On the Poisson Transform on the bounded domain of type IV.

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Abstract. Let \( \mathcal{D} \) be the Lie ball in \( \mathbb{C}^2 \) and let \( \mathcal{A}'(S) \) be the space of all hyperfunctions over the Shilov boundary \( S \) of \( \mathcal{D} \). The aim of this paper is to give a necessary and sufficient condition on the Poisson transform \( P_\lambda f \) of an element \( f \) in the space \( \mathcal{A}'(S) \) for \( f \) to be in \( L^2(S) \). More precisely, we establish for any \( \lambda \in \mathbb{R}\setminus\{0\} \) that:

(i) Let \( F = P_\lambda f, f \in L^2(S) \). Then we have

\[
\|F\|_{L^2(S)}^2 = \sup_{t > 0} \frac{1}{t} \int_0^t \left( \int_{S(O(2)\times O(2))} |F(ka_R, 0)|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) \, dk \, dr_2 \right) \, dr_1 < \infty.
\]

(ii) Let \( f \) be a hyperfunction on \( S \) such that its image \( F = P_\lambda f \) satisfies the growth condition \( |F|_\ast < \infty \), then necessarily such \( f \) is in \( L^2(S) \).

1 Introduction

Let \( \mathcal{D} = G/K \) be a Riemannian symmetric space of the non-compact type with Furstenberg boundary (maximal boundary) \( S_F \). Each eigenfunction of all invariant differential operators on \( \mathcal{D} \) can be represented by the Poisson integral of a hyperfunction on \( S_F \). This was conjectured by S. Helgason and proved by Kashiwara et al.

If \( \mathcal{D} = G/K \) is an irreducible bounded hermitian symmetric domain, it is well known that the Hua operator associated to \( \mathcal{D} \) characterizes the Poisson transform on the Shilov boundary (minimal boundary) \( S \) of \( \mathcal{D} \).

More precisely, the Poisson transform \( P_\lambda \) is a \( G \)-isomorphism from the space \( \mathcal{A}'(S) \) of all hyperfunctions on \( S \) onto an eigenspace of the Hua operator for \( \lambda \) varying in a subset of \( \mathbb{C} \).

Hence it becomes natural to look a characterization of the range of the Poisson transform on classical spaces of \( S \). This problem was handled by the authors in a series of papers for the case of the classical \( L^p \)-space on \( S \).

More precisely, they showed that for \( \lambda \) ranges a subset of \( \lambda \in \mathbb{C}\setminus\mathbb{R} \) the eigenfunctions of the Hua operator that are Poisson integral of \( L^p \)-functions on the Shilov boundary \( S \) are characterized by an \( H^p \)-condition.

In order to prove those result they established a Fatou-type theorem for the eigenfunctions of the Hua operator. More precisely, they gave the \( L^p \)-boundedness properties of the Poisson transform \( P_\lambda \) associated to the space \( \mathcal{D} \). And they established the asymptotic behaviour of the generalized spherical function \( \Phi_{\lambda,m} \)

\[
\Phi_{\lambda,m}(z) = \int_S P_\lambda(z, u) \phi_m(u) \, du
\]

where \( \phi_m \) is the zonal spherical function.

Thus, we address the question of the characterization of the \( L^p \)-range of the Poisson transform \( P_\lambda \) for \( \lambda \in \mathbb{R} \). In [1], [5] the authors gave an answer of this question in the case of rank one symmetric space with \( p = 2 \).

The aim of this paper is to give a necessary and sufficient condition on the Poisson transform \( P_\lambda f, \lambda \in \mathbb{R}\setminus\{0\} \) of an element \( f \) in the space \( \mathcal{A}'(S) \) for \( f \) to be in \( L^2(S) \) in the case of Lie ball \( \mathcal{D} = SO(2,2)/SO(2) \times SO(2) \). We are led:

First to establish the following lemma on the asymptotic behaviour of the generalized spherical function \( \phi_m \).
Lemma 1.1. There exists a constant $\gamma_1 > 0$ such that we have:

$$\lim_{t \to \infty} \frac{1}{t^2} \int_0^t \left( \int_0^{r_1} \left| \Phi_{\lambda,m}(aR,0) \right|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \right) dr_1 = \gamma_1 \left| \frac{\Gamma^2(i\lambda)}{\Gamma^2\left(\frac{i\lambda + 1}{2}\right)} \right|^2$$

for every $\lambda \in \mathbb{R} \setminus \{0\}$ and for every $m = (m_1, m_2) \in \Lambda$.

Second to investigate $L^2$-boundedness properties of the Poisson transform associated to $\mathcal{D}$. More precisely, we give the following lemma.

Lemma 1.2. Let $\lambda$ be a non zero real number. Then, there exists a positive constant $\gamma_2(\lambda)$ such that

$$\sup_{t > 0} \frac{1}{t^2} \int_0^t \left( \int_0^{r_1} \left| \Phi_{\lambda,m}(aR,0) \right|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \right) dr_1 < \gamma_2^2(\lambda).$$

The paper is organized as follows. In Section 2, we recall some preliminaries of harmonic analysis on the Lie ball in $\mathbb{C}^2$ and we state the main results of this paper. In Section 3, we give the precise action of the Poisson transform $L^2(S)$. Section 4 is devoted to the proof of Theorem 2.1 and Theorem 2.2. We conclude with an appendix in which we give the proof of Lemma 1.1.

2 Notation and statement of the main results

First, we recall some well known results of harmonic analysis in the Lie ball (see [3], [4]). For any matrix we denote by $a^t$ and $\bar{a}$ the transpose and conjugate of $a$ respectively.

Let

$$\mathcal{D} = \{ z = (z_1, z_2) \in \mathbb{C}^2 / 1 - 2\pi z^t + |zz|^2 > 0 \text{ and } |zz'| < 1 \},$$

be the Lie ball, where $|w|^2 = \bar{w}w$ for any $w \in \mathbb{C}^2$. The Shilov boundary $S$ of $\mathcal{D}$ is given by

$$S = \{ u = e^{i\theta} x \in \mathbb{C}^2; \ 0 \leq \theta < 2\pi, \ x \in S^1 \},$$

with

$$S^1 = \{ (x_1, x_2) \in \mathbb{R}^2; \ x_1^2 + x_2^2 = 1 \}.$$

Let $G = SO(2,2)$ be the group of all matrices $g$ in $SL(4, \mathbb{R})$ such that $g^t J g = J$, where $J = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$. Then, the group $G = SO(2,2)$ acts transitively on $\mathcal{D}$ by:

$$g : z \mapsto g.z = \left\{ \left( \left( \frac{zz^t + 1}{2}, i\left( \frac{zz^t - 1}{2} \right) \right) A^t + zB^t \right) \left( \begin{array}{c} 1 \\ i \end{array} \right) \right\}^{-1} \times \left\{ \left( \frac{zz^t + 1}{2}, i\left( \frac{zz^t - 1}{2} \right) \right) C^t + zD^t \right\}.$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G = SO(2,2)$. Thus as homogeneous space, we have the identification $\mathcal{D} = G/K$ where $K$ is the stabilizer in $G$ of $0$ given by $K = S(O(2) \times O(2))$. The action of $G$ extends naturally to $\mathcal{D}$ and under this action the group $K$ acts transitively on the Shilov boundary $S$ and we have $S = K/\{I_4\}$.

Finally recall that every $z$ in $\mathcal{D}$ can be written as $z = kaR,0$, with respect to the Cartan decomposition of $G$ is given by $SO(2,2) = KAK$. Here

$$aR \in A = \left\{ \begin{pmatrix} \text{diag}(\cosh r_1, \cosh r_2) & \text{diag}(\sinh r_1, \sinh r_2) \\ \text{diag}(\sinh r_1, \sinh r_2) & \text{diag}(\cosh r_1, \cosh r_2) \end{pmatrix}; \ R = (r_1, r_2) \in \mathbb{R}_2^2 \right\}.$$

Let $L^2(S)$ be the space of all square integrable $\mathbb{C}$-valued functions on $S$ with respect to the measure $du$. Then the group $K$ acts on $L^2(S)$ by:

$$f \mapsto \pi(k)f = f \circ k^{-1}, \ k \in K,$$
and under this action the space $L^2(S)$ has the following Peter–Weyl decomposition (see [3]):

$$L^2(S) = \bigoplus_{m \in \Lambda} V_m,$$

where $\Lambda$ is the set of all two-tuple $m = (m_1, m_2) \in \mathbb{Z}^2$ with $m_1 \geq m_2$. The $K$-irreducible component $V_m$ is the finite linear span $\{\phi_{m} \circ k, k \in K\}$, where $\phi_{m} \in V_m$ is the zonal spherical function given by

$$\phi_{m}(u) = (u_1 - iu_2)^{m_1 - m_2}(u_1^2 + u_2^2)^{m_2}, \quad u = (u_1, u_2) \in S, \quad m = (m_1, m_2).$$

Let $P(z, u)$ be the Poisson kernel of the Lie ball $D$ with respect to the Shilov boundary $S$ of $D$, given by (see [4])

$$P(z, u) = \frac{1 - 2zz^t + |zz^t|^2}{|(z - u)(z - u^t)|^2}.$$

Let $\lambda \in \mathbb{C}$ the Poisson transform $P_{\lambda}$ is defined for $f \in A'(S)$ by:

$$[P_{\lambda}f](z) = \int_{S} P_{\lambda}(z, u)f(u)du,$$

where

$$P_{\lambda}(z, u) = (P(z, u))^{\frac{1}{\lambda^2}}.$$

The main result of this paper is the following theorems.

**Theorem 2.1.** Let $\lambda \in \mathbb{R} \setminus \{0\}$. Then, we have:

1. Let $F = P_{\lambda}f, f \in L^2(S)$. Then

$$||F||_2^2 = \sup_{t > 0} \frac{1}{t^2} \int_{0}^{t} \left( \int_{S(O_2) \times S(O_2)} |F(kaR, 0)|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2)dkdr \right)dr_1 < \infty.$$

2. Let $f \in A'(S)$. If $F = P_{\lambda}f$ satisfies $||F||_* < \infty$, then $f \in L^2(S)$. Moreover, there exists a positive constants $\gamma_1$ and $\gamma_2(\lambda)$ such that for every function $f \in L^2(S)$ we have:

$$\gamma_1|C(\lambda)||f||_{L^2(S)} \leq ||P_{\lambda} f||_* \leq \gamma_2(\lambda)||f||_{L^2(S)}$$

**Theorem 2.2.** Let $F = P_{\lambda}f, f \in L^2(S)$. Then its $L^2$-boundary value $f$ is given by the following inversion formula

$$\lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} \left( \int_{k} \int_{S(O_2) \times S(O_2)} F(kaR, 0)P_{\lambda}(hakR, 0) \sinh(r_1 - r_2) \sinh(r_1 + r_2)dkdr \right)dr_1 = \gamma_2(\lambda)|C(\lambda)|^2f(h, e), \quad \text{in} \ L^2(S).$$

As a second result of this paper, we give an $L^2$-type inversion formula for the Poisson transform.

**Theorem 2.3.** Let $F = P_{\lambda}f, f \in L^2(S)$. Then its $L^2$-boundary value $f$ is given by the following inversion formula

$$\lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} \left( \int_{k} \int_{S(O_2) \times S(O_2)} F(kaR, 0)P_{\lambda}(hakR, 0) \sinh(r_1 - r_2) \sinh(r_1 + r_2)dkdr \right)dr_1 = \gamma_2(\lambda)|C(\lambda)|^2f(h, e), \quad \text{in} \ L^2(S).$$

The difficult part in proving our result is to show that every $F = P_{\lambda}f, f \in A'(S)$ such that $||F||_* < \infty$ is the poisson transform of an $L^2$-function on the Shilov boundary $S$. Indeed, expanding $F$ into a $C^\infty$ series (see corollary below)

$$F(kaR, 0) = \sum_{m \in \Lambda} a_m \Phi_{\lambda, m}(aR, 0)f_m(k, e)$$

next, applying the Lemma 1.1 of asymptotic behaviour of integral type of $\Phi_{\lambda, m}(aR, 0)$. 
3 The Poisson transform.

In this section, we give the precise action of the Poisson transform $P_\lambda$ on $L^2(S)$. For $\lambda \in \mathbb{C}$ and for $k \in \mathbb{Z}^+$, let $\varphi_{\lambda,k}(r)$ denote the following $\mathbb{C}$-valued function on $r \in [0,1]$

$$
\varphi_{\lambda,k}(r) = (1 - r^2)\tanh^{-1} k \frac{(i \lambda + 1)}{2} F\left(\frac{i \lambda + 1}{2}, \frac{i \lambda + 1}{2} + k, 1 + k; r^2\right),
$$

where $(a)_k = a(a+1)(a+2)\cdot \cdot \cdot (a+k-1)$ is the Pochhammer’s symbol and $F(a,b,c;z)$ is the classical Gauss hypergeometric function.

Proposition 3.1. [2] Let $m = (m_1, m_2) \in \wedge$ and let $f \in V_m$. Then, we have

$$
(P_\lambda f)(ka_0,0) = \Phi_{\lambda,m}(a_0,0)f(k,e),
$$

where the generalized spherical function $\Phi_{\lambda,m}$ is given by

$$
\Phi_{\lambda,m}(a_0,0) = 4\pi^2 \left[ \varphi_{\lambda|m_1}(\tanh \frac{r_1 - r_2}{2}) \varphi_{\lambda|m_2}(\tanh \frac{r_1 + r_2}{2}) \right].
$$

Corollary 3.2. Let $F = P_\lambda f, f \in A'(S)$. Then, there exists a sequence of spherical harmonic functions $(f_m)_{m \in \wedge}$ such that for every $z = ka_0,0 \in D$, $k \in K a_0 \in A$, $F$ may be written in the form as follows

$$
F(z) = \sum_{m \in \wedge} \Phi_{\lambda,m}(a_0,0)f_m(k,e), \quad f_m \in V_m.
$$

Proof. For $f$ in $A'(S)$. Let $f = \sum f_m$ it’s $K$-type decomposition. Then using Proposition 3.1 we get

$$
F(ka_0,0) = \sum_{m \in \wedge} \Phi_{\lambda,m}(a_0,0)f_m(k,e), \quad f_m \in V_m.
$$

4 Proof of main results

4.1 Proof of Theorem 2.1

For the proof Theorem 2.1, we will need the Lemma 1.2, which we recall below

Lemma 1.2 Let $\lambda$ be a non zero real number. Then, there exists a positive constant $\gamma_2(\lambda)$ such that

$$
\sup_{t > 0} \frac{1}{t^2} \int_0^t \left( \int_0^{t_1} \left| \Phi_{\lambda,m}(a_0,0) \right|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2)dr_2 \right)dr_1 < \gamma_2^2(\lambda).
$$

Proof. In order to get the proof for this lemma, we introduce the following lemma

Lemma (see [1]): Let $\lambda$ be a non zero real number. Then, there exists a positive constant $A(\lambda)$ such that for every $t > 0$, we have

$$
\sup_{k \in \mathbb{Z}^+} \left| \varphi_{\lambda,k}(\tanh t) \right| \leq A(\lambda) \cosh^{-1} t.
$$

For fixed $t > 0$, we have

$$
\frac{1}{t^2} \int_0^t \left( \int_0^{t_1} \left| \Phi_{\lambda,m}(a_0,0) \right|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2)dr_2 \right)dr_1
$$

$$
= 4 \frac{1}{t^2} \int_0^t \left( \int_0^{t_1} \coth^2\left(\frac{r_1 - r_2}{2}\right) \coth^2\left(\frac{r_1 + r_2}{2}\right) \left| \Phi_{\lambda,m}(a_0,0) \right|^2 \tanh\left(\frac{r_1 - r_2}{2}\right) \tanh\left(\frac{r_1 + r_2}{2}\right)dr_2 \right)dr_1
$$

$$
\leq 4 \frac{1}{t^2} \int_0^t \left( \int_0^{t_1} \coth^2\left(\frac{r_1 - r_2}{2}\right) \coth^2\left(\frac{r_1 + r_2}{2}\right) \left| \Phi_{\lambda,m}(a_0,0) \right|^2 dr_2 \right)dr_1.
$$
Then, we deduce from the above lemma that there exists a positive constant $\gamma_2(\lambda)$ such that
\[
\sup_{t > 0} \frac{1}{t^2} \int_0^t \left( \int_0^{r_1} |\Phi_{\lambda,m}(a_R,0)|^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \right) dr_1 \leq \frac{1}{4\pi^2} A^2(\lambda) = \gamma_2^2(\lambda).
\]

Then, for the necessary condition let $f \in L^2(S)$ and let $f = \sum m \in \Lambda_f$ be its K-type decomposition. By Proposition 3.1, with $\sum m \in \Lambda_f \Phi_{\lambda,m}(a_R,0)^2 ||f_m||^2_{L^2(S)} < \infty$, for every $R = (r_1, r_2) \in \mathbb{R}_+^2$, we have
\[
(P_\lambda f)(ka_R,0) = F(ka_R,0) = \sum m \in \Lambda_f \Phi_{\lambda,m}(a_R,0) f_m(k,e).
\]

Then, replacing $F$ by the above series expansion we get
\[
||F||_*^2 = \sup_{t > 0} \frac{1}{t^2} \int_0^t \left( \int_0^{r_1} \sum m \in \Lambda_f \Phi_{\lambda,m}(a_R,0)^2 ||f_m||_2^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \right) dr_1.
\]

Next, we use the Lemma 1.2 to obtain
\[
\sum m \in \Lambda_f \frac{||f_m||_2^2}{t^2} \int_0^t \left( \int_0^{r_1} \Phi_{\lambda,m}(a_R,0)^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \right) dr_1 \leq \gamma_2^2(\lambda) \sum m \in \Lambda_f ||f_m||_2^2 < \infty.
\]

Henceforth
\[
||P_\lambda f||_* \leq \gamma_2(\lambda)||f||_2,
\]

this gives the right hand side of estimate (2.3) in Theorem 2.1.

Now, to prove the sufficiency condition. Let $F = P_\lambda f, f \in A'(S)$ such that $||F||_* < \infty$. Let $f = \sum m \in \Lambda_f$ be its K-type decomposition, then using Proposition 3.1, we get
\[
F(ka_R,0) = \sum m \in \Lambda_f \Phi_{\lambda,m}(a_R,0) f_m(k,e).
\]

Since $||F||_* < \infty$, we have
\[
\sum m \in \Lambda_f \frac{||f_m||_2^2}{t^2} \int_0^t \left( \int_0^{r_1} \Phi_{\lambda,m}(a_R,0)^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \right) dr_1 < \infty.
\]

Let $\Lambda_0$ be a finite subset of $\Lambda$, then we have
\[
\sum m \in \Lambda_0 \frac{||f_m||_2^2}{t^2} \int_0^t \left( \int_0^{r_1} \Phi_{\lambda,m}(a_R,0)^2 \sinh(r_1 - r_2) \sinh(r_1 + r_2) dr_2 \right) dr_1 \leq ||F||_*^2 < \infty,
\]

for every $t > 0$.

Next, using the asymptotic behaviour of $\Phi_{\lambda,m}$ given by Lemma 1.1. we obtain
\[
\gamma_2^2 |C(\lambda)|^2 \sum m \in \Lambda_0 ||f_m||_2^2 \leq ||F||_*^2 < \infty.
\]

Since $\Lambda_0$ is arbitrary, we get
\[
\gamma_2^2 |C(\lambda)|^2 \sum m \in \Lambda ||f_m||_2^2 \leq ||F||_*^2 < \infty.
\]

Thus $\gamma_2^2 |C(\lambda)|^2 ||f||_2^2 \leq ||F||_*^2 < \infty$ and $f \in L^2(S)$ this finishes the proof of Theorem 2.1.
4.2 Proof of Theorem 2.2

In this section we try to prove the $L^2$-inversion formula. Let $F = Pf$, $f \in A(S)$ such that $\|F\|_{\ast} < \infty$. By the Theorem 2.1, we know that $f$ in $L^2(S)$. Expanding $f$ into its K-type series, $f = \sum_{m \in \Lambda} f_m$ and using Proposition 3.1, we get the series expansion of $F$,

$$F(ka_R, 0) = \sum_{m \in \Lambda} \Phi_{\lambda, m}(a_R, 0)f_m(k, e), \quad f_m \in V_m, \quad (4.1)$$

with $\sum_{m \in \Lambda} |\Phi_{\lambda, m}(a_R, 0)|^2 \|f_m\|^2 < \infty$, for all $R = (r_1, r_2)$, $r_1 > r_2 > 0$. Next, set for each $t > 0$, the following C-valued function on $S$

$$g_t(h, e) = \frac{1}{t^2} \int_0^t \left( \int_0^{\gamma_1} \int_K F(ka_R, 0)P_{\lambda}(ha_R, 0, k, e) |\sinh(r_1 - r_2)| \sinh(r_1 + r_2) dk dr \right) dr_1.$$

Then, replacing $F$ by its above series expansion in (4.1), the function $g_t$ can be rewritten as:

$$g_t(h, e) = \frac{1}{t^2} \int_0^t \left( \int_0^{\gamma_1} \int_K \sum_{m \in \Lambda} \Phi_{\lambda, m}(a_R, 0)f_m(k, e)P_{\lambda}(ha_R, 0, k, e) |\sinh(r_1 - r_2)| \sinh(r_1 + r_2) dk dr \right) dr_1.$$

Since, for every fixed $r_1 > r_2 > 0$, the series $\sum_{m \in \Lambda} \Phi_{\lambda, m}(a_R, 0)f_m(k, e)$ is uniformly convergent on $S$, we get

$$g_t(h, e) = \frac{1}{t^2} \sum_{m \in \Lambda} \int_0^t \left( \int_0^{\gamma_1} \left| \Phi_{\lambda, m}(a_R, 0) \right|^2 |\sinh(r_1 - r_2)| \sinh(r_1 + r_2) dr_1 \right) f_m(h, e).$$

Hence the $L^2(S)$-norm of the function $g_t$ is given by:

$$\|g_t\|_2^2 = \left( \frac{1}{t^2} \right)^2 \sum_{m \in \Lambda} \left[ \int_0^t \left( \int_0^{\gamma_1} \left| \Phi_{\lambda, m}(a_R, 0) \right|^2 |\sinh(r_1 - r_2)| \sinh(r_1 + r_2) dr_1 \right]^2 \|f_m\|^2 \right].$$

Now using the fact that

$$\frac{1}{t^2} \int_0^t \left( \int_0^{\gamma_1} \left| \Phi_{\lambda, m}(a_R, 0) \right|^2 |\sinh(r_1 - r_2)| \sinh(r_1 + r_2) dr_1 \right) = \frac{1}{t^2} \int_0^t \left( \int_0^{\gamma_1} \left| \Phi_{\lambda, m}(a_R, 0) \right|^2 |\sinh(r_1 - r_2)| \sinh(r_1 + r_2) dr_1 \right) \leq \gamma_1^2$$

we obtain that

$$\|g_t\|_2^2 |C(\lambda)|^{-2} \|f\|_2^2 = \sum_{m \in \Lambda} \left[ \frac{\gamma_1^2 |C(\lambda)|^{-2}}{t^2} \int_0^t \left( \int_0^{\gamma_1} \left| \Phi_{\lambda, m}(a_R, 0) \right|^2 |\sinh(r_1 - r_2)| \sinh(r_1 + r_2) dr_1 \right) - 1 \right]^2 \|f_m\|^2$$

and, using the uniform pointwises boundedness of $\Phi_{\lambda, m}$ given by Lemma 1.1, we see that

$$\lim_{t \to \infty} \|g_t\|_2^2 |C(\lambda)|^{-2} \|f\|_2^2 = 0$$

which given the desired result.
5 Appendix  The asymptotic behaviour of $\Phi_{\lambda,m}$.

We will now establish the asymptotic behavior of the generalized spherical function $\Phi_{\lambda,m}$. Recall that $\Phi_{\lambda,m}$ is given by

$$
\Phi_{\lambda,m}(aR,0) = 4\pi^2 \left[ \cosh \left( \frac{r_1 - r_2}{2} \right) \cosh \left( \frac{r_1 + r_2}{2} \right) \right]^{-(i\lambda + 1)} 
\times \tanh^{m_1}(r_1 - r_2)^{m_1} \tanh^{m_2}(r_1 + r_2) \left( \frac{\lambda + 1}{2} \right)^{m_1} \left( \frac{\lambda + 1}{2} \right)^{m_2} 
\times F(\frac{i\lambda + 1}{2}, \frac{i\lambda + 1}{2}, |m_1|, |m_2| + 1; \tanh^2(r_1 - r_2)) 
\times F(\frac{i\lambda + 1}{2}, \frac{i\lambda + 1}{2}, |m_1|, |m_2| + 1; \tanh^2(r_1 + r_2)).
$$

**Lemma 1.1** There exists a constant $\gamma_1 > 0$ such that we have:

$$
\lim_{t \to 0} \frac{1}{t^2} \int_0^t \left( \int_0^{r_1} \left| \Phi_{\lambda,m}(aR,0) \right|^2 |\sinh(r_1 - r_2)| |\sinh(r_1 + r_2)| dr_2 \right) dr_1 = \gamma_1^2 \left| \frac{\Gamma^2(i\lambda)}{\Gamma^2(\frac{i\lambda + 1}{2})} \right|^2
$$

for every $\lambda \in \mathbb{R} \setminus \{0\}$ and for every $m = (m_1, m_2) \in \Lambda$.

**Proof.** Using the following identity on hypergeometric function (see[6]):

$$
F(a, b, c; x) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b, a + b - c + 1, 1 - x) 
+ \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - x)^{c-a-b} 
\times F(c - a, c - b, c - a - b + 1, 1 - x)
$$

Then, the hypergeometric function $\varphi_{\lambda,k}(x)$ can be written as follows

$$
\varphi_{\lambda,k}( \tanh^2 x) = \cosh^{-(i\lambda + 1)}(x) \tanh^k(x) \left( \frac{i\lambda + 1}{2} \right)^k \frac{\Gamma(i\lambda + 1)}{\Gamma(\frac{i\lambda + 1}{2})} F(i\lambda + 1, i\lambda + 1, k, k + 1; \tanh^2(x)) 
= \cosh^{-(i\lambda + 1)}(x) \tanh^k(x) \left( \frac{i\lambda + 1}{2} \right)^k \frac{\Gamma(i\lambda)}{\Gamma(\frac{i\lambda + 1}{2}) \Gamma(\frac{i\lambda}{2})} F(i\lambda + 1, i\lambda + 1, k, i\lambda + 1, 1 - \tanh^2(x)) 
+ \cosh^{i\lambda - 1}(x) \tanh^k(x) \frac{\Gamma(i\lambda)}{\Gamma(\frac{i\lambda + 1}{2})} F(1 - i\lambda, 1 - i\lambda, k, 1 - i\lambda; 1 - \tanh^2(x)).
$$

Therefore for $\lambda \in \mathbb{R} \setminus \{0\}$, we have

$$
|\varphi_{\lambda,k}(\tanh^2 x)|^2 \cosh(x) \sinh(x) \simeq \left| \frac{\Gamma(i\lambda)}{\Gamma^2(\frac{i\lambda + 1}{2})} \right|^2 + A(\lambda, k) \cosh^{-2i\lambda}(x) + A(\lambda, k) \cosh^{2i\lambda}(x)
$$

with $A(\lambda, k) = \frac{\Gamma^2(\frac{i\lambda}{2} - \frac{i\lambda}{2})}{\Gamma^2(\frac{i\lambda + 1}{2})}$. To complete the proof, we are going to establish that

$$
\lim_{t \to \infty} \frac{I_1}{t^2} = \lim_{t \to \infty} \frac{I_2}{t^2} = \lim_{t \to \infty} \frac{I_3}{t^2} = 0
$$

where

$$
I_1 = \int_0^t \left[ \int_0^{r_1} \cosh^{2i\lambda}(r_1 - r_2) \cosh^{-2i\lambda}(r_1 + r_2) dr_2 \right] dr_1,
$$

$$
I_2 = \int_0^t \left[ \int_0^{r_1} \cosh^{2i\lambda}(r_1 - r_2) \cosh^{-2i\lambda}(r_1 + r_2) dr_2 \right] dr_1,
$$

$$
I_3 = \int_0^t \left[ \int_0^{r_1} \cosh^{2i\lambda}(r_1 \pm r_2) dr_2 \right] dr_1.
$$
Indeed,

For the integral $I_1$
By using the fact that

$$\int_0^{r_2} \cosh^{2\lambda} \left( \frac{r_1 - r_2}{2} \right) \cosh^{2\lambda} \left( \frac{r_1 + r_2}{2} \right) dr_2 = \frac{1}{2} \int_0^{r_2} \left( \cosh(r_1) + \cosh(r_2) \right)^{2\lambda} dr_2$$

and the fact that for every $s > 0$

$$\int_0^s \cosh(x)(\cosh(x) + \cosh(y))^{2\lambda - 1} dx = \frac{(\cosh(s) + \cosh(y))^{2\lambda} - (\cosh(y) + 1)^{2\lambda}}{2\lambda} + \int_0^s e^{-x}(\cosh(x) + \cosh(y))^{2\lambda - 1} dx,$$

which imply that

$$\left| \int_0^s e^{-x}(\cosh(x) + \cosh(y))^{2\lambda - 1} dx \right| \leq \int_0^s e^{-x}(\cosh(x) + \cosh(y))^{-1} dx \leq \int_0^s e^{-x} dx = 1 - e^{-s} < 1, \quad \lambda \in \mathbb{R} \setminus \{0\},$$

we have

$$\lim_{t \to \infty} \frac{I_1}{t^2} = 0.$$

For the integral $I_2$
The integral $I_2$ is equal

$$I_2 = \int_0^t \left[ \int_0^{r_1} \cosh^{2\lambda} \left( \frac{r_1 - r_2}{2} \right) \cosh^{-2\lambda} \left( \frac{r_1 + r_2}{2} \right) dr_2 \right] dr_1$$

$$= \frac{1}{2} \int_0^t \left[ \int_0^{r_1} \frac{\cosh(r_1) + \cosh(r_2)}{\cosh^{-2\lambda} \left( \frac{r_1 + r_2}{2} \right)} \left[ \cosh \left( \frac{r_1 - r_2}{2} \right) \right]^{2\lambda - 1} dr_2 \right] dr_1$$

$$= \frac{1}{2} \int_0^t \left[ \int_0^{r_1} \frac{\sinh(r_1) + e^{-r_1} + \cosh(r_2)}{\cosh^{-2\lambda} \left( \frac{r_1 + r_2}{2} \right)} \left[ \cosh \left( \frac{r_1 - r_2}{2} \right) \right]^{2\lambda - 1} dr_2 \right] dr_1$$

$$= \int_0^t \frac{1 - \left( \frac{1}{\cosh(r_1)} \right)^{2\lambda}}{2\lambda} dr_1$$

$$+ \frac{1}{2} \int_0^t \left[ \int_0^{r_1} \frac{e^{-r_1} + \cosh(r_2)}{\cosh^{-2\lambda} \left( \frac{r_1 + r_2}{2} \right)} \left[ \cosh \left( \frac{r_1 - r_2}{2} \right) \right]^{2\lambda - 1} dr_2 \right] dr_1.$$

Then, by using the fact that for every $\lambda \in \mathbb{R} \setminus \{0\}$

$$\left| \int_0^t \left[ \int_0^{r_1} \frac{e^{-r_1} + \cosh(r_2)}{\cosh^{-2\lambda} \left( \frac{r_1 + r_2}{2} \right)} \left[ \cosh \left( \frac{r_1 - r_2}{2} \right) \right]^{2\lambda - 1} dr_2 \right] dr_1 \right|$$

$$\leq \int_0^t \left[ \int_0^{r_1} \frac{e^{-r_1} + \cosh(r_2)}{\cosh(r_1) + \cosh(r_2)} dr_2 \right] dr_1 \leq \int_0^t \left[ \int_0^{r_1} (e^{-r_1} + \cosh(r_2)) dr_2 \right] dr_1 + \int_0^t \log \left( \frac{2\cosh(r_1)}{\cosh(r_1) + 1} \right) dr_1,$$

we have

$$\lim_{t \to \infty} \frac{I_2}{t^2} = 0.$$

For the integral $I_3^\pm$
By using the fact that
\[
\int_0^{r_1} \cosh^{2\lambda}\left(\frac{r_1 - r_2}{2}\right)dr_2 = \int_0^{r_1} \cosh\left(\frac{r_1 - r_2}{2}\right) \cosh^{2\lambda-1}\left(\frac{r_1 - r_2}{2}\right)dr_2 \\
= \int_0^{r_1} \left[ \sinh\left(\frac{r_1 - r_2}{2}\right) + e^{-\left(\frac{r_1 - r_2}{2}\right)} \right] \cosh^{2\lambda-1}\left(\frac{r_1 - r_2}{2}\right)dr_2 \\
= \frac{\cosh^{2\lambda}\left(\frac{r_1}{2}\right) - 1}{i\lambda} + 2 \int_0^{r_1} \left[ \frac{e\left(r_2 - r_1\right)}{1 + e\left(r_2 - r_1\right)} \right] \cosh^{2\lambda}\left(\frac{r_1 - r_2}{2}\right)dr_2
\]
and the fact that
\[
\left| \int_0^{r_2} \left[ \frac{e^{r_2 - r_1}}{1 + e^{r_2 - r_1}} \right] \cosh^{2\lambda}\left(\frac{r_1 - r_2}{2}\right)dr_2 \right|
\leq \int_0^{r_2} \frac{e^{r_2 - r_1}}{1 + e^{r_2 - r_1}} dr_2 = \log \left( \frac{2}{1 + e^{-r_1}} \right) = r_1 + \log \left( \frac{2}{1 + e^{-r_1}} \right)
\]
we have, for every \( \lambda \in \mathbb{R} \setminus \{0\} \) that
\[
\lim_{t \to \infty} \frac{I^-}{t^2} = 0.
\]
and analogously
\[
\lim_{t \to \infty} \frac{I^+}{t^2} = 0.
\]

References


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