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IDEALS OF TRANSITIVE BE-ALGEBRAS

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Abstract. The notion of ideals is introduced in transitive BE-algebras. Some characterization theorems of ideals of transitive BE-algebras are derived. The notion of semi-ideals is introduced and studied a relationship between semi-ideals and ideals. Properties of ideals are studied with the help of homomorphisms and congruences.

1 Introduction

The concept of BE-algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [6]. The class of BE-algebras was introduced as a generalization of the class of BCK-algebras of K. Iseki and S. Tanaka [5]. Some properties of filters of BE-algebras were studied by S.S. Ahn and Y.H. Kim in [1] and by B.L. Meng in [7]. In [10], A. Walendziak discussed some relationships between congruence relations and normal filters of a BE-algebra. In [9], P. Sun investigated homomorphism theorems via dual ideals of BCK-algebras.

In this work, the notion of ideals is introduced in transitive BE-algebras as a generalization of special type of down sets in many algebraic structures. Some necessary and sufficient conditions are derived for a non-empty subsets of BE-algebras to become ideals. The concepts of semi-ideals and strong semi-ideals are introduced and then some relations among these sets of ideals are studied. Some properties of ideals are derived in terms of homomorphisms and congruences.

2 Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [2], [3], [6], [7]and [8] for the ready reference.

Definition 2.1. [6] An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1$,
(2) $x * 1 = 1$,
(3) $1 * x = x$,
(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra X is called *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called *transitive* if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. Every self-distributive BE-algebra is transitive. A BE-algebra X is called *commutative* if $(x*y)*y = (y * x) * x$ for all $x, y \in X$. We introduce a relation $\leq$ on X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. If X is commutative, then the relation $\leq$ is a partial ordering on X.

Theorem 2.2. [7] *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1 \leq x$ *implies* $x = 1$,
(2) $y \leq z$ *implies* $x * y \leq x * z$.

Definition 2.3. [6] A non-empty subset F of a BE-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,
(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

A BE-algebra X is called bounded [3], if there exists an element 0 satisfying $0 \leq x$ (or $0 * x = 1$) for all $x \in X$. Define an unary operation N on a bounded BE-algebra X by $xN = x * 0$ for all $x \in X$.

Theorem 2.4. [3] *Let X be a transitive BE-algebra and $x, y \in X$. Then*

(1) $1N = 0$ *and* $0N = 1$,
(2) $x \leq xNN$,
(3) $x * yN = y * xN$.

An element x of a bounded BE-algebra X is called *dense* [8] if $xN = 0$ and $\mathcal{D}(X)$ denotes the class of all dense elements of the BE-algebra X. Let X and Y be two bounded BE-algebras, then a homomorphism $f : X \to Y$ is called bounded [2] if $f(0) = 0$. If f is a bounded homomorphism, then it is easily observed that $f(xN) = f(x)N$ for all $x \in X$.

Definition 2.5. [2] An element x of a bounded BE-algebra X is called an *involutory element* if $xNN = x$. If every element of a BE-algebra X is involutory, then X is called an *involutory*.

3 Ideals of Transitive BE-algebras

In this section, some properties of ideals of transitive BE-algebras are studied. Some characterization theorems of ideals are derived. The notions of semi-ideals and strong semi-ideals are introduced and obtained the relationship among the classes of ideals, semi-ideals and strong semi-ideals.

Definition 3.1. A non-empty subset I of a BE-algebra X is called an *ideal* of X if it satisfies the following conditions for all $x, y \in X$:

(I1) $0 \in I$,
(I2) $x \in I$ and $(xN * yN)N \in I$ imply that $y \in I$.

Obviously the single-ton set $\{0\}$ is an ideal of a BE-algebra X. For, suppose $x \in \{0\}$ and $(xN * yN)N \in \{0\}$ for $x, y \in X$. Then $x = 0$ and $yNN = (0N * yN)N \in \{0\}$. Hence $y \leq yNN = 0 \in \{0\}$. Thus $\{0\}$ is an ideal of X. In the following example, we observe non-trivial ideals of a BE-algebra.

Example 3.2. Let $X = \{1, a, b, c, d, 0\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra. It can be easily verified that the set $I = \{0, c, d\}$ is an ideal of X. However, the set $J = \{0, a, b, d\}$ is not an ideal of X, because of $a \in J$ and $(aN * cN)N = (d * b)N = aN = d \in J$ but $c \notin J$.

Some properties of ideals of transitive BE-algebras are now observed. Here after, by a BE-algebra X we mean a bounded BE-algebra $(X, *, 0, 1)$ unless and otherwise mentioned.
In the following lemma, we fist observe a few essential properties of transitive BE-algebras.

Lemma 3.3. *Let X be a transitive BE-algebra . For any $x, y \in X$, we have*

(1) $xNNN \leq xN$,

(2) $x * y \leq yN * xN$,
(3) $x * yN \leq xNN * yN$,
(4) $(x * yNN)NN \leq x * yNN$,
(5) $(xN * yN)NN \leq xN * yN$.

Proof. (1). Let $x \in X$. Then $1 = (x*0)*(x*0) = x*((x*0)*0) = x*xNN \leq x*xNNNN = xNNN * xN$. Hence $xNNN * xN = 1$, which gives $xNNN \leq xN$.
(2). Let $x, y \in X$. Since X is transitive, we get $yN = y*0 \leq (x*y)*(x*0) = (x*y)*xN$. Hence $1 = yN*yN \leq yN*((x*y)*xN) = (x*y)*(yN*xN)$. Thus, we get $(x*y)*(yN*xN) = 1$. Therefore $x * y \leq yN * xN$.
(3). Let $x, y \in X$. Then, we get $x * yN = y * xN \leq y * xNNN = xNN * yN$.
(4). Let $x, y \in X$. Clearly $(x * yNN)N \leq (x * yNN)NNN$. Since X is transitive, we get $yN * (x * yNN)N \leq yN * (x * yNN)NNN$ and so $x * (yN * (x * yNN)N) \leq x * (yN * (x * yNN)NNN)$. Hence, we get

$$\begin{aligned} 1 &= (x * yNN) * (x * yNN) \\ &= x * ((x * yNN) * yNN) \\ &= x * (yN * (x * yNN)N) \\ &\leq x * (yN * (x * yNN)NNN) \\ &= x * ((x * yNN)NN * yNN \\ &= (x * yNN)NN * (x * yNN). \end{aligned}$$

Thus $(x * yNN)NN * (x * yNN) = 1$. Therefore $(x * yNN)NN \leq (x * yNN)$.
(5). Form (4), it can be easily verified. □

Proposition 3.4. *Let I be an ideal of a transitive BE-algebra X. Then we have*

(1) *For any $x, y \in X, x \in I$ and $y \leq x$ imply $y \in I$,*
(2) *For any $x, y \in X, xN = yN, x \in I$ imply $y \in I$,*
(3) *For any $x \in X, x \in I$ if and only if $xNN \in I$,*
(4) *$I \cap \mathcal{D}(X) \neq \emptyset$ if and only if $I = X$.*

Proof. (1). Let $x, y \in X$. Suppose $x \in I$ and $y \leq x$. Then $xN \leq yN$, which implies $xN * yN = 1$. Hence $(xN * yN)N = 0 \in I$. Since $x \in I$, we get $y \in I$.
(2). Let $x, y \in X$. Assume that $xN = yN$. Suppose $x \in I$. Then we get $(xN * yN)N = 1N = 0 \in I$. Since I is an ideal of X, we get $y \in I$.
(3). Let $x \in X$. Suppose $x \in I$. Then we get $(xN * xNNN)N = (xNN * xNN)N = 1N = 0 \in I$. Since $x \in I$, it yields $xNN \in I$. Conversely, let $xNN \in I$ for any $x \in X$. Since $x \leq xNN$, by property (1) we get that $x \in I$.
(4). Assume that $I \cap \mathcal{D}(X) \neq \emptyset$. Let $x \in I \cap \mathcal{D}(X)$. Then by condition (2), we get that $xN = 0$ and $1 = xNN \in I$. Hence by (1), we get that $I = X$. Conversely, assume that $I = X$. Therefore $1 \in I$ and so $I \cap \mathcal{D}(X) \neq \emptyset$. □

Some equivalent condition are now derived for every non-empty subset of a transitive BE-algebra to become an ideal. For this purpose, we observe the essential properties of the relation $\leq$ of bounded and transitive BE-algebras.

Lemma 3.5. *Let X be a transitive BE-algebra X. For any $x, y, z \in X$, we have*

(1) *$x \leq y$ implies $yN \leq xN$,*
(2) *$x \leq y$ implies $y * zN \leq x * zN$.*

Proof. (1). Let $x, y \in X$ be such that $x \leq y$. Then by Lemma 3.3(2), we get $1 = x*y \leq yN*xN$. Hence $yN * xN = 1$. Therefore $yN \leq xN$.
(2). Let $x, y \in X$ be such that $x \leq y$. Then by (1), we get $yN \leq xN$. Since X is transitive, we get $z * yN \leq z * xN$. Therefore $y * zN \leq x * zN$. □

Theorem 3.6. *Let X be a transitive BE-algebra and $\emptyset \neq I \subseteq X$. Then I is an ideal of X if and only if it satisfies the following property:*

$$xN \leq yN * zN \text{ implies that } z \in I \text{ for all } x, y \in I \text{ and } z \in X.$$

Proof. Assume that I is an ideal of X. Let $x, y \in I$ and $z \in X$. Suppose $xN \leq yN * zN$. Then $xN \leq yN * zN \leq (yN * zN)NN$ and hence $(xN * (yN * zN)NN)N = 1N = 0 \in I$. Since $x \in I$ and I is an ideal of X, we get that $(yN * zN)N \in I$. Since $y \in I$, it yields that $z \in I$.

Conversely, assume that I satisfies the given condition. Since $I \neq \emptyset$, choose $x \in I$. Clearly $xN \leq 1 = xN * 0N$. Then by the given condition, we get $0 \in I$. Let $x, y \in X$ be such that $x \in I$ and $(xN * yN)N \in I$. By Lemma 3.3(5), we get $(xN * yN)NN \leq xN * yN$. Now, by Lemma 3.5(2), we get

$$(xN * yN) * yN \leq (xN * yN)NN * yN.$$

Since X is transitive, the above consequence gives rise to

$$\begin{aligned} 1 &= (xN * yN) * (xN * yN) \\ &= xN * ((xN * yN) * yN) \\ &\leq xN * ((xN * yN)NN * yN). \end{aligned}$$

Hence, we get $xN \leq (xN * yN)NN * yN$. Since $x \in I$ and $(xN * yN)N \in I$, from the assumed condition, it gives $y \in I$. Therefore I is an ideal of X. □

Theorem 3.7. *Let I be a non-empty subset of a transitive BE-algebra X. Then I is an ideal of X if and only if it satisfies the following condition for all $x \in X$:*

$$\text{for all } a, b \in I, \ (aN * (bN * xN)NN)N = 0 \text{ implies } x \in I$$

Proof. Let $\emptyset \neq I \subseteq X$. Assume that I is an ideal of X. Let $a, b \in I$. Suppose $(aN * (bN * xN)NN)N = 0 \in I$. Since $a \in I$ and I is an ideal of X, we get that $(bN * xN)N \in I$. Since $b \in I$, we get that $x \in I$.

Conversely, assume that I satisfies the above condition. For any $x \in I$, we have $(xN * (xN * 0N)NN)N = (xN * (xN * 1)NN)N = (xN * 1NN)N = 1N = 0$. Hence by the given condition, we get $0 \in I$. Let $x, y \in X$. Suppose $x \in I$ and $(xN * yN)N \in I$. By Lemma 3.3(5), we get $(xN * yN)NN \leq xN * yN$. Now, Lemma 3.5(2), provides

$$(xN * yN) * yN \leq (xN * yN)NN * yN.$$

Using Lemma 3.5(1) and the transitivity of X, we get the following consequence:

$$\begin{aligned} (xN * ((xN * yN)NN * yN))N &\leq (xN * ((xN * yN) * yN))N \\ &= ((xN * yN) * (xN * yN))N \\ &= 1N \\ &= 0 \end{aligned}$$

which means $(xN * ((xN * yN)NN * yN))N = 0$. Since $x \in I$ and $(xN * yN)N \in I$, by the assumed condition, we get $y \in I$. Therefore I is an ideal of X. □

In [2], R. Borzooei and A.B. Saeid extensively studied the properties of involutory BE-algebras. For any x, y of an involutory BE-algebra, they proved that $x * y = yN * xN$. Hence the following proposition is straightforward:

Proposition 3.8. *Let X be a transitive and involutory BE-algebra and $\emptyset \neq I \subseteq X$. Then I is an ideal of X if and only if it satisfies the following conditions:*

(1) *$0 \in I$,*
(2) *for $x, y \in X$, $x \in I$ and $(y * x)N \in I$ imply $y \in I$.*

In the following, the notion of semi-ideals is introduced in BE-algebras.

Definition 3.9. Let X be a BE-algebra and $\emptyset \neq I \subseteq X$. Then I is said to be a *semi-ideal* of X if it satisfies the following properties, for all $x \in X$:

(SI1) $0 \in I$,
(SI2) $xNN \in I$ implies $x \in I$.

Clearly every ideal of a transitive BE-algebra is a semi-ideal but not the converse. If X is an involutory BE-algebra, then it is also observed that every subset containing 0 is a semi-ideal of X.

Example 3.10. Let $X = \{1, a, b, 0\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	0
1	1	a	b	0
a	1	1	1	a
b	1	a	1	0
0	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra. It can be easily verified that the set $I = \{0, a\}$ is a semi-ideal of X. I is not an ideal of X, because of $a \in I$ and $(aN * bN)N = (a * 0)N = aNN = a \in I$ but $b \notin I$.

Definition 3.11. Let X be a BE-algebra and $\emptyset \neq I \subseteq X$. Then I is said to be a strong semi-ideal of X if it satisfies the following properties:

(SI3) $0 \in I$,
(SI4) $x \in I$ implies $(yN * xN)N \in I$ for all $x, y \in X$.

Proposition 3.12. *Every ideal of a transitive BE-algebra is a strong semi-ideal.*

Proof. Let I be an ideal of a transitive BE-algebra X. Let $x, y \in X$. Suppose $x \in I$. Clearly $yN * xN \leq (yN * xN)NN$. Then by Lemma 3.5(1), we get

$$\begin{aligned}(xN * (yN * xN)NN)N &\leq (xN * (yN * xN))N \\ &= (yN * (xN * xN))N \\ &= (yN * 1)N \\ &= 1N \\ &= 0\end{aligned}$$

which concludes that $(xN * (yN * xN)NN)N = 0 \in I$. Since $x \in I$ and I is an ideal of X, we get $(yN * xN)N \in I$. Therefore I is a strong semi-ideal of X. □

Example 3.13. In the bounded BE-algebra given in Example 3.2, it is easy to check that the set $J = \{0, a, b, d\}$ is a strong semi-ideal of X. But J is not an ideal of X, because of $a \in J$ and $(aN * cN)N = d \in J$ but $c \notin J$.

Theorem 3.14. *A semi-ideal I of a transitive BE-algebra X is an ideal of X if and only if it satisfies the following properties:*

(1) *$x \in I$ implies $(yN * xN)N \in I$,*
(2) *$x \in I$ and $y \leq x$ imply $y \in I$,*
(3) *$a, b \in I$ implies $((aN * (bN * xN)) * xN)N \in I$*

for all $x, y \in X$.

Proof. Let I be a semi-ideal of X. Assume that I is an ideal of X. Let $x \in I$ and $y \in X$. Clearly $yN * xN \leq (yN * xN)NN$. Then by Lemma 3.5(1), we get that $(xN * (yN * xN)NN)N \leq (xN * (yN * xN))N = (yN * (xN * xN))N = (yN * 1)N = 1N = 0 \in I$. Hence $(xN * (yN * xN)NN)N \in I$. Since $x \in I$, we get $(yN * xN)N \in I$. Condition (2) is obtained by

Proposition 3.4(1).
Let $a, b \in I$. Then by putting $bN * xN = t$, we get

$$\begin{aligned}(aN * ((aN * tNN)NN * tNN)NN)N &\leq (aN * ((aN * tNN)NN * tNN))N \\ &\leq (aN * ((aN * tNN) * tNN))N \\ &= ((aN * tNN) * (aN * tNN))N \\ &= 1N \\ &= 0\end{aligned}$$

which yields $(aN * ((aN * tNN)NN * tNN)NN)N = 0 \in I$. Since $a \in I$, we get $((aN * tNN)NN * tNN)N \in I$. By Lemma 3.3(5) and Lemma 3.5(1), we get

$$\begin{aligned}(bN * ((aN * tNN)NN * xN)NN)N &\leq (bN * ((aN * tNN)NN * xN))N \\ &= ((aN * tNN)NN * (bN * xN))N \\ &\leq ((aN * tNN)NN * (bN * xN)NN)N \\ &= ((aN * tNN)NN * tNN)N \in I\end{aligned}$$

which gives $(bN * ((aN * tNN)NN * xN)NN)N \in I$. Since $b \in I$, we get $((aN * tNN)NN * xN)N \in I$. Now, we observe

$$\begin{aligned}((aN * t) * xN)N &\leq ((aN * tNN) * xN)N \\ &\leq ((aN * tNN)NN * xN)N \in I\end{aligned}$$

which concludes that $((aN * (bN * xN)) * xN)N = ((aN * t) * xN)N \in I$.

Conversely, assume that I satisfies the given conditions. By taking $x = y$ in the condition (1), it can be seen that $0 \in I$. Let $x, y \in X$. Suppose that $x \in I$ and $(xN * yN)N \in I$. Then we have the consequence condition (3):

$$\begin{aligned}yNN &= (1 * yN)N \\ &= (((xN * yN) * (xN * yN)) * yN)N \\ &= ((xN * ((xN * yN) * yN)) * yN)N \\ &\leq ((xN * ((xN * yN)NN * yN)) * yN)N \in I\end{aligned}$$

because of since $x \in I$ and $(xN * yN)N \in I$. By condition (2), we obtain $yNN \in I$. Since I is a semi-ideal, it yields $y \in I$. Thus I is an ideal of X. □

Corollary 3.15. *A strong semi-ideal I of a transitive BE-algebra X is an ideal of X if and only if it satisfies the following conditions for any $x, y \in X$:*

(1) *$x \in I$ and $y \leq x$ imply $y \in I$,*

(2) *$a, b \in I$ implies $((aN * (bN * xN)) * xN)N \in I$.*

Proposition 3.16. *The set-theoretic intersection of ideals (strong semi-ideals) of a transitive BE-algebra is again an ideal (strong semi-ideal).*

Proof. Let $\{I_\alpha\}_{\alpha\in\Delta}$ be a family of ideals of X. Clearly $0 \in I_\alpha$ for each $\alpha \in \Delta$. Hence $0 \in \bigcap_{\alpha\in\Delta} I_\alpha$. Let $x \in \bigcap_{\alpha\in\Delta} I_\alpha$ and $(xN * yN)N \in \bigcap_{\alpha\in\Delta} I_\alpha$. Then $x \in I_\alpha$ and $(xN * yN)N \in I_\alpha$ for each $\alpha \in \Delta$. Since each I_α is an ideal of X, we get $y \in I_\alpha$ for each $\alpha \in \Delta$. Hence $y \in \bigcap_{\alpha\in\Delta} I_\alpha$. Therefore $\bigcap_{\alpha\in\Delta} I_\alpha$ is an ideal of X. □

Example 3.17. Let $X = \{1, a, b, c, d, 0\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	d	b
b	1	a	1	c	c	a
c	1	1	b	1	b	d
d	1	1	1	1	1	c
0	1	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a transitive BE-algebra. It is easy to check that $I_1 = \{0, a, b\}$ and $I_2 = \{0, b, d\}$ are ideals of X. The set $I = I_1 \cup I_2 = \{0, a, b, d\}$ is not an ideal of X, because of $a \in I$ and $(aN * cN)N = d \in I$ but $c \notin I$.

As a generalization of Proposition 3.16, we can derive that the set-theoretic union of ideals (semi-ideals) of a transitive BE-algebra is again an ideal (semi-ideal) when ever the family of ideals form a chain (totally ordered set).

A homomorphism $f : X \to Y$ of bounded BE-algebras is called *bounded* if $f(0) = 0$. If f is bounded, then $f(xN) = f(x * 0) = f(x) * f(0) = f(x) * 0 = (f(x))N$ for all $x \in X$. For any bounded homomorphism $f : X \to Y$, define the *dual kernel* of the homomorphism f as $Dker(f) = \{x \in X \mid f(x) = 0\}$. It is easy to check that $Dker(f) = \{0\}$ whenever f is an injective homomorphism.

Lemma 3.18. *Let X and Y be two bounded BE-algebras. For any bounded homomorphism $f : X \to Y$, the dual kernel is an ideal of X.*

Proof. Clearly $0 \in Dker(f)$. Let $x \in Dker(f)$ and $(xN * yN)N \in Dker(f)$. Then $f(x) = 0$ and $(f(x)N * f(y)N)N = f((xN * yN))N = 0$. Thus $f(y)NN = 0$ and so $f(y) \le f(yNN) = f(y)NN = 0$. Hence $f(y) = 0$ and so $y \in Dker(f)$. Therefore $Dker(f)$ is an ideal of X. □

Proposition 3.19. *Let X and Y be two BE-algebras and $f : X \to Y$ a bounded homomorphism. Then $f^{-1}(I)$ is an ideal of X for any ideal I of Y.*

Proof. Let $f : X \to Y$ be a bounded homomorphism. Suppose I is an ideal of Y. Let $x, y \in X$ be such that $x \in f^{-1}(I)$ and $(xN * yN)N \in f^{-1}(I)$. Then $f(x) \in I$ and $(f(x)N * f(y)N)N = f((xN * yN)N) \in I$. Since $f(x) \in I$ and I is an ideal, we get $f(y) \in I$. Hence $y \in f^{-1}(I)$. Thus $f^{-1}(I)$ is an ideal of X. □

For any filter F of a self-distributive BE-algebra X, it was observed in [11] that θ_F defined by $(x, y) \in \theta_F \Leftrightarrow x * y \in F$ and $y * x \in F$ is the unique congruence whose kernel is F. If X is bounded, then the quotient algebra $X/F = \{F_x \mid x \in X\}$(where F_x is the congruence class of x) is also a bounded BE-algebra with smallest element F_0 in which $F_x * F_y = F_{x*y}$ and $(F_x)N = F_{xN}$ for all $x, y \in X$.

Proposition 3.20. *For any filter F of a self-distributive BE-algebra X, the congruence class F_0 is an ideal of X.*

Proof. Let F be a filter of X. Since X is self-distributive, θ_F is a congruence on X. Clearly $0 \in F_0$. Let $x \in F_0$ and $(xN * yN)N \in F_0$. Hence $xN = x * 0 \in F$ and $(xN * yN)NN = (xN * yN)N * 0 \in F$. Since $(xN * yN)NN \le xN * yN$, we get $xN * yN \in F$. Since $xN \in F$, we get $y * 0 = yN \in F$. Since $0 * y = 1 \in F$, we get $(y, 0) \in \theta_F$. Hence $y \in F_0$. Therefore F_0 is an ideal of X. □

4 Homomorphism theorems

In this section, we introduced a congruence on BE-algebras with the help of ideals. Some homomorphism theorems are derived with the help of these congruences, ideal and cartesian products of quotient algebras.

Definition 4.1. Let I be an ideal of a BE-algebra X. For any $x, y \in X$, define a relation θ_I on X as follows:

$$(x, y) \in \theta_I \text{ if and only if } (x * y)N \in I \text{ and } (y * x)N \in I.$$

Proposition 4.2. *If X is a transitive BE-algebra and I an ideal of X, then the above relation θ_I is an equivalence relation on X.*

Proof. Clearly θ_I is reflexive and symmetric. Let $(x, y), (y, z) \in \theta_I$. Then $(x * y)N \in I, (y * x)N \in I$ and $(y * z)N \in I, (z * y)N \in I$. By Lemma 3.3(2), we get

$$y * z \le (x * y) * (x * z) \le (x * y)NN * (x * z)NN$$

Hence $((x * y)NN * (x * z)NN)N \le (y * z)N$. Since $(y * z)N \in I$, we get that $((x * y)NN * (x * z)NN)N \in I$. Since $(x * y)N \in I$, we get $(x * z)N \in I$. Similarly, we can obtain $(z * x)N \in I$. Hence $(x, z) \in \theta_I$. Therefore θ_I is an equivalence relation on X. □

Theorem 4.3. *If X is a transitive BE-algebra and I an ideal of X, then the above relation θ_I is a congruence on X. Moreover θ_I is a unique congruence such that $I_0 = I$, where I_0 is the congruence class of 0 with respect to θ_I.*

Proof. Let $(x, y) \in \theta_I$ and $(u, v) \in \theta_I$. Then $(x * y)N \in I, (y * x)N \in I, (u * v)N \in I$ and $(v * u)N \in I$. Since X is transitive, we get $x * y \le (u * x) * (u * y)$ and so $((u * x) * (u * y))N \le (x * y)N$. Since $(x * y)N \in I$, we get $((u * x) * (u * y))N \in I$. Similarly, we can get $((u*y)*(u*x))N \in I$ because of $(y*x)N \in I$. Hence both together provide us $(u*x, u*y) \in \theta_I$. Again, since X is transitive, we get $v * y \le (u * v) * (u * y)$. Thus we get the following:

$$u * v \le (v * y) * (u * y) \le ((v * y) * (u * y))NN$$

Hence $((v*y)*(u*y))N \le (u*v)N$. Since $(u*v)N \in I$, we get $((v*y)*(u*y))N \in I$. Similarly, we can obtain $((u * y) * (v * y))N \in I$ because of $(v * u)N \in I$. Thus we get $(u * y, v * y) \in \theta_I$. Therefore θ_I is a congruence on X. Now, let $x \in I_0$. Then $xNN = (x * 0)N \in I$. Since $x \le xNN$, we get $x \in I$. Therefore $I_0 \subseteq I$. Again, let $x \in I$. Then $(x * 0)N = xNN \in I$. Clearly $(0 * x)N = 1N = 0 \in I$. Hence $(x, 0) \in \theta_I$, which implies $x \in I_0$. Thus $I \subseteq I_0$. Therefore $I_0 = I$. □

From the above result, it is easy to see that the quotient algebra $X/I = \{I_x \mid x \in X\}$(where I_x is the congruence class of x modulo θ_I) is a bounded BE-algebra in which the binary operation $*$ is defined as $I_x * I_y = I_{x*y}$ for $x, y \in X$. Moreover, the quotient algebra X/I contains the smallest element I_0. For any ideal I of a transitive BE-algebra X, it is natural to obtain the epimorphism $\nu : X \to X/I$ given by $\nu(x) = I_x$.

Theorem 4.4. *The following are equivalent in a commutative BE-algebra.*

(1) X has a unique dense element;
(2) for $x, y \in X$, $(x * y)N = 0$ and $(y * x)N = 0$ imply that $x = y$;
(3) X isomorphic to $X/\theta_{\{0\}}$.

Proof. (1) ⇒ (2): Assume that X has a unique dense element, precisely 1. Then $\mathcal{D}(X) = \{1\}$. Let $x, y \in X$. Suppose that $(x * y)N = 0$ and $(y * x)N = 0$. Then, we get $x * y \in \mathcal{D}(X) = \{1\}$ and $y * x \in \mathcal{D}(X) = \{1\}$. Hence $x \le y$ and $y \le x$. Since X is commutative, it concludes that $x = y$.
(2) ⇒ (3): Assume that the condition (2) holds. We know that the natural map $\nu : X \to X/\theta_{\{0\}}$ defined by $\nu(x) = \{0\}_x$, for all $x \in X$, is an epimorphism. Let $\nu(x) = \nu(y)$ for $x, y \in X$. Then $\{0\}_x = \{0\}_y$. Thus, it immediately infers that $(x * y)N \in \{0\}$ and $(y * x)N \in \{0\}$. Hence by condition (2), we get $x = y$. Therefore ν is an injective and so X is isomorphic to $X/\theta_{\{0\}}$.
(3) ⇒ (1): Assume that X is isomorphic to $X/\theta_{\{0\}}$. Let $a \ne 1$ and $a \in \mathcal{D}(X)$. Then we get $(1 * a)N = aN = 0 \in \{0\}$ and $(a * 1)N = 1N = 0 \in \{0\}$. Hence $(a, 1) \in \theta_{\{0\}}$, which implies $\nu(a) = \{0\}_a = \{0\}_1 = \nu(1)$. Since ν is injective, we get $a = 1$, which is a contradiction. Therefor X has a unique dense element. □

Theorem 4.5. *Let X be a transitive BE-algebra and I is an ideal of X. Then the quotient algebra X/θ_I contains an unique dense element.*

Proof. Since X is transitive, it is cleared that X/θ_I is a transitive BE-algebra. Always I_1 is a dense element of X/θ_I. For $1 \neq x \in X$, suppose $I_{xN} = (I_x)N = I_0$. Then $(xN, 0) \in \theta_I$. Hence $xN \leq xNNN = (0N * xNN)N \in I$. Thus $(1 * x)N \in I$ and $(x * 1)N \in I$. Hence $(1, x) \in \theta_I$, which implies that $I_x = I_1$. Therefore I_1 is the unique dense element of X/θ_I. □

Theorem 4.6. *Let I, J be two ideals of a transitive BE-algebra X. Then*

$$I \vee J = \{x \in X \mid aN * (bN * xN) = 1 \text{ for some } a \in I \text{ and } b \in J\}$$

is the smallest ideal of X which is containing both I and J.

Proof. Clearly, $0 \in I \vee J$. Let $x \in I \vee J$ and $(xN * yN)N \in I \vee J$. Then there exists $a, c \in I$ and $b, d \in J$ such that $aN * (bN * xN) = 1$ and $cN * (dN * (xN * yN)NN) = 1$. Then by Lemma 3.3(4), we deduce that

$$1 = cN * (dN * (xN * yN)NN) \leq cN * (dN * (xN * yN)) = xN * (cN * (dN * yN)).$$

Hence $xN \leq cN * (dN * yN)$. Since X is transitive, we get

$$1 = aN * (bN * xN) \leq aN * (bN * (cN * (dN * yN))) = aN * (cN * (bN * (dN * yN))).$$

Hence $aN * (cN * (bN * (dN * yN))) = 1$. Thus by Lemma 3.3(4), we get

$$\begin{aligned} (aN * (cN * (bN * (dN * yN)NN)NN)NN)N &\leq (aN * (cN * (bN * (dN * yN))))N \\ &= 1N \\ &= 0 \in I \end{aligned}$$

Hence $(aN * (cN * (bN * (dN * yN)NN)NN)NN)N \in I$ where $a, c \in I$ and $b, d \in J$. Since $a, c \in I$, we get $(bN * (dN * yN)NN)N \in I$. Put $f = (bN * (dN * yN)NN)N$. Then $fN = (bN * (dN * yN)NN)NN$. By Lemma 3.3(5), we have

$$fN = (bN * (dN * yN)NN)NN \leq bN * (dN * yN)NN \leq bN * (dN * yN).$$

Hence $bN * (dN * (fN * yN)) = fN * (bN * (dN * yN)) = 1$. Thus, we get

$$(bN * (dN * (fN * yN)))N = 0 \in J.$$

Hence $(bN * (dN * (fN * yN)NN)NN)N \leq (bN * (dN * (fN * yN)))N \in J$. Since $b, d \in J$, we get $(fN * yN)N \in J$. Put $g = (fN * yN)N$. Then $gN = (fN * yN)NN \leq fN * yN$. Hence

$$1 = (fN * yN) * (fN * yN) \leq gN * (fN * yN) = fN * (gN * yN)$$

Since $f \in I, g \in J$, we get $y \in I \vee J$. Therefore $I \vee J$ is an ideal of X. Let $x \in I$. Clearly $xN * (0N * xN) = xN * xN = 1$. Since $0 \in J$, we get $x \in I \vee J$. Hence $I \subseteq I \vee J$. Similarly, we get $J \subseteq I \vee J$.

Let K be an ideal of X such that $I \subseteq K$ and $J \subseteq K$. Let $x \in I \vee J$. Then there exists $a \in I \subseteq K$ and $b \in J \subseteq K$ such that $aN * (bN * xN) = 1$. Hence $aN * (bN * xN)NN = 1$, which implies $(aN * (bN * xN)NN)N = 0 \in K$. Since $a \in K$, we get $(bN * xN)N \in K$. Since $b \in K$, we get $x \in K$. Hence $I \vee J \subseteq K$. Therefore $I \vee J$ is the smallest ideal which contains both I and J. □

Since the intersection of ideals is again an ideal, the following is direct:

Corollary 4.7. *For any transitive BE-algebra X, the set $\mathcal{I}(X)$ of all ideals of X forms a complete lattice.*

Theorem 4.8. *Let I and J be two ideals of a transitive BE-algebra X. Then the mapping $f : X \to (X/I) \times (X/J)$ defined by $f(x) = (I_x, J_x)$ for all $x \in X$ is a homomorphism. Moreover, the following hold:*

(1) *If f is injective, then $I \cap J = \{0\}$,*
(2) *If f is surjective, then $I \vee J = X$.*

Proof. Clearly f is well-defined. Let $x, y \in X$. Then $f(x * y) = (I_{x*y}, J_{x*y}) = (I_x * I_y, J_x * J_y) = (I_x, J_x) * (I_y, J_y) = f(x) * f(y)$. Therefore f is a homomorphism.
(1). Suppose f is injective. Then clearly $Dker\ f = \{0\}$. Now

$$\begin{aligned} x \in Dker(f) &\Leftrightarrow f(x) = \overline{0}, \text{ the smallest element in } (X/I) \times (X/J) \\ &\Leftrightarrow (I_x, J_x) = (I_0, J_0) \\ &\Leftrightarrow I_x = I_0 \text{ and } J_x = J_0 \\ &\Leftrightarrow xNN \in I \text{ and } xNN \in J \\ &\Leftrightarrow x \in I \text{ and } x \in J \qquad \text{since } x \leq xNN \\ &\Leftrightarrow x \in I \cap J \end{aligned}$$

Thus $Dker(f) = I \cap J$. Therefore $I \cap J = \{0\}$ whenever f is injective.
(2). Assume that f is surjective. Clearly $(I_0, J_1) \in (X/I) \times (X/J)$. Since f is surjective, there exists $x \in X$ such that $f(x) = (I_0, J_1)$. Hence

$$\begin{aligned} f(x) = (I_0, J_1) &\Leftrightarrow (I_x, J_x) = (I_0, J_1) \\ &\Leftrightarrow I_x = I_0 \text{ and } J_x = J_1 \\ &\Leftrightarrow xNN \in I \text{ and } xN \in J \\ &\Leftrightarrow x \in I \text{ and } xN \in J \end{aligned}$$

Clearly $xN * (xNN * 1N) = xN * xNNN = 1$. Since $x \in I$ and $xN \in J$, it imply that $1 \in I \vee J$. Therefore $I \vee J = X$ whenever f is surjective. □

The following is an extension of the above theorem.

Corollary 4.9. *Let $I^i, i = 1, 2, 3, \ldots, n$ be the ideals of a transitive BE-algebra X. Then the mapping $f : X \to (X/I^1) \times (X/I^2) \times (X/I^3) \times \cdots \times (X/I^n)$ defined by $f(x) = (I^1_x, I^2_x, I^3_x, \ldots, I^n_x)$ for all $x \in X$ is a homomorphism. Moreover,*

(1) *If f is injective, then $\bigcap_{i=1}^{n} I^i = \{0\}$,*

(2) *If f is surjective, then $I^i \vee I^j = X$ for $i \neq j$.*

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