# Trigonometric approximation of (signals) functions by Nörlund type means in the variable space $L^{p(x)}$

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Communicated by Ayman Badawi

MSC 2010 Classifications: 41A25, 42A10, 46E30.

Keywords and phrases: Numerical sequences, classes  $Lip(\alpha,p(x))$ , trigonometric approximation,  $L^{p(x)}$ -norm, Fourier series.

**Abstract**. In this paper the results obtained in [7] are extended in three directions: they are extended for a wider class of numerical sequences, are obtained sharper degrees of approximation, and are used some recent new means.

### 1 Introduction

Let  $p: \mathbb{R} \to [1, \infty)$  be a measurable  $2\pi$  periodic function. Denote by  $L^{p(x)} = L^{p(x)}([0, 2\pi])$  the set of all measurable  $2\pi$  periodic functions f such that  $m_p(\mu f) < \infty$  for  $\mu = \mu(f) > 0$ , where

$$m_p(f) := \int_0^{2\pi} |f(x)|^{p(x)} dx.$$

 $L^{p(x)}$  becomes a Banach space with respect to the norm

$$\|f\|_{p(x)}:=\inf\left\{\mu>0:m_p\left(\frac{f}{\mu}\right)\leq 1\right\}.$$

If the function p(x) = p is a constant one  $(1 \le p < \infty)$ , then the space  $L^{p(x)}$  is isometrically isomorphic to the Lebesgue space  $L^p$ .

Moreover, if the function p satisfies

$$1 < p_{-} := \operatorname{ess} \inf_{x \in [0, 2\pi]} p(x), \quad p_{+} := \operatorname{ess} \sup_{x \in [0, 2\pi]} p(x) < \infty, \tag{1.1}$$

then the function

$$p'(x) := \frac{p(x)}{p(x) - 1}$$

is well defined and satisfies (1.1) itself.

The space  $L^{p(x)}$  consists of all measurable  $2\pi$  periodic functions f such that

$$\int_0^{2\pi} |f(x)g(x)| dx < \infty$$

for all measurable functions g with  $m_{p'}(g) \leq 1$ .

Denote by M(f) the Hardy-Littlewood maximal operator, defined for  $f \in L^1$  by

$$M(f)(x) = \sup_{I} \frac{1}{|I|} \int_{I} |f(t)| dt, \quad x \in [0, 2\pi],$$

where the supremum is taken over all intervals with  $x \in I$ .

It was proved in [6] that if the function p(x) satisfies (1.1) and the condition

$$|p(x) - p(y)| \le \frac{C}{-\ln|x - y|}, \quad 0 < |x - y| \le \frac{1}{2},$$
 (1.2)

then the maximal operator M(f) is bounded on  $L^{p(x)}$ , that is,

$$||M(f)||_{p(x)} \le A||f||_{p(x)} \tag{1.3}$$

for all  $f \in L^{p(x)}$ , where A is a constant depending only on p.

The set of all measurable  $2\pi$  periodic functions  $p: \mathbb{R} \to [0, \infty)$  satisfies the conditions (1.1) and (1.2) will be denoted by  $\mathcal{M}$ .

Let  $p \in \mathcal{M}$  and  $f \in L^{p(x)}$ . The modulus of continuity of the function f is defined by equality

$$\Omega_{p(x)}(f,\delta) = \sup_{|h| \leq \delta} \|T_h(f)\|_{p(x)}, \quad \delta > 0,$$

where

$$T_h(f;x) := \frac{1}{h} \int_0^h |f(x+t) - f(x)| dt.$$

The modulus of continuity  $\Omega_{p(x)}(f,\delta)$  and the classical integral modulus of continuity  $\omega_p(f,\delta)$  in the Lebesgue space  $L^p$  are equivalent (for details see [8]).

Let  $f \in L$  has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$
 (1.4)

with its n-th partial sums at the point x

$$S_n(f;x) = \sum_{k=0}^n U_k(f;x),$$

where

$$U_0(f;x) := \frac{a_0}{2}; \quad U_k(f;x) := a_k \cos kx + b_k \sin kx, \quad k = 1, 2, \dots$$

Let  $(p_n)_{n=0}^{\infty}$  be a sequence of positive real numbers. We consider the so-called Nörlund means of the sums  $S_n(f;x)$  defined by

$$N_n(f;x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} S_m(f;x),$$

where  $P_n := \sum_{m=0}^n p_m, p_{-1} := P_{-1} := 0$ . In the case  $p_m = 1$  for all  $m \ge 0$ , the means  $N_n(f; x)$  reduced to the Cesàro mean given by equality

$$\sigma_n(f;x) = \frac{1}{n+1} \sum_{m=0}^n S_m(f;x).$$

The approximation properties of the mean  $\sigma_n(f;x)$  in classes  $Lip(\alpha,p)$ ,  $1 \le p < +\infty$ ,  $0 < \alpha \le 1$  were established first by E. S. Quade [13]. His results are generalized by R. N. Mohapatra and D. C. Russell [11], P. Chandra [2]-[5] and L. Leindler [9].

Let  $p \in \mathcal{M}$  and  $0 < \alpha \le 1$ . Very recently, A. Guven and D. Israfilov [7] defined the Lipschitz class  $Lip(\alpha, p(x))$  as

$$Lip(\alpha,p(x)) = \left\{ f \in L^{p(x)} : \Omega_{p(x)}(f,\delta) = \mathcal{O}(\delta^{\alpha}), \ \delta > 0 \right\},$$

and gave  $L^{p(x)}$  counterparts of the results obtained by L. Leindler [9] and P. Chandra [5].

Before we write their results we need first to recall some known notions.

A sequence of positive real numbers  $(p_n)_0^\infty$  is called almost monotone decreasing (increasing) if there exists a constant K, depending only on  $(p_n)_0^\infty$  such that for all  $n \ge m$  the inequality

$$p_n \le Kp_m \quad (p_n \ge Kp_m)$$

holds. Such sequences will be denoted by  $(p_n)_0^\infty \in AMDS$   $((p_n)_0^\infty \in AMIS)$ . Among others they have proved the following.

**Theorem 1.1** ([7]). Let  $p \in \mathcal{M}$ ,  $0 < \alpha < 1$ ,  $f \in Lip(\alpha, p(x))$  and let  $(p_n)_{n=0}^{\infty}$  be a sequence of positive real numbers. If

$$(p_n)_{n=0}^{\infty} \in AMDS$$

or

$$(p_n)_{n=0}^{\infty} \in AMIS$$
 and  $(n+1)p_n = \mathcal{O}(P_n)$ ,

then

$$||f - N_n(f)||_{n(r)} = \mathcal{O}(n^{-\alpha}).$$

holds.

**Theorem 1.2** ([7]). Let  $p \in \mathcal{M}$ ,  $f \in Lip(1, p(x))$  and let  $(p_n)_{n=0}^{\infty}$  be a sequence of positive real numbers. If

$$\sum_{k=1}^{n-1} k |\triangle p_k| = \mathcal{O}(P_n)$$

or

$$\sum_{k=1}^{n-1} |\triangle p_k| = \mathcal{O}\left(\frac{P_n}{n}\right),\,$$

then

$$||f - N_n(f)||_{p(x)} = \mathcal{O}(n^{-1}).$$

holds for  $n = 1, 2, \ldots$ 

Let  $\mathbb F$  be an infinite subset of the set of natural numbers  $\mathbb N$  and  $\mathbb F$  as the range of strictly increasing sequence of positive integers, say  $\mathbb F=(\lambda(n))_{n=1}^\infty$ . The Cesàro submethod  $C_\lambda$  is defined by

$$(C_{\lambda}x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \quad (n = 1, 2, ...),$$

where  $(x_k)_{k=1}^{\infty}$  is a sequence of real or complex numbers.

The  $C_{\lambda}$ -method yields a subsequence of the Cesàro method  $C_1$  and thus it is regular for any  $\lambda$ . A very important fact to point out here is that  $C_{\lambda}$ -method is obtained by deleting a set of rows from Cesàro matrix. An interested reader could find basic properties of  $C_{\lambda}$ -method in [1] and [12].

Next we shall consider trigonometric polynomials  $N_n^{\lambda}(f;x)$  defined by (see [10])

$$N_n^{\lambda}(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{n=0}^{\lambda(n)} p_{\lambda(n)-m} S_m(f;x),$$

where  $P_{\lambda(n)}:=\sum_{m=0}^{\lambda(n)}p_m,\,p_{-1}:=P_{-1}:=0$ . In the case  $p_m=1$  for all  $m\geq 0$ , the means  $N_n^\lambda(f;x)$  reduced to the  $\lambda$ -Cesàro mean given by equality

$$\sigma_n^{\lambda}(f;x) = \frac{1}{\lambda(n)+1} \sum_{m=0}^{\lambda(n)} S_m(f;x),$$

where

$$S_m(f;x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t) D_m(t) dt$$
 and  $D_m(t) = \sum_{j=1}^m \sin jx = \frac{\sin(m+1/2)t}{2\sin(t/2)}$ .

Motivated from [14] we introduce two new classes of numerical sequences.

Let  $B_{\lambda(n),k} = \frac{1}{(k+1)P_{\lambda(n)}} \sum_{i=\lambda(n)-k}^{\lambda(n)} p_i$ . If  $(B_{\lambda(n),k}) \in AMDS$   $((B_{\lambda(n),k}) \in AMIS)$ , then it is said that  $(p_k)$  is an  $\lambda$ -almost monotone decreasing (increasing) upper mean sequence, briefly  $(p_k) \in \lambda - AMDUMS$   $((p_k) \in \lambda - AMIUMS)$ .

**Remark 1.3.** Note that in particular cases for  $\lambda(n)=n, \ n=1,2,\ldots$ , we obtain classes  $\lambda-AMDUMS\equiv AMDUMS$  and  $\lambda-AMIUMS\equiv AMIUMS$  defined in [14]. So, the classes  $\lambda-AMDUMS$  and  $\lambda-AMIUMS$  are generalizations of the classes AMDUMS and AMIUMS respectively.

The main object of this paper is to prove the Theorems 1.1 and 1.2 using new means  $N_n^{\lambda}(f;x)$  and considering new classes  $\lambda - AMIUMS$  and  $\lambda - AMIUMS$ , which give better degrees of approximations than those means that are considered previously by others.

## 2 Helpful Lemmas

To achieve the aim, which we mentioned above, we need some helpful statements given below.

**Lemma 2.1** ([7]). Let  $p \in \mathcal{M}$ . Then the estimate

$$\|\sigma_n(f) - S_n(f)\|_{p(x)} = \mathcal{O}(n^{-1}), \ n = 1, 2, \dots,$$

holds for every  $f \in Lip(1, p(x))$ .

**Lemma 2.2** ([7]). Let  $p \in \mathcal{M}$  and  $0 < \alpha \le 1$ . Then the estimate

$$||f - S_n(f)||_{p(x)} = \mathcal{O}(n^{-\alpha}), \ n = 1, 2, \dots,$$

holds for every  $f \in Lip(\alpha, p(x))$ .

**Lemma 2.3.** Let  $(p_n)$  be a positive sequence so that

(i) 
$$(p_n) \in \lambda - AMDUMS \text{ or,}$$

(ii) 
$$(p_n) \in \lambda - AMIUMS$$
, and  $(\lambda(n) + 1)p_{\lambda(n)} = \mathcal{O}(P_{\lambda(n)})$ 

are satisfied. Then

$$\Lambda := \sum_{k=0}^{\lambda(n)} \frac{p_{\lambda(n)-k}}{(k+1)^{\alpha}} = \mathcal{O}_{\alpha}\left(\frac{P_{\lambda(n)}}{(\lambda(n)+1)^{\alpha}}\right)$$

*holds for all*  $0 < \alpha < 1$ .

*Proof.* Let  $r = [\lambda(n)/2]$  be the integer part of  $\lambda(n)/2$ . Then under assumptions of the lemma, then applying the summation by parts and using the inequality

$$(j+1)^{\beta}-j^{\beta}\leq \beta j^{\beta-1}, \quad \text{for} \quad j\in \mathbb{N} \quad \text{and} \quad 0<\beta<1,$$

we have

$$\begin{split} \Lambda & \leq \sum_{k=0}^{r} \frac{p_{\lambda(n)-k}}{(k+1)^{\alpha}} + \frac{1}{(r+1)^{\alpha}} \sum_{k=r+1}^{\lambda(n)} p_{\lambda(n)-k} \\ & = \sum_{k=0}^{r-1} \left[ \frac{1}{(k+1)^{\alpha}} - \frac{1}{(k+2)^{\alpha}} \right] \sum_{i=0}^{k} p_{\lambda(n)-i} + \frac{1}{(r+1)^{\alpha}} \sum_{k=0}^{r} p_{\lambda(n)-k} + \frac{P_{\lambda(n)}}{(r+1)^{\alpha}} \\ & = P_{\lambda(n)} \sum_{k=0}^{r-1} \frac{(k+2)^{\alpha} - (k+1)^{\alpha}}{(k+1)^{\alpha-1}(k+2)^{\alpha}} B_{\lambda(n),k} + \frac{P_r}{(r+1)^{\alpha}} + \frac{P_{\lambda(n)}}{(r+1)^{\alpha}} \\ & \leq P_{\lambda(n)} \left[ \sum_{k=0}^{r-1} \frac{\alpha B_{\lambda(n),k}}{(k+2)^{\alpha}} + \frac{2}{(r+1)^{\alpha}} \right]. \end{split}$$

If  $(p_n) \in \lambda - AMDUMS$ , then

$$\Lambda \leq P_{\lambda(n)} \left[ \alpha B_{\lambda(n),r} \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\alpha}} + \frac{2}{(r+1)^{\alpha}} \right] \\
\leq P_{\lambda(n)} \left[ \frac{\alpha}{(r+1)P_{\lambda(n)}} \sum_{i=\lambda(n)-r}^{\lambda(n)} p_{i} \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\alpha}} + \frac{2}{(r+1)^{\alpha}} \right] \\
\leq P_{\lambda(n)} \left[ \frac{\alpha}{(r+1)P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)} p_{i} \cdot (r+1)^{1-\alpha} + \frac{2}{(r+1)^{\alpha}} \right] \\
= \frac{C_{\alpha}P_{\lambda(n)}}{(r+1)^{\alpha}} \leq \frac{C_{\alpha}P_{\lambda(n)}}{(\lambda(n)+1)^{\alpha}},$$

where  $C_{\alpha}$  is a positive constant that depends only on  $\alpha$ .

If  $(p_n) \in \lambda - AMIUMS$  and  $(\lambda(n) + 1)p_{\lambda(n)} = \mathcal{O}(P_{\lambda(n)})$ , we obtain

$$\Lambda \leq P_{\lambda(n)} \left[ \alpha B_{\lambda(n),0} \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\alpha}} + \frac{2}{(r+1)^{\alpha}} \right] 
\leq P_{\lambda(n)} \left[ \frac{\alpha p_{\lambda(n)}}{P_{\lambda(n)}} \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\alpha}} + \frac{2}{(r+1)^{\alpha}} \right] 
\leq P_{\lambda(n)} \left[ \frac{\alpha p_{\lambda(n)}}{P_{\lambda(n)}} (r+1)^{1-\alpha} + \frac{2}{(r+1)^{\alpha}} \right] \leq \frac{C_{\alpha} P_{\lambda(n)}}{(\lambda(n)+1)^{\alpha}}.$$

The proof of the lemma is completed.

Next section will be devoted to the main results.

## 3 Main Results

First we verify the following statement.

**Theorem 3.1.** The following properties hold true:

- (i) If  $(p_m) \in AMDS$ , then  $(p_m) \in \lambda AMIUMS$ ,
- (ii) If  $(p_m) \in AMIS$ , then  $(p_m) \in \lambda AMDUMS$ ,

(iii) If 
$$\sum_{i=0}^{\lambda(n)-1} \left| \triangle \left( \frac{p_i}{P_{\lambda(n)}} \right) \right| = \mathcal{O}\left( \frac{1}{\lambda(n)} \right)$$
, then  $\sum_{i=0}^{\lambda(n)-1} \left| \triangle \left( B_{\lambda(n),i} \right) \right| = \mathcal{O}\left( \frac{1}{\lambda(n)} \right)$ ,

(iv) If 
$$\sum_{i=1}^{\lambda(n)-1} i \left| \triangle \left( \frac{p_i}{P_{\lambda(n)}} \right) \right| = \mathcal{O}(1)$$
, then  $\sum_{i=0}^{\lambda(n)-2} \left| \triangle \left( B_{\lambda(n),i} \right) \right| = \mathcal{O}\left( \frac{1}{\lambda(n)} \right)$ .

*Proof.* (i) If  $(p_m) \in AMDS$ , then  $Kp_m \ge p_\ell$  for  $m \le \ell$ . For  $m = \ell$  the implication (i) is true. Let  $m < \ell$ . Then

$$\begin{split} (\ell+1) \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i &= (m+1) \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i + (\ell-m) \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i \\ &\leq (m+1) \left[ \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i + K(\ell-m) p_{\lambda(n)-m} \right] \\ &\leq (m+1) \left[ \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i + K^2 \sum_{i=\lambda(n)-\ell}^{\lambda(n)-m-1} p_i \right] \\ &\leq \max \left\{ 1, K^2 \right\} (m+1) \sum_{i=\lambda(n)-\ell}^{\lambda(n)} p_i. \end{split}$$

Multiplying the above inequality by  $P_{\lambda(n)}$  we clearly obtain

$$B_{\lambda(n),m} \leq \max\left\{1,K^2\right\} B_{\lambda(n),\ell}$$
.

(ii) If  $(p_m) \in AMIS$ , then  $p_m \leq Kp_\ell$  for  $m \leq \ell$ . For  $m = \ell$  the implication (ii) is true. Let  $m < \ell$ . Then

$$(\ell+1) \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i = (m+1) \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i + (\ell-m) \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i$$

$$\geq (m+1) \left[ \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i + \frac{1}{K} (\ell-m) p_{\lambda(n)-m} \right]$$

$$\geq (m+1) \left[ \sum_{i=\lambda(n)-m}^{\lambda(n)} p_i + \frac{1}{K^2} \sum_{i=\lambda(n)-\ell}^{\lambda(n)-m-1} p_i \right]$$

$$\geq \min \left\{ 1, \frac{1}{K^2} \right\} (m+1) \sum_{i=\lambda(n)-\ell}^{\lambda(n)} p_i.$$

Multiplying the above inequality by  $P_{\lambda(n)}$  we clearly obtain

$$\frac{1}{\min\left\{1,\frac{1}{K^2}\right\}}B_{\lambda(n),m} \geq B_{\lambda(n),\ell}.$$

(iii) After some calculations we have

$$\sum_{i=0}^{\lambda(n)-1} \left| \triangle \left( B_{\lambda(n),i} \right) \right| = \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \frac{1}{(i+1)(i+2)} \times \left| (i+2) \sum_{j=\lambda(n)-i}^{\lambda(n)} p_i - (i+1) \sum_{j=\lambda(n)-i-1}^{\lambda(n)} p_i \right| \\
= \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \frac{1}{(i+1)(i+2)} \times \left| \sum_{j=\lambda(n)-i}^{\lambda(n)} p_i - (i+1) p_{\lambda(n)-i-1} \right|. \tag{3.1}$$

On the other side we have

$$\sum_{i=\lambda(n)-i}^{\lambda(n)} p_j - (i+1)p_{\lambda(n)-i-1} = \sum_{j=0}^i (j+1)(p_{\lambda(n)-j} - p_{\lambda(n)-j-1})$$
(3.2)

for any  $0 \le k \le n$ .

Now using (3.1) and (3.2) we obtain

$$\begin{split} \sum_{i=0}^{\lambda(n)-1} \left| \triangle \left( B_{\lambda(n),i} \right) \right| &= \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \frac{1}{(i+1)(i+2)} \times \left| \sum_{j=0}^{i} (j+1)(p_{\lambda(n)-j} - p_{\lambda(n)-j-1}) \right| \\ &\leq \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \frac{1}{(i+1)(i+2)} \times \sum_{j=0}^{i} (j+1) \left| p_{\lambda(n)-j} - p_{\lambda(n)-j-1} \right| \\ &\leq \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} (i+1) \left| p_{\lambda(n)-i} - p_{\lambda(n)-i-1} \right| \times \sum_{j=i}^{\infty} \frac{1}{(j+1)(j+2)} \\ &= \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \left| p_{\lambda(n)-i} - p_{\lambda(n)-i-1} \right| = \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-1} \left| p_{i+1} - p_{i} \right|. \end{split}$$

Therefore, if  $\sum_{i=0}^{\lambda(n)-1} \left| \triangle \left( \frac{p_i}{P_{\lambda(n)}} \right) \right| = \mathcal{O}\left( \frac{1}{\lambda(n)} \right)$ , then we also have  $\sum_{i=0}^{\lambda(n)-1} \left| \triangle \left( B_{\lambda(n),i} \right) \right| = \mathcal{O}\left( \frac{1}{\lambda(n)} \right)$ .

(iv) Let  $r = [\lambda(n)/2]$ . Taking into consideration (3.2) we get

$$\sum_{i=0}^{\lambda(n)-2} \left| \triangle \left( B_{\lambda(n),i} \right) \right| \leq \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{\lambda(n)-2} \frac{1}{(i+1)(i+2)} \sum_{j=0}^{i} (j+1) \left| p_{\lambda(n)-j} - p_{\lambda(n)-j-1} \right| 
= \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{r-1} \frac{1}{(i+1)(i+2)} \sum_{j=0}^{i} (j+1) \left| p_{\lambda(n)-j} - p_{\lambda(n)-j-1} \right| 
+ \frac{1}{P_{\lambda(n)}} \sum_{i=r}^{\lambda(n)-2} \frac{1}{(i+1)(i+2)} \sum_{j=0}^{i} (j+1) \left| p_{\lambda(n)-j} - p_{\lambda(n)-j-1} \right| 
= J_1 + J_2.$$

Under assumption of the theorem we have

$$J_{1} \leq \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{r-1} (i+1) \left| p_{\lambda(n)-i} - p_{\lambda(n)-i-1} \right| \sum_{j=i}^{\infty} \frac{1}{(i+1)(i+2)}$$

$$\leq \frac{1}{P_{\lambda(n)}} \sum_{i=0}^{r-1} \left| p_{\lambda(n)-i} - p_{\lambda(n)-i-1} \right| = \frac{1}{P_{\lambda(n)}} \sum_{i=\lambda(n)-r}^{\lambda(n)-1} \left| p_{i} - p_{i+1} \right|$$

$$\leq \frac{2}{\lambda(n) P_{\lambda(n)}} \sum_{i=\lambda(n)-r}^{\lambda(n)-1} i \left| \triangle p_{i} \right| \leq \frac{2}{\lambda(n) P_{\lambda(n)}} \sum_{i=1}^{\lambda(n)-1} i \left| \triangle p_{i} \right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right).$$

Now we write

$$J_{2} = \frac{1}{P_{\lambda(n)}} \sum_{i=r}^{\lambda(n)-2} \frac{1}{(i+1)(i+2)} \sum_{j=0}^{r-1} (j+1) \left| p_{\lambda(n)-j} - p_{\lambda(n)-j-1} \right|$$

$$+ \frac{1}{P_{\lambda(n)}} \sum_{i=r}^{\lambda(n)-2} \frac{1}{(i+1)(i+2)} \sum_{j=r}^{i} (j+1) \left| p_{\lambda(n)-j} - p_{\lambda(n)-j-1} \right| = J_{2}^{(1)} + J_{2}^{(2)}.$$

Then we have

$$J_{2}^{(1)} \leq \frac{1}{P_{\lambda(n)}} \sum_{i=r}^{\lambda(n)-2} \frac{1}{(i+1)\frac{\lambda(n)}{2}} \sum_{j=0}^{r-1} \frac{\lambda(n)}{2} \left| p_{\lambda(n)-j} - p_{\lambda(n)-j-1} \right|$$

$$\leq \frac{1}{P_{\lambda(n)}} \frac{\lambda(n) - r - 1}{r+1} \sum_{j=\lambda(n)-r}^{\lambda(n)-1} \left| p_{j} - p_{j+1} \right| \leq \frac{1}{P_{\lambda(n)}} \sum_{j=\lambda(n)-r}^{\lambda(n)-1} \left| \triangle p_{j} \right|$$

$$= \frac{\lambda(n)}{\lambda(n)P_{\lambda(n)}} \sum_{j=\lambda(n)-r}^{\lambda(n)-1} \left| \triangle p_{j} \right| \leq \frac{2}{\lambda(n)P_{\lambda(n)}} \sum_{j=\lambda(n)-r}^{\lambda(n)-1} i \left| \triangle p_{j} \right|$$

$$\leq \frac{2}{\lambda(n)P_{\lambda(n)}} \sum_{j=1}^{\lambda(n)-1} i \left| \triangle p_{j} \right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$$

and

$$J_{2}^{(2)} \leq \frac{1}{P_{\lambda(n)}} \sum_{i=r}^{\lambda(n)-2} \frac{1}{(i+1)\frac{\lambda(n)}{2}} \sum_{j=r}^{i} \frac{\lambda(n)}{2} \left| p_{\lambda(n)-j} - p_{\lambda(n)-j-1} \right|$$

$$\leq \frac{1}{P_{\lambda(n)}(r+1)} \sum_{i=r}^{\lambda(n)-2} \sum_{j=r}^{i} \left| p_{\lambda(n)-j} - p_{\lambda(n)-j-1} \right|$$

$$= \frac{1}{P_{\lambda(n)}(r+1)} \sum_{i=r}^{\lambda(n)-2} \sum_{j=\lambda(n)-i}^{\lambda(n)-r} \left| p_{j} - p_{j+1} \right|$$

$$\leq \frac{1}{P_{\lambda(n)}(r+1)} \sum_{i=r}^{\lambda(n)-2} \sum_{j=1}^{\lambda(n)-r} \left| \triangle p_{j} \right| = \frac{\lambda(n)-r-1}{P_{\lambda(n)}(r+1)} \sum_{j=1}^{\lambda(n)-r} \left| \triangle p_{j} \right|$$

$$\leq \frac{1}{P_{\lambda(n)}} \sum_{j=1}^{\lambda(n)-1} \left| \triangle p_{j} \right| \leq \frac{2}{\lambda(n)P_{\lambda(n)}} \sum_{j=0}^{\lambda(n)-1} i \left| \triangle p_{j} \right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right).$$

Inserting  $J_2^{(1)}$  and  $J_2^{(2)}$  into  $J_2$  we obtain  $J_2=\mathcal{O}\left(\frac{1}{\lambda(n)}\right)$ , which along with  $J_1=\mathcal{O}\left(\frac{1}{\lambda(n)}\right)$  we clearly find that  $\sum_{i=0}^{\lambda(n)-2}\left|\triangle\left(B_{\lambda(n),i}\right)\right|=\mathcal{O}\left(\frac{1}{\lambda(n)}\right)$ . The proof of the lemma is completed.  $\square$ 

**Theorem 3.2.** Let  $p \in \mathcal{M}$ ,  $f \in Lip(\alpha, p(x))$ ,  $0 < \alpha < 1$ , and  $(p_n)_{n=0}^{\infty}$  be a sequence of positive real numbers. Let

$$(p_n)_{n=0}^{\infty} \in \lambda - AMDUMS$$
 or

$$(p_n)_{n=0}^{\infty} \in \lambda - AMIUMS \quad and \quad (\lambda(n)+1)p_{\lambda(n)} = \mathcal{O}(P_{\lambda(n)}),$$
 (3.3)

then

$$||f - N_n^{\lambda}(f)||_{p(x)} = \mathcal{O}\left(\frac{1}{\left(\lambda(n) + 1\right)^{\alpha}}\right)$$

*holds for all*  $n \in \mathbb{N} \cup \{0\}$ .

Proof. Since

$$f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} f(x),$$

then we can write

$$f(x) - N_n^{\lambda}(f; x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \{ f(x) - S_m(f; x) \}.$$

Whence, using Lemma 2.2, Lemma 2.3, and conditions (3.3) we get

$$||f - N_n^{\lambda}(f)||_{p(x)} \leq \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} ||f(x) - S_m(f; x)||_{p(x)}$$

$$= \frac{1}{P_{\lambda(n)}} \mathcal{O}\left(\sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} (m+1)^{-\alpha}\right)$$

$$= \frac{1}{P_{\lambda(n)}} \mathcal{O}\left(\frac{P_{\lambda(n)}}{(\lambda(n)+1)^{\alpha}}\right) = \mathcal{O}\left(\frac{1}{(\lambda(n)+1)^{\alpha}}\right).$$

Next theorem gives the same degree of approximation with different conditions from those of Theorem 3.1, considering the case  $\alpha = 1$ .

**Theorem 3.3.** Let  $p \in \mathcal{M}$ ,  $f \in Lip(1, p(x))$  and let  $(p_n)_{n=0}^{\infty}$  be a sequence of positive real numbers. If

$$\sum_{m=0}^{\lambda(n)-2} \left| B_{\lambda(n),m} - B_{\lambda(n),m+1} \right| = \mathcal{O}\left(\frac{1}{\lambda(n)}\right),\,$$

then for n = 1, 2, ... the estimate

$$||f - N_n^{\lambda}(f)||_{p(x)} = \mathcal{O}\left(\frac{1}{\lambda(n)}\right)$$

holds.

*Proof.* According to the definition of  $N_n^{\lambda}(f;x)$  the following equality is true

$$E_n^{\lambda}(f;x) := N_n^{\lambda}(f;x) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{n=0}^{\lambda(n)} p_{\lambda(n)-m} \{ S_m(f;x) - f(x) \}.$$

Applying the summation by parts twice we get

$$\begin{split} E_n^{\lambda}(f;x) &= \sum_{m=0}^{\lambda(n)-1} \left(S_m(f;x) - S_{m+1}(f;x)\right) \frac{1}{P_{\lambda(n)}} \sum_{i=0}^m p_{\lambda(n)-i} + S_{\lambda(n)}(f;x) - f(x) \\ &= -\sum_{m=0}^{\lambda(n)-1} \left(m+1\right) U_{m+1}(f;x) B_{\lambda(n),m} + S_{\lambda(n)}(f;x) - f(x) \\ &= -\sum_{m=0}^{\lambda(n)-2} \left(B_{\lambda(n),m} - B_{\lambda(n),m+1}\right) \sum_{j=0}^m (j+1) U_{j+1}(f;x) \\ &- \frac{1}{\lambda(n) P_{\lambda(n)}} \sum_{i=1}^{\lambda(n)} p_i \sum_{i=0}^{\lambda(n)-1} (j+1) U_{j+1}(f;x) + S_{\lambda(n)}(f;x) - f(x). \end{split}$$

Thus,

$$||E_{n}^{\lambda}(f)||_{p(x)} \leq \sum_{m=0}^{\lambda(n)-2} |B_{\lambda(n),m} - B_{\lambda(n),m+1}| \left\| \sum_{j=1}^{m+1} jU_{j}(f) \right\|_{p(x)} + \frac{1}{\lambda(n)} \left\| \sum_{j=1}^{\lambda(n)} jU_{j}(f) \right\|_{p(x)} + \left\| S_{\lambda(n)}(f) - f \right\|_{p(x)}.$$

Based on Lemma 2.1 and the equality

$$\sum_{j=1}^{\lambda(n)} jU_j(f;x) = (\lambda(n)+1)(S_{\lambda(n)}(f;x) - \sigma_{\lambda(n)}(f;x)),$$

we have

$$\left\| \sum_{j=1}^{\lambda(n)} j U_j(f) \right\|_{p(x)} = \mathcal{O}(1).$$

Hence, using Lemma 2.2 and the latter estimation we get

$$||E_n(f)||_{p(x)} = \mathcal{O}\left(\sum_{m=0}^{\lambda(n)-2} \left|B_{\lambda(n),m} - B_{\lambda(n),m+1}\right|\right) + \mathcal{O}\left(\frac{1}{\lambda(n)}\right).$$

Finally, if the condition  $\sum_{m=0}^{\lambda(n)-2}\left|B_{\lambda(n),m}-B_{\lambda(n),m+1}\right|=\mathcal{O}\left(\frac{1}{\lambda(n)}\right)$  is satisfied, then we obtain

$$||N_n^{\lambda}(f) - f(x)||_{p(x)} = \mathcal{O}\left(\frac{1}{\lambda(n)}\right).$$

The proof of theorem is completed.

**Remark 3.4.** If  $\lambda(n) = n$ , n = 1, 2, ..., then Theorems 3.2 and 3.3 reduce to Theorems 1.1 and 1.2, respectively. Moreover, for the same sequence Theorem 4 from [14] is an immediate result of Theorem 3.1.

**Remark 3.5.** Since  $(\lambda(n))^{-\alpha} \le n^{-\alpha}$  for  $0 < \alpha \le 1$ , then Theorems 3.2 and 3.3 give shaper estimates than those of Theorems 1.1 and 1.2.

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Received: July 19, 2015.

Accepted: Februery 12, 2016.