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THE PRIME GRAPH $PG_1(R)$ OF A RING

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Abstract. We define a simple undirected graph $PG_1(R)$ with all the elements of a ring R as vertices, and two distinct vertices x, y are adjacent if and only if either $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in U(R)$, the set of all units of R. We have proved that $PG_1(\mathbb{Z}_n)$ is not Eulerian for any positive integer n. Also we discuss the Planarity and girth of $PG_1(R)$ and some cases which gives the degree of all vertices in $PG_1(R)$, over a ring $\mathbb{Z}_n$, for $n \leq 100$.

1 Introduction

The study of graph theory for a commutative ring was began when Beck in [1] introduced the notion of zero divisor graph. The graph $\Gamma_1(R)$ defined by R. Sen Gupta et al. [2] as: Let R be a ring with unity. Let $G = (V, E)$ be an undirected graph in which V = R - $\{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a unit. Another graph structure associated to a ring called prime graph was introduced by Satyanarayana et al. [3]. Prime graph is defined as a graph whose vertices are all elements of the ring and any two distinct vertices $x, y \in R$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by $PG(R)$. Here, we introduced a newer type of graph called $PG_1(R)$ as a generalisation of [2] and try to find the degree of vertices in $PG_1(R)$ by distributing the vertex set $V(G)$ into two sets viz. the set of all zero-divisors and the set of all units. Also we discuss the Eulerianity, Planarity and girth of $PG_1(R)$ and some cases which gives the degree of all vertices in $PG_1(R)$, over a ring $\mathbb{Z}_n$, for $n \leq 100$.

2 Preliminary Definitions

Here we are listing some preliminary definitions of graph theory and Algebra. For more details the reader is referred to [3]-[7].

Definition 2.1. [6] A ring R is a set together with two binary operations $+$ and $\cdot$ (called addition and multiplication) satisfying the following axioms:

(i) $(R, +)$ is an abelian group.

(ii) $\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in R$.

(iii) The distributive law holds in $R:$ for all $a, b, c \in R$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ for all $a, b, c \in R$.

Moreover, if a ring R satisfies the condition $a \cdot b = b \cdot a$, for all $a, b \in R$, then we say that R is a commutative ring. If R contains the multiplicative identity i.e. $(1 \cdot a = a \cdot 1 = a, for\ all\ a \in R)$ then we say that R is a ring with identity.

Definition 2.2. [6] Let R be a ring. A non-zero element a of R is called a zero-divisor if there is a non-zero element b in R such that $a \cdot b = 0$ or $b \cdot a = 0$. The set of all zero-divisors in a ring R is denoted by $Z(R)$.

Definition 2.3. [6] The elements which are not zero-divisors are called units. The set of all units in a ring R is denoted by $U(R)$.

Definition 2.4. A graph that has neither self-loops nor parallel edges is called a simple graph.

Definition 2.5. A simple graph in which every pair of vertices is adjacent is called as a complete graph. Complete graph on n vertices is denoted by K_n.

Definition 2.6. Let $G = (V, E)$ be a simple graph. The complement of G is denoted by G^c and is defined as the simple graph whose vertex set is same as that of G and two vertices are adjacent in G^c if and only if they are not adjacent in G.

Definition 2.7. The number of edges incident to a vertex v is called the degree of the vertex v, and it is denoted by $d(v)$. The degree of vertex is also known as valancy.

Definition 2.8. For any integer $n > 1$, we define a function $\phi(n)$ to be the number of positive integers less than n and relatively prime to n called Euler's totient function. If $p_1, p_2, p_3, \ldots, p_n$ are distinct prime factors of $(n > 1)$, then $\phi(n) = n \cdot (1 - 1/p_1) \cdot (1 - 1/p_2) \cdots (1 - 1/p_n)$.

Definition 2.9. Let R be a commutative ring with unity. Let $Z(R)$ be the set of all zero-divisors of R. For $x \in Z(R)$, let $ann_R(x) = \{y \in R / y \cdot x = 0\}$ is called Annihilator of an element of a ring.

Definition 2.10. A walk is defined as a finite alternating sequence of vertices and edges, (no repetition of edge allowed) beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. A walk is said to be a closed walk if the terminal vertices are same. A walk that is not closed (i.e. the terminal vertices are distinct) is called an open walk.

Definition 2.11. An open walk in which no vertex appears more than once is called a path (or a simple path or an elementary path). The number of edges in a path is called the length of a path.

Definition 2.12. The girth of a graph G, denoted by $gr(G)$, is the length of the shortest cycle. If no such a cycle exists then $gr(G) = \infty$. The shortest possible cycle consists of three pairwise adjacent vertices.

Definition 2.13. Let G be a graph. A closed walk running through every edge of a graph G exactly once is called as Euler line. A graph contains Euler line is called as Euler graph.

Definition 2.14. A graph G is said to be planer if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect. A graph that cannot be drawn on a plane without a crossover between its edges is called nonplaner.

Definition 2.15. [2] For a ring R, a simple undirected graph $G = (V, E)$ is said to be a graph $\Gamma_2(R)$ if all the nonzero elements of R as vertices, and two distinct vertices a and b are adjacent if and only if either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a zero divisor (including zero).

Remark 2.16. When p is prime the complement of $PG_1(\mathbb{Z}_p)$, contains a graph $\Gamma_2(\mathbb{Z}_p)$ as a subgraph.

Remark 2.17. If $n \neq 2$ then $PG_1(\mathbb{Z}_n)$ is always incomplete graph.

3 Degree, Eulerianity, Planarity and girth of $PG_1(\mathbb{Z}_n)$

Theorem 3.1. *For any $n \in \mathbb{N}$, the degree of vertex zero in $PG_1(\mathbb{Z}_n)$ is $n - 1$.*

Theorem 3.2. *Let u be the unit element in a ring $R = \mathbb{Z}_n$, for any $n \in \mathbb{N}$, the degree of u in $PG_1(R)$ is*

$$\begin{aligned} deg(u) &= \phi(n), && \text{if } n \text{ is even} \\ deg(u) &= \phi(n) - 1, && \text{if } n \text{ is odd.} \end{aligned}$$

Theorem 3.3. *Let z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_n$, for any $n \in \mathbb{N}$ and $n = pq$, where p and q are distinct primes, then the degree of z in $PG_1(R)$ is*

$$deg(z) = \phi(n) + 1.$$

Theorem 3.4. *Let z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_n$, for any $n \in \mathbb{N}$ such that $z^2 \equiv 0 \ (mod\ n)$ then the degree of z in $PG_1(R)$ is*

$$deg(z) = |ann(z)| + \phi(n).$$

Theorem 3.5. *Let u be the unit element and z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_{p^2}$, where p is a prime positive integer. Then the degree of u and z is*

$$\begin{aligned} deg(u) &= \phi(n) && \text{[From Theorem 3.2]} \\ deg(u) &= \phi(n) - 1 && \text{[From Theorem 3.2]} \\ and\ \ deg(z) &= |ann(z)| + \phi(n) && \text{[From Theorem 3.4].} \end{aligned}$$

Theorem 3.6. *Let z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_{2^n p}$, for any $n \in \mathbb{N}$, where p is prime*

(a) If $p = 2$, then

$$deg(z) = |ann(z)| + \phi(n)$$

(b) If $p \neq 2$, then

$$\begin{aligned} deg(z) &= |ann(z)| + \phi(n), && \text{if } z \text{ is multiple of } 2p \text{ or } z^2 \equiv 0 \ (mod\ n) \\ &= \phi(n) + 1, && \text{if } z \text{ is multiple of } p \\ &= \phi(n) + 2^1, && \text{if } z \text{ is multiple of } 2^1 \\ &= \phi(n) + 2^2, && \text{if } z \text{ is multiple of } 2^2 \\ &\ \vdots && \vdots \\ &= \phi(n) + 2^{n-2}, && \text{if } z \text{ is multiple of } 2^{n-2} \\ &= \phi(n) + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^{n-1} \text{ or } 2^n. \end{aligned}$$

Theorem 3.7. *Let z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_{2^n p^2}$, for any $n \in \mathbb{N}$, where p is prime and $p \neq 2$*

(a) If $n = 1$, then

$$\begin{aligned} deg(z) &= |ann(z)| + \phi(n) && \text{if } z \text{ is multiple of } 2p \text{ or } z^2 \equiv 0 \ (mod\ n) \\ &= \phi(n) + p, && \text{if } z \text{ is multiple of } p \text{ or } p^2 \\ &= \phi(n) + 1, && \text{if } z \text{ is multiple of } 2. \end{aligned}$$

(b) If $n \neq 1$, then

$$\begin{aligned} deg(z) &= |ann(z)| + \phi(n), && \text{if } z \text{ is multiple of } 2p \text{ or } z^2 \equiv 0 \ (mod\ n) \\ &= \phi(n) + p, && \text{if } z \text{ is multiple of } p \text{ or } p^2 \\ &= \phi(n) + 2^1, && \text{if } z \text{ is multiple of } 2^1 \\ &= \phi(n) + 2^2, && \text{if } z \text{ is multiple of } 2^2 \\ &\ \vdots && \vdots \\ &= \phi(n) + 2^{n-2}, && \text{if } z \text{ is multiple of } 2^{n-2} \\ &= \phi(n) + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^{n-1} \text{ or } 2^n. \end{aligned}$$

Theorem 3.8. *Let z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_{2^n pq}$, for any $n \in \mathbb{N}$, where p and q are distinct odd primes and $p < q$*

(a) If $n = 1$, then

$$\begin{aligned} deg(z) &= \phi(n) + p + 1, && \text{if } z \text{ is multiple of } 2p \\ &= \phi(n) + q + 1, && \text{if } z \text{ is multiple of } 2q \\ &= pq, && \text{if } z \text{ is multiple of } pq \\ &= \phi(n) + 1, && \text{if } z \text{ is multiple of } p \text{ or } q. \end{aligned}$$

(b) If $n \neq 1$, then

$$\begin{aligned} deg(z) &= |ann(z)| + \phi(n), && \text{if } z^2 \equiv 0 \ (mod\ n) \\ &= \phi(n) + q + 2, && \text{if } z \text{ is multiple of } pq \\ &= \phi(n) + 2, && \text{if } z \text{ is multiple of } 2 \\ &= \phi(n) + 1, && \text{if } z \text{ is multiple of } p \text{ or } q \\ &= \phi(n) + 2^{n-1}p + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n p \\ &= \phi(n) + 2^{n-1}q + 2^{n-1}, && \text{if } z \text{ is multiple of } 2^n q \\ &= \phi(n) + 2^{n-1}p, && \text{if } z \text{ is multiple of } 2^{n-1} p \\ &= \phi(n) + 2^{n-1}q, && \text{if } z \text{ is multiple of } 2^{n-1} q \\ &\vdots && \vdots \\ &= \phi(n) + 2^{n-(n-1)}p, && \text{if } z \text{ is multiple of } 2^{n-(n-1)} p \\ &= \phi(n) + 2^{n-(n-1)}q, && \text{if } z \text{ is multiple of } 2^{n-(n-1)} q. \end{aligned}$$

Theorem 3.9. *$PG_1(\mathbb{Z}_n)$ is not Eulerian for any positive integer n.*

Proof. Let n be even, so from Theorem 3.1, we have that $deg(0) = n - 1$ = an odd number. Again if n is odd, then by Theorem 3.2, degree of any unit in $\mathbb{Z}_n$ is $\phi(n) - 1$, an odd number. Hence, $PG_1(\mathbb{Z}_n)$ is not Eulerian for any positive integer n. □

Theorem 3.10. *$PG_1(\mathbb{Z}_n)$ is planar if and only if $n = 4, 6$ or n is a prime.*

Proof. $PG_1(\mathbb{Z}_2)$ is a complete graph K_2, so it is a planar graph. Also, $PG_1(\mathbb{Z}_3)$ is a union of two copies of K_2 in which vertex zero is common. So, it is also a planar graph. When n is prime, $n > 3$, $PG_1(\mathbb{Z}_n)$ is a union of copies of K_3 in which again zero is a common vertex. So, the graph is planar when n is prime. When $n = 4$, $PG_1(\mathbb{Z}_4)$ is union of two copies of K_3, having an edge $\{0, 2\}$ is common hence, planar. When $n = 6$, $PG_1(\mathbb{Z}_6)$ is again union of four copies of K_3 hence, planar. When $n = 8$, the graph $PG_1(\mathbb{Z}_8)$ contains a subgraph $K_{3,3}$. So, it cannot be a planar graph. Now, let $n = 2^m$, $m > 2$ contains a $K_{3,3}$ and hence cannot be planar. For $p \geq 3$, $PG_1(\mathbb{Z}_{p^m})$, where $m > 1$ contains a K_5, hence it cannot be planar. Now, we come to the cases, where n has more than one prime factor. First let n be even. If $n = 10$, then the subgraph induced by the vertices $\{0, 2, 4, 5, 6, 7\}$ forms a $K_{3,3}$ and for $n = 12$, the subgraph induced by the vertices $\{2, 3, 4, 6, 9, 10\}$ forms again $K_{3,3}$. So, the subgraph of $PG_1(\mathbb{Z}_n)$ where n is even form $K_{3,3}$ and hence the graph is not planar. Now, let n be odd. If $n = 15$ then the subgraph induced by $\{0, 1, 6, 7, 10\}$ forms a K_5 also, for $n = 21$ the subgraph induced by $\{1, 3, 7, 10, 16\}$ forms again K_5. So, the subgraph of $PG_1(\mathbb{Z}_n)$, where n is odd forms a K_5 and hence the graph is again nonplanar. Hence the result. □

Theorem 3.11. *The girth of $PG_1(\mathbb{Z}_n)$ is,*

$$\begin{aligned} girth(PG_1(\mathbb{Z}_n)) &= \infty, && \text{if } n = 2,\ 3 \\ &= 3, && \text{otherwise.} \end{aligned}$$

Proof. $PG_1(\mathbb{Z}_2)$ is a complete graph K_2, hence $girth(PG_1(\mathbb{Z}_2)) = \infty$. In $PG_1(\mathbb{Z}_3)$ there is no edge between the vertices 1 and 2 so 3-cycle does not exist. Hence, again $girth(PG_1(\mathbb{Z}_3)) = \infty$. Now, let $n \geq 4$, then in $PG_1(\mathbb{Z}_n)$ always a 3-cycle exist and hence $girth(PG_1(\mathbb{Z}_n)) = 3$ for $n \geq 4$. □

4 Degree of vertices in $PG_1(\mathbb{Z}_n)$

In this section, we discuss some more cases in continuation to Theorem 3.1 to Theorem 3.8 which calculates the degree of vertices in $PG_1(\mathbb{Z}_n)$, for $n \leq 100$.

Case-1: (a) Let z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_n$, $n = 3pq$ where, $p = 3, q = 5, 7, 11, 13, \ldots$ then

$$\begin{aligned} deg(z) &= |ann(z)| + \phi(n) \text{ Or } \phi(n) + 3q, && \text{if } z \text{ is multiple of } pq \text{ and } 3q \\ &= \phi(n) + p, && \text{if } z \text{ is multiple of } p \\ &= \phi(n) + 1, && \text{otherwise.} \end{aligned}$$

(b) In this case when $p = q = 3$, then

$$\begin{aligned} deg(z) &= |ann(z)| + \phi(n), && \text{if } z \text{ is multiple of } 3p,\ 3q \text{ and } p^2 \\ &= \phi(n) + p, && \text{if } z \text{ is multiple of } p. \end{aligned}$$

Case-2: Let z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_n$, $n = 3p^2$, $p = 3, 5, 7, \ldots$ then,

$$\begin{aligned} deg(z) &= |ann(z)| + \phi(n), && \text{if } z \text{ is multiple of } 3p \\ &= \phi(n) + p, && \text{if } z \text{ is multiple of } p \\ &= \phi(n) + 1, && \text{otherwise.} \end{aligned}$$

Case-3: Let z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_n$, $n = 2p^3$, $p = 3, 5, 7, \ldots, p > 2$ then

$$\begin{aligned} deg(z) &= \phi(n) + 2p^2 - 1, && \text{if } z \text{ is multiple of } 2p^2 \\ &= \phi(n) + p^2, && \text{if } z \text{ is multiple of } p^2 \\ &= \phi(n) + 2p, && \text{if } z \text{ is multiple of } 2p \\ &= \phi(n) + p, && \text{if } z \text{ is multiple of } p \\ &= \phi(n) + 1, && \text{otherwise.} \end{aligned}$$

Case-4: Let z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_n$, $n = p^4$, $p = 2, 3, 5, 7, \ldots$ then

$$\begin{aligned} deg(z) &= \phi(n) + p^3 - 1, && \text{if } z \text{ is multiple of } p^3 \\ &= \phi(n) + p^2 - 1, && \text{if } z \text{ is multiple of } p^2 \\ &= \phi(n) + p, && \text{if } z \text{ is multiple of } p. \end{aligned}$$

Case-5: Let z be a non-zero zero-divisor in a ring $R = \mathbb{Z}_n$, $n = 2p^2q$, $p = 3$, $q = 5, 7, 11, \ldots$ then

$$\begin{aligned} deg(z) &= \phi(n) + 2p^2, && \text{if } z \text{ is multiple of } 2p^2 \\ &= \phi(n) + 2p, && \text{if } z \text{ is multiple of } 2p \text{ and } 2q \\ &= \phi(n) + p, && \text{if } z \text{ is multiple of } p^2 \text{ and } q \\ &= \phi(n) + pq, && \text{if } z \text{ is multiple of } pq \\ &= p^2q, && \text{if } z \text{ is multiple of } p^2q \\ &= \phi(n) + 1, && \text{if } z \text{ is multiple of 2 and } q. \end{aligned}$$

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