M)}$ i.e. either $a_1 \dots a_nm \in N$ or $\hat{a}_ia_{n+1}m \in N$ for some $2 \leq i \leq n$ or $\hat{a}_1a_{n+1} \in \sqrt{(N : M)}$. Therefore either $a_1 \dots a_n \in (N : m)$ or $\hat{a}_ia_{n+1} \in (N : m)$ for some $2 \leq i \leq n$ or $\hat{a}_1a_{n+1} \in \sqrt{(N : m)}$. Since by assumption, $\hat{a}_ia_{n+1} \notin \sqrt{(N : m)}$ for every $1 \leq i \leq n$, both the latter cases are not possible. Therefore $a_1 \dots a_n \in (N : m)$. Thus $(N : m)$ is an n-absorbing primary ideal of R. □

The set of zero divisors of an R-module M is denoted by $Zd(M)$ and is defined as $Zd(M) = \{r \in R : \text{there exists } 0 \neq m \in M \text{ such that } rm = 0\}$.

Theorem 2.7. *Let N be an n-absorbing primary submodule of M. If the set of all zero divisors of M/N, $Zd(M/N)$, forms an ideal in R, then it is an n-absorbing primary ideal of R.*

Proof. Assume $Zd(M/N)$ is an ideal in R. Let $a_1 \dots a_{n+1} \in Zd(M/N)$ for some $a_1, \dots, a_{n+1} \in R$. We know from [3] that if M is an R-module and N is a proper submodule of M, then $Zd(M/N) = \bigcup_{x \in M \setminus N} (N : x)$. Therefore $a_1 \dots a_{n+1} \in (N : m)$ for some $m \in M \setminus N$. Since N is an n-absorbing primary submodule and $m \in M \setminus N$, by Theorem 2.6, $(N : m)$ is an n-absorbing primary ideal of R. This implies either $a_1 \dots a_n \in (N : m)$ or $\hat{a}_ia_{n+1} \in \sqrt{(N : m)}$ for some $1 \leq i \leq n$. If $a_1 \dots a_n \in (N : m)$, then $a_1 \dots a_n \in Zd(M/N)$ and we are done. We know from [2] that if R is a ring and E_α is a family of subsets of R, then $\sqrt{\bigcup_\alpha E_\alpha} = \bigcup_\alpha \sqrt{E_\alpha}$. Therefore $\sqrt{Zd(M/N)} = \sqrt{\bigcup_{x \in M \setminus N} (N : x)} = \bigcup_{x \in M \setminus N} \sqrt{(N : x)}$. If $\hat{a}_ia_{n+1} \in \sqrt{(N : m)}$ for some $1 \leq i \leq n$, then $\hat{a}_ia_{n+1} \in \sqrt{Zd(M/N)}$ for some $1 \leq i \leq n$. Thus we get that $Zd(M/N)$ is an n-absorbing primary ideal of R. □

Theorem 2.8. *Every n-absorbing primary submodule of an R-module is an m-absorbing primary submodule for $m \geq n$.*

Proof. It is sufficient to prove that every n-absorbing primary submodule of an R-module is an $(n + 1)$-absorbing primary submodule. Suppose N is an n-absorbing primary submodule of an R-module M. Let $a_1 \dots a_na_{n+1}m \in N$ for some $a_1, \dots, a_n, a_{n+1} \in R$ and $m \in M$. Let $a_na_{n+1} := a_{n'}$. Then we have $a_1a_2 \dots a_{n'}m \in N$ and N is an n-absorbing primary submodule. This implies either $a_1a_2 \dots a_{n'} \in \sqrt{(N : M)}$ or $\hat{a}_im \in N$ for some $i \in \{1, 2, 3, \dots, n-1, n'\}$. If $i \neq n'$, then we are done. If $i = n'$, then we have $a_1 \dots a_{n-1}m \in N$ and by definition of an ideal, we get that $a_1 \dots a_{n-1}a_nm \in N$ or $a_1 \dots a_{n-1}a_{n+1}m \in N$. Hence N is an $(n + 1)$-absorbing primary submodule of M. □

We now examine the structure of the intersection of k submodules that are each n_j-absorbing primary submodule of an R-module. For this, we first prove the following lemma.

Lemma 2.9. *Let N_j be submodules of an R-module M for every $1 \leq j \leq k$. Then* $\bigcap_{j=1}^{k} \sqrt{(N_j : M)} = \sqrt{(\bigcap_{j=1}^{k} N_j : M)}$.

Proof. Let $r \in \bigcap_{j=1}^{k} \sqrt{(N_j : M)}$. Then $r \in \sqrt{(N_j : M)}$ for every $1 \leq j \leq k$. Therefore $r^{l_j} M \subseteq N_j$ for every $1 \leq j \leq k$, where l_j is some positive integer. Let $l = max\{l_1, \ldots, l_k\}$. Then $r^l M \subseteq N_j$ for every $1 \leq j \leq k$ and so $r^l M \subseteq \bigcap_{j=1}^{k} N_j$. Thus $r \in \sqrt{(\bigcap_{j=1}^{k} N_j : M)}$. For the reverse inclusion, let $s \in \sqrt{(\bigcap_{j=1}^{k} N_j : M)}$. Then $s^n M \subseteq \bigcap_{j=1}^{k} N_j$ for some positive integer n. This implies $s^n M \subseteq N_j$ for every $1 \leq j \leq k$, that is, $s \in \sqrt{(N_j : M)}$ for every $1 \leq j \leq k$. Therefore $s \in \bigcap_{j=1}^{k} \sqrt{(N_j : M)}$. Hence $\bigcap_{j=1}^{k} \sqrt{(N_j : M)} = \sqrt{(\bigcap_{j=1}^{k} N_j : M)}$. □

Theorem 2.10. *Let M be an R-module. If N_j is an n_j-absorbing primary submodule of M for every $1 \leq j \leq k$, then $N_1 \cap \cdots \cap N_k$ is an n-absorbing primary submodule of M for $n = n_1 + \cdots + n_k$. In particular, if $N_1, \ldots, N_n$ are primary submodules of M, then $N_1 \cap \cdots \cap N_n$ is an n-absorbing primary submodule of M.*

Proof. Let $a_1, \ldots, a_n \in R$ and $m \in M$ with $a_1 \ldots a_n m \in N_1 \cap \cdots \cap N_k := N$ such that $\hat{a_i} m \notin N$ for every $1 \leq i \leq n$. Since $a_1 \ldots a_n m \in N_1 \cap \cdots \cap N_k$, $a_1 \ldots a_n m \in N_j$ for every $1 \leq j \leq k$. Now, for every $1 \leq j \leq k$, N_j is an n_j-absorbing primary submodule of M and $n_j \leq n$. Therefore by Theorem 2.8, each N_j is an n-absorbing primary submodule of M. This implies $a_1 \ldots a_n \in \sqrt{(N_j : M)}$ for every $1 \leq j \leq k$, which gives that $a_1 \ldots a_n \in \bigcap_{j=1}^{k} \sqrt{(N_j : M)} = \sqrt{(\bigcap_{j=1}^{k} N_j : M)}$ by Lemma 2.9. Thus $a_1 \ldots a_n \in \sqrt{(N : M)}$, proving that, N is an n-absorbing primary submodule of M. The "In particular" statement is clear. □

Theorem 2.11. *Let N be an n-absorbing primary submodule of an R-module M and K be a submodule of M. Then $N \cap K$ is an n-absorbing primary submodule of K.*

Proof. Clearly, $N \cap K$ is a proper submodule of K. Let $a_1 \ldots a_n k \in N \cap K$ for some $a_1, \ldots, a_n \in R$ and $k \in K$. Then $a_1 \ldots a_n k \in N$ and N is an n-absorbing primary submodule of M. This implies either $\hat{a_i} k \in N$ for some $1 \leq i \leq n$ or $a_1 \ldots a_n \in \sqrt{(N : M)}$. If $\hat{a_i} k \in N$ for some $1 \leq i \leq n$, then $\hat{a_i} k \in N \cap K$ for some $1 \leq i \leq n$ and we are done. If $a_1 \ldots a_n \in \sqrt{(N : M)}$, then $(a_1 \ldots a_n)^m M \subseteq N$ for some positive integer m. In particular, $(a_1 \ldots a_n)^m K \subseteq N$. Therefore $(a_1 \ldots a_n)^m K \subseteq N \cap K$, which implies $a_1 \ldots a_n \in \sqrt{(N \cap K : K)}$. Hence $N \cap K$ is an n-absorbing primary submodule of K. □

Theorem 2.12. *Let $M = M_1 \oplus M_2$ where M_1 and M_2 are R-modules. Let P and Q be proper submodules of M_1 and M_2 respectively. Then the following statements hold.*

(1) $P \oplus M_2$ is an n-absorbing primary submodule of M if and only if P is an n-absorbing primary submodule of M_1.

(2) $M_1 \oplus Q$ is an n-absorbing primary submodule of M if and only if Q is an n-absorbing primary submodule of M_2.

Proof. (1) Let $P \oplus M_2$ be an n-absorbing primary submodule of M. Let $a_1 \ldots a_n m \in P$ for some $a_1, \ldots, a_n \in R$ and $m \in M_1$ such that $\hat{a_i} m \notin P$ for every $1 \leq i \leq n$. Then $a_1 \ldots a_n (m, 0) \in P \oplus M_2$ but $(\hat{a_i} m, 0) \notin P \oplus M_2$ for every $1 \leq i \leq n$. As $P \oplus M_2$ is an

n-absorbing primary submodule of M, we get that $a_1 \dots a_n \in \sqrt{(P \oplus M_2 : M_1 \oplus M_2)}$. This implies $(a_1 \dots a_n)^k (M_1 \oplus M_2) \subseteq P \oplus M_2$ for some positive integer k. Therefore $(a_1 \dots a_n)^k M_1 \subseteq P$, that is, $a_1 \dots a_n \in \sqrt{(P : M_1)}$. Hence P is an n-absorbing primary submodule of M_1.

Conversely, let P be an n-absorbing primary submodule of M_1. Let $a_1, \dots, a_n \in R$ and $(m_1, m_2) \in M$ with $a_1 \dots a_n(m_1, m_2) \in P \oplus M_2$. Then $a_1 \dots a_n m_1 \in P$. Assume that $\hat{a_i}(m_1, m_2) \notin P \oplus M_2$ for every $1 \leq i \leq n$, which gives that $\hat{a_i} m_1 \notin P$ for every $1 \leq i \leq n$. As P is an n-absorbing primary submodule of M_1, this implies that $a_1 \dots a_n \in \sqrt{(P : M_1)}$, that is, $(a_1 \dots a_n)^k M_1 \subseteq P$ for some positive integer k. Therefore $(a_1 \dots a_n)^k (M_1 \oplus M_2) \subseteq P \oplus M_2$. Hence $P \oplus M_2$ is an n-absorbing primary submodule of M.

(2) Proof is smiliar to (1). □

Let M be an R-module and N be a submodule of M. For $r \in R$, $(N : r)$, also denoted by N_r is defined as $N_r = (N : r) = \{m \in M : rm \in N\}$. Clearly, N_r is a submodule of M containing N.

Theorem 2.13. *Let N be an n-absorbing primary submodule of an R-module M. Then $N_r = (N : r)$ is an n-absorbing primary submodule of M containing N for all $r \in R \setminus (N : M)$.*

Proof. Let $r \in R \setminus (N : M)$. Let $a_1 \dots a_n m \in (N : r)$ for some $a_1, \dots, a_n \in R$ and $m \in M$. Then $a_1 \dots a_n(rm) \in N$ and N is an n-absorbing primary submodule of M. This implies either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a_i} rm \in N$ for some $1 \leq i \leq n$. If $a_1 \dots a_n \in \sqrt{(N : M)}$, then $(a_1 \dots a_n)^k M \subseteq N$ for some positive integer k. Therefore $(a_1 \dots a_n)^k M \subseteq N_r$ as $N \subseteq N_r$. This gives that $a_1 \dots a_n \in \sqrt{(N_r : M)}$ and we are done. If for some $1 \leq i \leq n$, $\hat{a_i} rm \in N$, then $\hat{a_i} m \in (N : r)$ for some $1 \leq i \leq n$. Thus $(N : r)$ is an n-absorbing primary submodule of M containing N. □

Theorem 2.14. *Let N be a submodule of an R-module M. Then the following are equivalent.*

(1) N is an n-absorbing primary submodule of M.

(2) For $a_1, \dots, a_n \in R$ such that $a_1 \dots a_n \notin \sqrt{(N : M)}$, $N_{a_1 \dots a_n} = \bigcup_{i=1}^{n} N_{\hat{a_i}}$ where $\hat{a_i} = a_1 \dots a_{i-1} a_{i+1} \dots a_n$.

Proof. (1) $\Rightarrow$ (2) Assume that N is an n-absorbing primary submodule of M. For $a_1, \dots, a_n \in R$, let $a_1 \dots a_n \notin \sqrt{(N : M)}$. Let $m \in N_{a_1 \dots a_n}$. Then $a_1 \dots a_n m \in N$ and N is an n-absorbing primary submodule. This implies either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a_i} m \in N$ for some $1 \leq i \leq n$. Since by assumption, $a_1 \dots a_n \notin \sqrt{(N : M)}$, $\hat{a_i} m \in N$ for some $1 \leq i \leq n$, that is, $m \in N_{\hat{a_i}}$ for some $1 \leq i \leq n$. Thus $m \in \bigcup_{i=1}^{n} N_{\hat{a_i}}$. Now, let $k \in \bigcup_{i=1}^{n} N_{\hat{a_i}}$. Then $\hat{a_i} k \in N$ for some $1 \leq i \leq n$. Therefore $a_i \hat{a_i} k = a_1 \dots a_n k \in N$. Thus $k \in N_{a_1 \dots a_n}$. Hence we get that $N_{a_1 \dots a_n} = \bigcup_{i=1}^{n} N_{\hat{a_i}}$.

(2) $\Rightarrow$ (1) Let $a_1 \dots a_n m \in N$ for some $a_1, \dots, a_n \in R$ and $m \in M$ such that $a_1 \dots a_n \notin \sqrt{(N : M)}$. Then by assumption, $N_{a_1 \dots a_n} = \bigcup_{i=1}^{n} N_{\hat{a_i}}$. As $a_1 \dots a_n m \in N$, $m \in N_{a_1 \dots a_n} = \bigcup_{i=1}^{n} N_{\hat{a_i}}$. Therefore $m \in N_{\hat{a_i}}$ for some $1 \leq i \leq n$, that is, $\hat{a_i} m \in N$ for some $1 \leq i \leq n$. Thus N is an n-absorbing primary submodule of M. □

A submodule N of an R-module M is said to be irreducible if it cannot be expressed as the intersection of two submodules of M. We now give a characterisation of an n-absorbing primary submodule when it is irreducible.

Theorem 2.15. *Let N be an irreducible proper submodule of an R-module M. Then N is an n-absorbing primary submodule of M if and only if $(N : r^{n-1}) = (N : r^n)$ for all $r \in R \setminus \sqrt{(N : M)}$.*

Proof. Assume that N is an n-absorbing primary submodule of M. Let $r \in R \setminus \sqrt{(N : M)}$. Clearly, $(N : r^{n-1}) \subseteq (N : r^n)$. For the reverse inclusion, let $m \in (N : r^n)$. Then $r^n m \in N$ and

N is an n-absorbing primary submodule of M. Therefore either $r^{n-1}m \in N$ or $r^n \in \sqrt{(N:M)}$. If $r^{n-1}m \in N$, then $m \in (N : r^{n-1})$ and we are done. If $r^n \in \sqrt{(N:M)}$, then $r \in \sqrt{(N:M)}$, which is a contradiction.

Conversely, assume that $(N : r^{n-1}) = (N : r^n)$ for all $r \in R \setminus \sqrt{(N:M)}$. Let $a_1 \dots a_n m \in N$ for some $a_1, \dots, a_n \in R$ and $m \in M$ such that $a_1 \dots a_n \notin \sqrt{(N:M)}$. Then we have to show that $\hat{a}_i m \in N$ for some $1 \le i \le n$. On the contrary, we assume that $\hat{a}_i m \notin N$ for every $1 \le i \le n$. If $a_i \in \sqrt{(N:M)}$ for some $1 \le i \le n$, then $a_1 \dots a_n \in \sqrt{(N:M)}$, which is a contradiction. Therefore $a_i \notin \sqrt{(N:M)}$ for every $1 \le i \le n$. Hence by assumption $(N : a_i^{n-1}) = (N : a_i^n)$ for every $1 \le i \le n$. Clearly, $N + Ra_1^{n-1}m$ and $N + R\hat{a}_1 m$ are submodules of M and $N \subseteq (N + Ra_1^{n-1}m) \cap (N + R\hat{a}_1 m)$. Let $n \in (N + Ra_1^{n-1}m) \cap (N + R\hat{a}_1 m)$. Then $n = n_1 + r_1 a_1^{n-1} m = n_2 + r_2 \hat{a}_1 m$ where $r_1, r_2 \in R$ and $n_1, n_2 \in N$. Therefore $a_1 n = a_1 n_1 + r_1 a_1^n m = a_1 n_2 + r_2 a_1 \dots a_n m$ and $r_2 a_1 \dots a_n m,\ a_1 n_2,\ a_1 n_1 \in N$, so $r_1 a_1^n m \in N$, which implies $r_1 m \in (N : a_1^n)$. But $(N : a_1^n) = (N : a_1^{n-1})$. Therefore $r_1 a_1^{n-1} m \in N$ and hence $n \in N$. Therefore $(N + Ra_1^{n-1}m) \cap (N + R\hat{a}_1 m) \subseteq N$. Thus we get that $N = (N + Ra_1^{n-1}m) \cap (N + R\hat{a}_1 m)$, which is a contradiction as N is an irreducible submodule of M. Hence N is an n-absorbing primary submodule of M. □

Theorem 2.16. *Let $f : M \to M'$ be an epimorphism of R-modules. Then the following statements hold.*

(1) If N is an n-absorbing primary submodule of M such that $Ker f \subseteq N$, then $f(N)$ is an n-absorbing primary submodule of M'.

(2) If N' is an n-absorbing primary submodule of M', then $f^{-1}(N')$ is an n-absorbing primary submodule of M.

Proof. (1) Assume N is an n-absorbing primary submodule of M such that $Ker f \subseteq N$. Let $a_1 \dots a_n m' \in f(N)$ for some $a_1, \dots, a_n \in R$ and $m' \in M'$. Then $a_1 \dots a_n m' = f(t)$ for some $t \in N$. As $m' \in M'$ and f is an epimorphism, there exists $m \in M$ such that $f(m) = m'$. Therefore $a_1 \dots a_n f(m) = f(t)$, that is, $f(a_1 \dots a_n m - t) = 0$. This implies $a_1 \dots a_n m - t \in Ker f \subseteq N$. Thus $a_1 \dots a_n m \in N$ and N is an n-absorbing primary submodule of M. Therefore either $a_1 \dots a_n \in \sqrt{(N:M)}$ or $\hat{a}_i m \in N$ for some $1 \le i \le n$. This implies either $a_1 \dots a_n \in \sqrt{(f(N):M')}$ or $\hat{a}_i m' \in f(N)$ for some $1 \le i \le n$. Hence $f(N)$ is an n-absorbing primary submodule of M'.

(2) Assume N' is an n-absorbing primary submodule of M'. Let $a_1 \dots a_n m \in f^{-1}(N')$ for some $a_1, \dots, a_n \in R$ and $m \in M$. Then $a_1 \dots a_n f(m) \in N'$ and N' is an n-absorbing primary submodule of M'. This implies either $a_1 \dots a_n \in \sqrt{(N':M')}$ or $\hat{a}_i f(m) \in N'$ for some $1 \le i \le n$. Therefore either $a_1 \dots a_n \in \sqrt{(f^{-1}(N'):M)}$ or $\hat{a}_i m \in f^{-1}(N')$ for some $1 \le i \le n$. Hence $f^{-1}(N')$ is an n-absorbing primary submodule of M. □

Theorem 2.17. *Let N and K be submodules of an R-module M such that $K \subseteq N$. Then N is an n-absorbing primary submodule of M if and only if N/K is an n-absorbing primary submodule of M/K.*

Proof. Define $f : M \to M/K$ by $f(m) = m + K$. Then f is an epimorphism of R-modules M and M/K. Assume that N is an n-absorbing primary submodule of M. Now, $Ker f = K \subseteq N$. Then by Theorem 2.16 (1), $f(N)$ is an n-absorbing primary submodule of M/K. Hence N/K is an n-absorbing primary submodule of M/K.

Conversely, assume that N/K is an n-absorbing primary submodule of M/K. Then by Theorem 2.16 (2), $f^{-1}(N/K)$ is an n-absorbing primary submodule of M. Hence N is an n-absorbing primary submodule of M. □

Theorem 2.18. *Suppose S is a multiplicatively closed subset of R and $S^{-1}M$ is the module of fraction of M. Then the following statements hold.*

(1) If N is an n-absorbing primary submodule of M such that $(N : M) \cap S = \emptyset$, then $S^{-1}N$ is an n-absorbing primary submodule of $S^{-1}M$.

(2) If $S^{-1}N$ is an n-absorbing primary submodule of $S^{-1}M$ such that $Zd(M/N) \cap S = \emptyset$, then N is an n-absorbing primary submodule of M.

Proof. (1) Assume N is an n-absorbing primary submodule of M such that $(N : M) \cap S = \emptyset$. Let $\frac{a_1}{s_1} \dots \frac{a_n}{s_n} \frac{m}{l} \in S^{-1}N$ for some $a_1, \dots, a_n \in R,\ s_1, \dots, s_n, l \in S$ and $m \in M$. Then there exists $s' \in S$ such that $s'a_1 \dots a_n m \in N$. Since N is an n-absorbing primary submodule of M, we get that either $a_1 \dots a_n \in \sqrt{(N : M)}$ or $\hat{a}_i s' m \in N$ for some $1 \leq i \leq n$. This implies either $\frac{a_1}{s_1} \dots \frac{a_n}{s_n} \in S^{-1}\sqrt{(N : M)} = \sqrt{(S^{-1}N : S^{-1}M)}$ or $\frac{\hat{a}_i}{\hat{s}_i} \frac{m}{l} \in S^{-1}N$ for some $1 \leq i \leq n$. Thus $S^{-1}N$ is an n-absorbing primary submodule of $S^{-1}M$.

(2) Assume $S^{-1}N$ is an n-absorbing primary submodule of $S^{-1}M$ such that $Zd(M/N) \cap S = \emptyset$. Let $a_1 \dots a_n m \in N$ for some $a_1, \dots, a_n \in R$ and $m \in M$. Then $\frac{a_1 \dots a_n m}{1} \in S^{-1}N$ and $S^{-1}N$ is an n-absorbing primary submodule of $S^{-1}M$. Therefore either $\frac{a_1 \dots a_n}{1} \in \sqrt{(S^{-1}N : S^{-1}M)}$ or $\frac{\hat{a}_i m}{1} \in S^{-1}N$ for some $1 \leq i \leq n$. If $\frac{a_1 \dots a_n}{1} \in \sqrt{(S^{-1}N : S^{-1}M)} = S^{-1}\sqrt{(N : M)}$, then there exists $s \in S$ such that $(sa_1 \dots a_n)^k M \subseteq N$ for some positive integer k, that is, $s^k (a_1 \dots a_n)^k M \subseteq N$. As $Zd(M/N) \cap S = \emptyset$, this implies $(a_1 \dots a_n)^k M \subseteq N$ and we are done. If $\frac{\hat{a}_i m}{1} \in S^{-1}N$ for some $1 \leq i \leq n$, then there exists $t \in S$ such that $t\hat{a}_i m \in N$ for some $1 \leq i \leq n$, which gives that $\hat{a}_i m \in N$ for some $1 \leq i \leq n$. Hence N is an n-absorbing primary submodule of M. □

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Author information

Nilofer I. Shaikh, Department of Mathematics, University of Mumbai, Mumbai-400098, Maharashtra, India.
E-mail: `shaikhnilofer23@gmail.com`

Rajendra P. Deore, Department of Mathematics, University of Mumbai, Mumbai-400098, Maharashtra, India.
E-mail: `rpdeore@gmail.com`

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