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Approximation of functions by Stancu variant of genuine Gupta-Srivastava operators

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Abstract. In the present paper, we introduce the non-negative parametric variant of the genuine Gupta-Srivastava operators, which preserve constant as well as linear functions. We obtain the moments of the operators and then prove the basic convergence theorem. Next, the Voronovskaja type asymptotic formula and some direct results for the above operators are discussed. Also, the rate of convergence and weighted approximation by these operators in terms of modulus of continuity are studied. Then, we obtain pointwise estimates using the Lipschitz type maximal function and two parameter Lipschitz-type space.

1 Introduction

In the field of mathematical analysis, Karl Weierstrass established an elegant theorem, the first Weierstrass approximation theorem, in 1885. This theorem has specially a big role in polynomial interpolation corresponding to every continuous function $f(x)$ on interval $[a, b]$. The proof given by Weierstrass was rigorous and difficult to understand. In 1912, Bernstein gave a simple proof of this theorem by introducing the Bernstein polynomials with the aid of the binomial distribution, hence for $f \in C[0, 1]$, we have

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), n \in N,$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ $x \in [0, 1]$ is the Bernstein basis function. Many mathematicians researched in this direction and studied various modifications in several functional spaces using different error optimization techniques.
In the year 2003, Srivastava and Gupta [30] introduced a general family of summation-integral type operators $\{G_{n,c}\}$ which includes some well-known operators as special cases. They obtained the rate of convergence for functions of bounded variation.
For $f \in C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(t)| \le M(1+t)^\gamma$ for some $M > 0, \gamma > 0\}$, Srivastava and Gupta proposed a certain family of positive linear operators defined by

$$G_{n,c}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x,c) \int_0^\infty p_{n+c,k-1}(t,c) f(t)dt + p_{n,0}(x,c) f(0), \tag{1.1}$$

where

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x) \tag{1.2}$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0, \\ (1+cx)^{-n/c}, & c \in N, \\ (1-x)^{-n}, & c = -1. \end{cases}$$

Verma and Agrawal [32] introduced the generalized form of the operators (1.1) and studied some of its approximation properties. Deo [3] gave a modification of these operators and established the rate of convergence and Voronovskaja type asymptotic result. Recently, Acar et al. [1] introduced Stancu type generalization of the operators (1.1) and obtained an estimate of the rate of convergence for functions having derivatives of bounded variation and also studied the simultaneous approximation for these operators.
It is well-known that if operators preserve the linear function, one may get a better approximation. In this direction very recently Gupta and Srivastava [7] proposed a general family of positive linear operators, which preserve constant as well as linear functions for all $c \in N \cup \{0\} \cup \{-1\}$, which may be termed as Gupta-Srivastava operators and for all integers m are defined as:

$$\begin{aligned} L_{n,c}(f;x) &= [n+(m+1)c]\sum_{k=1}^{\infty} p_{n+mc,k}(x,c)\int_0^{\infty} p_{n+(m+2)c,k-1}(t,c)f(t)dt \\ &\quad + p_{n+mc,0}(x,c)f(0). \end{aligned} \tag{1.3}$$

Very recently, Pratap et al. [29] studied several interesting approximation properties of the operator (1.3).
It is very well known that the polynomial approximation of continuous functions has an important role in numerical analysis. The Lagrange interpolating polynomials have a great practical interest in approximation theory of continuous functions, but they do not provide always uniform convergence of approximating sequences for any continuous function on a compact interval of the real axis, no matter how the nodes are chosen.
In 1905, Borel proposed a way to obtain an approximation polynomial of a function $f \in C[0,1]$ by using an interpolation polynomial having a similar form with the Lagrange ones and using the nodes $x_{n,k} = \frac{k}{n}, k = 0, 1...n$ and with an appropriate selection of the basic polynomials $p_{n,k}(x)$. In 1912, Bernstein had the wonderful idea to select $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, inspired by the binomial probability distribution. He considered the binomial probability distribution assuming that the discrete random variable has the value $f(\frac{k}{n})$ with probability $p_{n,k}(x)$ and then he calculate the mean value. In 1969, [31], Stancu wanted to choose the nodes in another different way, in order to obtain more flexibility. So, he considered the nodes such as, when $n \to \infty$ the distance between two consecutive nodes and the distance between 0 and first node and also between last node and 1 to tend all to zero. Thus, Stancu introduced the following linear positive operators which are known as Bernstein-Stancu polynomials in literature

$$P_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right),$$

acting from $C[0,1]$ into $C[0,1]$, the space of all real valued continuous functions defined on $[0,1]$, where $n \in N$, $f \in C[0,1], x \in [0,1]$ and α, β are any two real numbers which satisfy the condition that $0 \le \alpha \le \beta$.
In the recent years, Stancu type generalization of the certain operators introduced by several researchers and obtained different type of approximation properties of many operators, we refer some of the important papers in this direction as [1], [2], [9], [11], [15], [28] etc.
Inspired by the above work, we introduce the Stancu type generalization of the operators (1.3):

$$\begin{aligned} L_{n,c}^{(\alpha,\beta)}(f;x) &= [n+(m+1)c]\sum_{k=1}^{\infty} p_{n+mc,k}(x,c)\int_0^{\infty} p_{n+(m+2)c,k-1}(t,c) f\left(\frac{nt+\alpha}{n+\beta}\right)dt \\ &\quad + p_{n+mc,0}(x,c) f\left(\frac{\alpha}{n+\beta}\right). \end{aligned} \tag{1.4}$$

For $\alpha = \beta = 0,$ we denote $L_{n,c}^{(\alpha,\beta)}(f;x)$ by $L_{n,c}(f;x)$.
The goal of the present paper is to study the basic convergence theorem, Voronovskaja type asymptotic result, local approximation theorem, rate of convergence, weighted approximation and pointwise estimation of the operators (1.4).

2 Moment and central moment estimates

In approximation theory, moments play an important role in uniform approximation by linear positive operators. The r-th order moment of an operator $L_n(f,x)$ is given by $L_n(e_r,x)$, where $e_r(t)=t^r, r=0,1,2,...$ Additionally, the r-th order central moment of the operator $L_n(f,x)$ is represented as follows

$$L_n\left((e_1-e_0x)^r;x\right)=L_n\left((t-x)^r;x\right).$$

Lemma 2.1. *[7] For $L_{n,c}(t^m;x)$, $m=0,1,2$, and $c\in N\cup\{0\}\cup\{-1\}$, we have*

(i) $L_{n,c}(1;x)=1$;

(ii) $L_{n,c}(t;x)=x$;

(iii) $L_{n,c}(t^2;x)=\frac{(n+(m+1)c)}{(n+(m-1)c)}x^2+\frac{2}{(n+(m-1)c)}x$.

Lemma 2.2. *For the operators $L_{n,c}^{(\alpha,\beta)}(f;x)$ as defined in (1.4), the following equalities hold:*

(i) $L_{n,c}^{(\alpha,\beta)}(1;x)=1$;

(ii) $L_{n,c}^{(\alpha,\beta)}(t;x)=\frac{nx+\alpha}{n+\beta}$;

(iii) $L_{n,c}^{(\alpha,\beta)}(t^2;x)=\left\{\frac{n^2(n+(m+1)c)}{(n+\beta)^2(n+(m-1)c)}\right\}x^2+\left\{\frac{2n(n+\alpha(n+(m-1)c)}{(n+\beta)^2(n+(m-1)c)}\right\}x+\frac{\alpha^2}{(n+\beta)^2}$.

Proof. For $x\in[0,\infty)$, in view of Lemma 2.1, we have

$$L_{n,c}^{(\alpha,\beta)}(1;x) = 1.$$

Next, for $f(t)=t$, again applying Lemma 2.1, we get

$$L_{n,c}^{(\alpha,\beta)}(t;x) = \frac{n}{n+\beta}L_{n,c}(t;x)+\frac{\alpha}{n+\beta}=\frac{nx+\alpha}{n+\beta}.$$

Proceeding similarly, we have

$$\begin{aligned}L_{n,c}^{(\alpha,\beta)}(t^2;x) &= \left(\frac{n}{n+\beta}\right)^2L_{n,c}(t^2;x)+\frac{2n\alpha}{(n+\beta)^2}L_{n,c}(t;x)+\left(\frac{\alpha}{n+\beta}\right)^2\\ &= \left\{\frac{n^2(n+(m+1)c)}{(n+\beta)^2(n+(m-1)c)}\right\}x^2+\left\{\frac{2n(n+\alpha(n+(m-1)c)}{(n+\beta)^2(n+(m-1)c)}\right\}x+\frac{\alpha^2}{(n+\beta)^2}.\end{aligned}$$

□

Lemma 2.3. *For $f\in C_B[0,\infty)$ (space of all bounded and continuous functions on $[0,\infty)$ endowed with norm $\| f \|=\sup\{|f(x)|:x\in[0,\infty)\}$), $\| L_{n,c}^{(\alpha,\beta)}(f;x)\|\leq\| f\|$.*

Proof. In view of (1.4) and Lemma 2.2, the proof of this lemma easily follows. □

Remark 2.4. For every $x\geq 0$, we have

$$L_{n,c}^{(\alpha,\beta)}\left((t-x);x\right) = \frac{\alpha-\beta x}{n+\beta},$$

and

$$\begin{aligned}L_{n,c}^{(\alpha,\beta)}\left((t-x)^2;x\right) &= \left\{\frac{2n^2c+n\beta^2+\beta^2mc-\beta^2c}{(n+\beta)^2(n+(m-1)c)}\right\}x^2\\ &\quad+\left\{\frac{2n^2-2\alpha\beta n-2\alpha\beta mc+2\alpha\beta c}{(n+\beta)^2(n+(m-1)c)}\right\}x+\frac{\alpha^2}{(n+\beta)^2}\\ &= \gamma_{n,c}^{(\alpha,\beta)}(x), \text{(say)}.\end{aligned}$$

3 Main results

In this section, we prove the basic convergence theorem for the operators (1.4) by using Bohman-Korovkin criterion. We have also found estimates of the rate of convergence involving modulus of continuity and Lipschitz function. In addition, we have studied weighted approximation and pointwise convergence of the operators (1.4). For the readers convenience we split up this section in more subsections.
Let $e_i(t) = t^i,\ i = 0, 1, 2.$

Theorem 3.1. *Let* $f \in C_B[0, \infty)$. *Then* $\lim_{n\to\infty} L_{n,c}^{(\alpha,\beta)}(f; x) = f(x)$, *uniformly in each compact subset of* $[0, \infty)$.

Proof. In view of Lemma 2.2, we get

$$\lim_{n\to\infty} L_{n,c}^{(\alpha,\beta)}(e_i; x) = x^i,\ i = 0, 1, 2,$$

uniformly in each compact subset of $[0, \infty)$. Applying Bohman-Korovkin theorem, it follows that $\lim_{n\to\infty} L_{n,c}^{(\alpha,\beta)}(f; x) = f(x)$, uniformly in each compact subset of $[0, \infty)$. □

3.1 Voronovskaja type result

A general Voronovskaja type theorem for a sequence of linear positive operators $(L_n)_n$, is a limit of the form:

$$\lim_{n\to\infty} \alpha_n \left(L_n(f; x) - f(x)\right) = E(x, f'(x), f'', ...).$$

For classical operators of approximation the usual value for α_n is $\alpha_n = n$.
Now, we prove Voronvoskaja type theorem for the operators $L_{n,c}^{(\alpha,\beta)}$.

Theorem 3.2. *Let f be bounded and integrable on* $[0, \infty)$ *, second derivative of f exists at a fixed point* $x \in [0, \infty)$*, then*

$$\lim_{n\to\infty} n\left(L_{n,c}^{(\alpha,\beta)}(f; x) - f(x)\right) = (\alpha - \beta x)f'(x) + x(1 + cx)f''(x).$$

Proof. Let $x \in [0, \infty)$ be fixed. From the Taylor's theorem, we may write

$$f(t) = f(x) + (t - x)f'(x) + \frac{1}{2}f''(x)(t - x)^2 + \xi(t, x)(t - x)^2, \tag{3.1}$$

where $\xi(t, x)$ is the peano form of the remainder and $\lim_{t\to x} \xi(t, x) = 0$.
Applying $L_{n,c}^{(\alpha,\beta)}(f; x)$ on both sides of (3.1), we have

$$\begin{aligned} n\left(L_{n,c}^{(\alpha,\beta)}(f; x) - f(x)\right) &= nf'(x)L_{n,c}^{(\alpha,\beta)}\left((t - x); x\right) + \frac{1}{2}nf''(x)L_{n,c}^{(\alpha,\beta)}\left((t - x)^2; x\right) \\ &\quad + nL_{n,c}^{(\alpha,\beta)}\left((t - x)^2\xi(t, x); x\right). \end{aligned}$$

In view of Remark 2.4, we have

$$\lim_{n\to\infty} nL_{n,c}^{(\alpha,\beta)}\left((t - x); x\right) = \alpha - \beta x \tag{3.2}$$

and

$$\lim_{n\to\infty} nL_{n,c}^{(\alpha,\beta)}\left((t - x)^2; x\right) = 2x(1 + cx). \tag{3.3}$$

Now, we shall show that

$$\lim_{n\to\infty} nL_{n,c}^{(\alpha,\beta)}\left(\xi(t, x)(t - x)^2; x\right) = 0.$$

By using Cauchy-Schwarz inequality, we have

$$L_{n,c}^{(\alpha,\beta)}\left(\xi(t,x)(t-x)^2;x\right) \leq \sqrt{L_{n,c}^{(\alpha,\beta)}(\xi^2(t,x);x)}\sqrt{L_{n,c}^{(\alpha,\beta)}((t-x)^4;x)}. \tag{3.4}$$

We observe that $\xi^2(x,x)=0$ and $\xi^2(.,x)\in C_B[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty} L_{n,c}^{(\alpha,\beta)}(\xi^2(t,x);x) = \xi^2(x,x) = 0. \tag{3.5}$$

Now, from (3.4) and (3.5) we obtain

$$\lim_{n\to\infty} nL_{n,c}^{(\alpha,\beta)}\left(\xi(t,x)(t-x)^2;x\right) = 0. \tag{3.6}$$

From (3.2), (3.3) and (3.6), we get the required result. □

3.2 Local approximation

For $C_B[0,\infty)$, let us consider the following K-functional:

$$K_2(f;\delta) = \inf_{x\in W_\infty^2}\{\| f-g \| + \delta \| g'' \|\},$$

where $\delta>0$ and $W_\infty^2=\{g\in C_B[0,\infty): g',g''\in C_B[0,\infty)\}$. By p. 177, Theorem 2.4 in [4], there exists an absolute constant $M>0$ such that

$$K_2(f;\delta)\leq M\omega_2(f;\sqrt{\delta}), \tag{3.7}$$

where $\omega_2(f;\sqrt{\delta})$ is second order modulus of continuity defined by

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<|h|\leq\sqrt{\delta}}\ \sup_{x\in[0,\infty)} | f(x+2h)-2f(x+h)+f(x) | .$$

The usual modulus of smoothness (or simply modulus of continuity of first order) for $f\in C_B[0,\infty)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta>0$ and is defined as

$$\omega(f,\delta) = \sup_{0<|h|\leq\delta}\ \sup_{x\in[0,\infty)} | f(x+h)-f(x) | .$$

Theorem 3.3. *Let $f\in C_B[0,\infty)$. Then, for every $x\in[0,\infty)$, we have*

$$| L_{n,c}^{(\alpha,\beta)}(f;x)-f(x) | \leq C\omega_2\left(f,\delta_{n,c}^{(\alpha,\beta)}(x)\right)+\omega\left(f,\frac{|\alpha-\beta x|}{n+\beta}\right),$$

where C is an absolute constant and

$$\delta_{n,c}^{(\alpha,\beta)}(x) = \left(L_{n,c}^{(\alpha,\beta)}((t-x)^2;x)+\left(\frac{\alpha-\beta x}{n+\beta}\right)^2\right)^{1/2}.$$

Proof. For $x\in[0,\infty)$, we consider the auxiliary operators $\overline{L}_{n,c}^{(\alpha,\beta)}$ defined by

$$\overline{L}_{n,c}^{(\alpha,\beta)}(f;x) = L_{n,c}^{(\alpha,\beta)}(f;x)-f\left(\frac{nx+\alpha}{n+\beta}\right)+f(x). \tag{3.8}$$

From Lemma 2.2, we observe that the operators $\overline{L}_{n,c}^{(\alpha,\beta)}$ are linear and reproduce the linear functions.
Hence

$$\overline{L}_{n,c}^{(\alpha,\beta)}((t-x);x) = 0. \tag{3.9}$$

Let $g \in W^2_\infty$. By Taylor's theorem, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv, \;\; t \in [0,\infty).$$

Applying $\overline{L}_{n,c}^{(\alpha,\beta)}$ on both sides of the above equation and using (3.9), we have

$$\overline{L}_{n,c}^{(\alpha,\beta)}(g;x) \;\;=\;\; g(x) + \overline{L}_{n,c}^{(\alpha,\beta)}\left(\int_x^t (t-v)g''(v)dv; x\right).$$

Thus, by (3.8) we get

$$\begin{aligned}
&|\overline{L}_{n,c}^{(\alpha,\beta)}(g;x) - g(x)| \\
&\leq L_{n,c}^{(\alpha,\beta)}\left(\left|\int_x^t (t-v)g^{''}(v)dv\right|; x\right) + \left|\int_x^{\frac{nx+\alpha}{n+\beta}}\left(\frac{nx+\alpha}{n+\beta} - v\right)g^{''}(v)dv\right| \\
&\leq L_{n,c}^{(\alpha,\beta)}\left(\int_x^t |t-v||g^{''}(v)|dv; x\right) + \int_x^{\frac{nx+\alpha}{n+\beta}}\left|\frac{nx+\alpha}{n+\beta} - v\right||g^{''}(v)|dv \\
&\leq \left[L_{n,c}^{(\alpha,\beta)}((t-x)^2;x) + \left(\frac{\alpha-\beta x}{n+\beta}\right)^2\right] \parallel g^{''} \parallel \\
&\leq \left(\delta_{n,c}^{(\alpha,\beta)}(x)\right)^2 \parallel g^{''} \parallel . \qquad (3.10)
\end{aligned}$$

On other hand, by (3.8) and Lemma 2.3, we have

$$|\overline{L}_{n,c}^{(\alpha,\beta)}(f;x)| \;\;\leq\;\; \parallel f \parallel . \qquad (3.11)$$

Using (3.10) and (3.11) in (3.8), we obtain

$$\begin{aligned}
|L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| \;\;&\leq\;\; |\overline{L}_{n,c}^{(\alpha,\beta)}(f-g;x)| + |(f-g)(x)| + |\overline{L}_{n,c}^{(\alpha,\beta)}(g;x) - g(x)| \\
&\quad + \left|f\left(\frac{nx+\alpha}{n+\beta}\right) - f(x)\right| \\
&\leq\;\; 2 \parallel f-g \parallel + \left(\delta_{n,c}^{(\alpha,\beta)}(x)\right)^2 \parallel g'' \parallel + \left|f\left(\frac{nx+\alpha}{n+\beta}\right) - f(x)\right|.
\end{aligned}$$

Hence, taking infimum on the right hand side over all $g \in W^2_\infty$, we get

$$|\; L_{n,c}^{(\alpha,\beta)}(f;x) - f(x) \;| \;\;\leq\;\; K_2\left(f, (\delta_{n,c}^{(\alpha,\beta)}(x))^2\right) + \omega\left(f, \frac{|\alpha-\beta x|}{n+\beta}\right).$$

In view of (3.7), we get

$$|\; L_{n,c}^{(\alpha,\beta)}(f;x) - f(x) \;| \;\;\leq\;\; C\omega_2\left(f, \delta_{n,c}^{(\alpha,\beta)}(x)\right) + \omega\left(f, \frac{|\alpha-\beta x|}{n+\beta}\right).$$

Hence, the proof is completed. □

3.3 Rate of convergence

Let $\omega_b(f,\delta)$ denote the modulus of continuity of f on the closed interval $[0,b], b > 0$, and defined as

$$\omega_b(f,\delta) = \sup_{|t-x|\leq\delta} \; \sup_{x,t\in[0,b]} |f(t) - f(x)|.$$

We observe that for a function $f \in C_B[0,\infty)$, the modulus of continuity $\omega_b(f,\delta)$ tends to zero. Now, we give a rate of convergence theorem for the operators $L_{n,c}^{(\alpha,\beta)}$.

Theorem 3.4. *Let $f \in C_B[0,\infty)$ and $\omega_{b+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,b+1] \subset [0,\infty)$, where $b > 0$. Then, we have*

$$|L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq 4M_f(1+b^2)\gamma_{n,c}^{(\alpha,\beta)}(x) + 2\omega_{b+1}\left(f, \sqrt{\gamma_{n,c}^{(\alpha,\beta)}(x)}\right),$$

where $\gamma_{n,c}^{(\alpha,\beta)}(x)$ is defined in Remark 2.4 and M_f is a constant depending only on f.

Proof. For $x \in [0,b]$ and $t > b+1$. Since $t - x > 1$, we have

$$|f(t) - f(x)| \leq M_f(2+t^2+x^2) \leq M_f(t-x)^2(2+2x+2x^2) \leq 4M_f(1+b^2)(t-x)^2.$$

For $x \in [0,b]$ and $t \leq b+1$, we have

$$|f(t) - f(x)| \leq \omega_{b+1}(f,|t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta), \delta > 0.$$

From the above, we have

$$|f(t) - f(x)| \leq 4M_f(1+b^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta), \delta > 0.$$

Thus, by applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} |L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| &\leq 4M_f(1+b^2)(L_{n,c}^{(\alpha,\beta)}(t-x)^2;x) \\ &\quad + \omega_{b+1}(f,\delta)\left(1 + \frac{1}{\delta}(L_{n,c}^{*(\alpha,\beta)}(t-x)^2;x)^{\frac{1}{2}}\right) \\ &\leq 4M_f(1+b^2)\gamma_{n,c}^{(\alpha,\beta)}(x) + 2\omega_{b+1}\left(f, \sqrt{\gamma_{n,c}^{(\alpha,\beta)}(x)}\right), \end{aligned}$$

on choosing $\delta = \sqrt{\gamma_{n,c}^{(\alpha,\beta)}(x)}$. This completes the proof of the theorem. □

3.4 Weighted approximation.

Since the uniform norm is not valid to estimate the rate of convergence in the case of unbounded function defined on the non-compact interval $[0,\infty)$, in this section we study the approximation properties of the operators (1.4) in the weighted spaces of continuous and boundless functions defined on the internal $[0,\infty)$.

Let C_ν be the space of all continuous functions on $[0,\infty)$ with the norm $\| f \|_\nu = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\nu(x)}$ and $C_\nu^* = \{f \in C_\nu : \lim_{x\to\infty} \frac{|f(x)|}{\nu(x)} < \infty\}$, where $\nu(x)$ is a weight function.
In what follows we consider $\nu(x) = 1 + x^2$.

Theorem 3.5. *For each $f \in C_\nu^*$, we have*

$$\lim_{n\to\infty} \| L_{n,c}^{(\alpha,\beta)}(f) - f \|_\nu = 0.$$

Proof. From [5], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \| L_{n,c}^{(\alpha,\beta)}(t^k;x) - x^k \|_\nu = 0, \ k = 0,1,2. \tag{3.12}$$

Since $L_{n,c}^{(\alpha,\beta)}(1;x) = 1$, the condition in (3.12) holds for $k = 0$.

By Lemma 2.2, we have

$$\begin{aligned}\| L_{n,c}^{(\alpha,\beta)}(t;x)-x) \|_\nu &= \sup_{x\in[0,\infty)} \frac{|L_{n,c}^{(\alpha,\beta)}(t;x)-x|}{1+x^2}\\ &\leq \frac{\beta}{n+\beta}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{\alpha}{n+\beta}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}\\ &\leq \frac{\alpha+\beta}{n+\beta},\end{aligned}$$

which implies that the condition in (3.12) holds for $k=1$.
Similarly, we have

$$\begin{aligned}\| L_{n,c}^{(\alpha,\beta)}(t^2;x)-x^2 \|_\nu &= \sup_{x\in[0,\infty)} \frac{|L_{n,c}^{(\alpha,\beta)}(t^2;x)-x^2|}{1+x^2}\\ &\leq \left|\frac{n^2(n+(m+1)c)}{(n+\beta)^2(n+(m-1)c)}-1\right|+\left|\frac{2n(n+\alpha(n+(m-1)c))}{(n+\beta)^2(n+(m-1)c)}\right|\\ &+\frac{\alpha^2}{(n+\beta)^2},\end{aligned}$$

which implies that $\lim\limits_{n\to\infty} \| L_{n,c}^{(\alpha,\beta)}(t^2;x)-x^2 \|_\nu = 0$, the equation (3.12) holds for $k=2$.
This completes the proof of theorem. □

3.5 Pointwise Estimates

In this section, we establish some pointwise estimates of the rate of convergence of the operators $L_{n,c}^{(\alpha,\beta)}$. First, we give the relationship between the local smoothness of f and local approximation.
We know that a function $f\in C[0,\infty)$ is in Lip$_M$ (α) on E, $\alpha\in(0,1]$, E$\subset[0,\infty)$ if it satisfies the condition

$$|f(t)-f(x)|\leq M|t-x|^\alpha,\ t\in[0,\infty)\ \ and\ \ x\in E,$$

where M is a constant depending only on α and f.

Theorem 3.6. *Let $f\in C[0,\infty)\cap Lip_M\ (\alpha)$, $E\subset[0,\infty)$ and $\alpha\in(0,1]$. Then, we have*

$$|L_{n,c}^{(\alpha,\beta)}(f;x)-f(x)| \leq M\bigg(\Big(\gamma_{n,c}^{(\alpha,\beta)}(x)\Big)^{\alpha/2}+2d^\alpha(x,E)\bigg),\quad x\in[0,\infty),$$

where M is a constant depending on α and f and $d(x,E)$ is the distance between x and E defined as

$$d(x,E)=\inf\{|t-x|:t\in E\}.$$

Proof. Let $\overline{E}$ be the closure of E in $[0,\infty)$. Then, there exists at least one point $x_0\in\overline{E}$ such that

$$d(x,E)=|x-x_0|.$$

By our hypothesis and the monotonicity of $L_{n,c}^{(\alpha,\beta)}$, we get

$$\begin{aligned}|L_{n,c}^{(\alpha,\beta)}(f;x)-f(x)| &\leq L_{n,c}^{(\alpha,\beta)}(|f(t)-f(x_0)|;x)+L_{n,c}^{(\alpha,\beta)}(|f(x)-f(x_0)|;x)\\ &\leq M\left(L_{n,c}^{(\alpha,\beta)}(|t-x_0|^\alpha;x)+|x-x_0|^\alpha\right)\\ &\leq M\left(L_{n,c}^{(\alpha,\beta)}(|t-x|^\alpha;x)+2|x-x_0|^\alpha\right).\end{aligned}$$

Now, applying Hölder's inequality with $p = \frac{2}{\alpha}$ and $\frac{1}{q} = 1 - \frac{1}{p}$, we obtain

$$|L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq M\left(\{L_{n,c}^{(\alpha,\beta)}(|t-x|^2;x)\}^{\alpha/2} + 2d^{\alpha}(x,E)\right),$$

from which the desired result immediate. □

Next, we obtain the local direct estimate of the operators defined in (1.4), using the Lipschitz-type maximal function of order α introduced by B. Lenze [18] as

$$\widetilde{\omega}_{\alpha}(f,x) = \sup_{t\neq x,\ t\in[0,\infty)} \frac{|f(t)-f(x)|}{|t-x|^{\alpha}}, \quad x \in [0,\infty) \text{ and } \alpha \in (0,1]. \tag{3.13}$$

Theorem 3.7. *Let $f \in C_B[0,\infty)$ and $0 < \alpha \leq 1$. Then, for all $x \in [0,\infty)$ we have*

$$|L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq \widetilde{\omega}_{\alpha}(f,x)\left(\gamma_{n,c}^{(\alpha,\beta)}(x)\right)^{\alpha/2}.$$

Proof. From the equation (3.13), we have

$$|L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq \widetilde{\omega}_{\alpha}(f,x) L_{n,c}^{(\alpha,\beta)}(|t-x|^{\alpha};x).$$

Applying the Hölder's inequality with $p = \frac{2}{\alpha}$ and $\frac{1}{q} = 1 - \frac{1}{p}$, we get

$$|L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq \widetilde{\omega}_{\alpha}(f,x) L_{n,c}^{(\alpha,\beta)}((t-x)^2;x)^{\frac{\alpha}{2}} \leq \widetilde{\omega}_{\alpha}(f,x)\left(\gamma_{n,c}^{(\alpha,\beta)}(x)\right)^{\alpha/2}.$$

Thus, the proof is completed. □

For $a, b > 0$, Özarslan and Aktuğlu [26] consider the Lipschitz-type space with two parameters:

$$Lip_M^{(a,b)}(\alpha) = \left(f \in C[0,\infty) : |f(t)-f(x)| \leq M\frac{|t-x|^{\alpha}}{(t+ax^2+bx)^{\alpha/2}};\ x,t \in [0,\infty)\right),$$

where M is any positive constant and $0 < \alpha \leq 1$.

Theorem 3.8. *For $f \in Lip_M^{(a,b)}(\alpha)$. Then, for all $x > 0$, we have*

$$|L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq M\left(\frac{\gamma_{n,c}^{(\alpha,\beta)}(x)}{ax^2+bx}\right)^{\alpha/2}.$$

Proof. First we prove the theorem for $\alpha = 1$. Then, for $f \in Lip_M^{(a,b)}(1)$, and $x \in [0,\infty)$, we have

$$\begin{aligned} |L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| &\leq L_{n,c}^{(\alpha,\beta)}(|f(t)-f(x)|;x) \\ &\leq ML_{n,c}^{(\alpha,\beta)}\left(\frac{|t-x|}{(t+ax^2+bx)^{1/2}};x\right) \\ &\leq \frac{M}{(ax^2+bx)^{1/2}} L_{n,c}^{(\alpha,\beta)}(|t-x|;x). \end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} |L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| &\leq \frac{M}{(ax^2+bx)^{1/2}}\left(L_{n,c}^{(\alpha,\beta)}((t-x)^2;x)\right)^{1/2} \\ &\leq M\left(\frac{\gamma_{n,c}^{(\alpha,\beta)}(x)}{ax^2+bx}\right)^{1/2}. \end{aligned}$$

Thus the result holds for $\alpha = 1$.
Now, we prove that the result is true for $0 < \alpha < 1$. Then, for $f \in Lip_M^{(a,b)}(\alpha)$, and $x \in [0,\infty)$, we get

$$|L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| \quad \leq \quad \frac{M}{(ax^2+bx)^{\alpha/2}} L_{n,c}^{(\alpha,\beta)}(|t-x|^{\alpha};x).$$

Taking $p = \frac{1}{\alpha}$ and $q = \frac{p}{p-1}$, applying the Hölders inequality, we have

$$|L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| \leq \frac{M}{(ax^2+bx)^{\alpha/2}} \left(L_{n,c}^{(\alpha,\beta)}(|t-x|;x) \right)^{\alpha}.$$

Finally by Cauchy-Schwarz inequality, we get

$$|L_{n,c}^{(\alpha,\beta)}(f;x) - f(x)| \quad \leq \quad M \left(\frac{\gamma_{n,c}^{(\alpha,\beta)}(x)}{ax^2+bx} \right)^{\alpha/2}.$$

Thus, the proof is completed. □

4 Conclusion

In this paper, we introduce the Stancu type generalization of the operators defined in (1.3). For $\alpha = \beta = 0$, reduced results are proved in [29]. The results of our lemmas and theorems are more general rather than the results of any other previous proved lemmas and theorems, which will be enrich the literate of classical approximation theory. The researchers and professionals working or intend to work in areas of analysis and its applications will find this research article to be quite useful. Consequently, the results so established may be found useful in several interesting situation appearing in the literature on Mathematical Analysis and Applied Mathematics.

References

[1] T. Acar, L.N. Mishra and V.N. Mishra, Simultaneous approximation for generalized Srivastava-Gupta operators, J. Funct. Spaces, Article ID 936308, 11 pages, (2015).

[2] P.N. Agrawal, A. Sathish Kumar and T.A.K. Sinha, Stancu type generalization of modified Schurer operators based on q-integers, Appl. Math. Comput., **226**, 765–776 (2014).

[3] N. Deo, Faster rate of convergence on Srivastava-Gupta operators, Appl. Math. Comput., **218**, 10486–10491 (2012).

[4] R.A. DeVore and G.G. Lorentz, Constructive Approximation, Springer, Berlin (1993).

[5] A.D. Gadjiev, Theorems of the type of P. P. korovkin's theorems, Matematicheskie Zametki, **20 (5)**, 781–786 (1976).

[6] A.D. Gadjiev, R.O. Efendiyev and E. Ibikli, On Korovkin type theorem in the space of locally integrable functions, Czechoslovak Math. J. **1 (128)**, 45–53 (2003).

[7] V. Gupta, H.M. Srivastava, A general family of the Srivastava-Gupta operators preserving linear functions, Eur. J. Pure Appl. Math., **11 (3)**, 575–579 (2018).

[8] R.B. Gandhi, Deepmala and V.N. Mishra, Local and global results for modified Szász-Mirakjan operators, Math. Method. Appl. Sci., (2016). DOI: 10.1002/mma.4171.

[9] A. Kumar, Approximation by Stancu type generalized Srivastava-Gupta operators based on certain parameter, Khayyam J. Math., Vol. 3, no. 2, pp. 147–159 (2017). DOI: 10.22034/kjm.2017.49477

[10] A. Kumar, Artee, Direct estimates for certain summation-integral type Operators, Palestine Journal of Mathematics, Vol. **8 (2)**, 365–379 (2019).

[11] A. Kumar, Vandana, Some approximation properties of generalized integral type operators, Tbilisi Mathematical Journal, **11 (1)**, pp. 99–116 (2018). DOI 10.2478/tmj-2018-0007.

[12] A. Kumar, Vandana, Approximation properties of modified Srivastava-Gupta operators based on certain parameter, Bol. Soc. Paran. Mat., v. **38 (1)**, 41–53 (2020). doi:10.5269/bspm.v38i1.36907

[13] A. Kumar, Vandana, Approximation by Stancu type Jakimovski-Leviatan-Paltanea operators, TWMS J. App. Eng. Math., V.9, N.4, pp. 936–948 (2019).

[14] A. Kumar, Vandana, Approximation by genuine Lupas-Beta-Stancu operators, J. Appl. Math. and Informatics, Vol. 36, No. 1-2, pp. 15–28 (2018). https://doi.org/10.14317/jami.2018.015

[15] A. Kumar, L.N. Mishra, Approximation by modified Jain-Baskakov-Stancu operators, Tbilisi Mathematical Journal, **10 (2)**, pp. 185–199 (2017).

[16] A. Kumar, V.N. Mishra, Dipti Tapiawala, Stancu type generalization of modified Srivastava-Gupta operators, Eur. J. Pure Appl. Math., Vol. 10, No. 4, 890–907 (2017).

[17] A. Kumar, Dipti Tapiawala, L.N. Mishra, Direct estimates for certain integral type Operators, Eur. J. Pure Appl. Math, **11 (4)**, 958–975 (2018). DOI: https://doi.org/10.29020/nybg.ejpam.v11i4.3305

[18] B. Lenze, On Lipschitz type maximal functions and their smoothness spaces, Nederl. Akad. Indag. Math., **50**, 53–63 (1988).

[19] C.P. May, On Phillips operators, J. Approx. Theory, **20**, 315–332 (1977).

[20] P. Maheshwari(Sharma), On modified Srivastava-Gupta operators, Filomat, **29:6**, 1173–1177 (2015).

[21] L.N. Mishra, A. Kumar, Direct estimates for Stancu variant of Lupas-Durrmeyer operators based on Polya distribution, Khayyam J. Math., 5 no. 2, 51–64 (2019). DOI: 10.22034/kjm.2019.85886

[22] V.N. Mishra, K. Khatri, L.N. Mishra and Deepmala, Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators, Journal of Inequalities and Applications 2013, 2013:586. doi:10.1186/1029-242X-2013-586.

[23] V.N. Mishra, K. Khatri and L.N. Mishra, On Simultaneous Approximation for Baskakov-Durrmeyer-Stancu type operators, Journal of Ultra Scientist of Physical Sciences, Vol. 24, No. (3) A, pp. 567–577 (2012).

[24] V.N. Mishra, K. Khatri and L.N. Mishra, Some approximation properties of q-Baskakov-Beta-Stancu type operators, Journal of Calculus of Variations, Volume 2013, Article ID 814824, 8 pages. http://dx.doi.org/10.1155/2013/814824

[25] T. Neer, N. Ispir and P.N. Agrawal, Bezier variant of modified Srivastava-Gupta operators, Revista de la Unión Matemática Argentina, (2017).

[26] M. A. Özarslan and H. Aktuğlu, Local approximation for certain King type operators, Filomat, **27:1**, 173–181 (2013).

[27] R.S. Phillips, An inversion formula for semi-groups of linear operators, Ann. of Math. (Ser-2) 352–356, (1954).

[28] P. Patel and V.N. Mishra, Approximation properties of certain summation integral type operators, Demonstratio Mathematica Vol. XLVIII no. 1, (2015).

[29] Ram Pratap, N. Deo, Approximation by genuine Gupta-Srivastava operators, RACSAM. https://doi.org/10.1007/s13398-019-00633-4

[30] H.M. Srivastava and V. Gupta, A Certain family of summation-integral type operators, Math. Comput. Modelling, **37** 1307–1315 (2003).

[31] D.D. Stancu, Approximation of functions by a new class of linear polynomial operators, Rev. Roum. Math. Pures Appl., **13 (8)**, 1173–1194 (1968).

[32] D.K. Verma and P.N. Agrawal, Convergence in simultaneous approximation for Srivastava-Gupta operators, Math. Sci., 6–22 (2012).

[33] R. Yadav, Approximation by modified Srivastava-Gupta operators, Appl. Math. Comput., **226**, 61–66 (2014).

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