Palestine Journal of Mathematics

Vol. 6(1)(2017) , 173–178

SOME NEW MODULAR EQUATIONS OF RATIO'S OF RAMANUJAN QUANTITY

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Communicated by P.K. Banerji

MSC 2010 Classifications: Primary 33D10, 40A15,; Secondary 11A55, 30B70.

Keywords and phrases: Continued fraction, quantities, theta functions.

The First Author is thankful to the University Grants Commission (UGC), New Delhi, India for the Financial Support under the Minor Research Project No. MRP(S)0045/12-13/KABA095/UGC-SWRO, Dated 23.09.2013.

Abstract. Recently, In [5],[6] Nikos Bagis defined Ramanujan Quantities $R(a,b,p;q)$ as

$$R(a,b,p;q) = q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{\prod_{n=0}^{\infty}(1-q^a q^{np})(1-q^{p-a}q^{np})}{\prod_{n=0}^{\infty}(1-q^b q^{np})(1-q^{p-b}q^{np})}, \tag{0.1}$$

where a, b and p are positive rationals such that $a+b<p$. In this paper, we establish some modular equations of ratios for Ramanujan quantity $R(q^n) := R(1,2,4;q^n)$ for $n = 2,3,4,5,7,9$ and some of their evaluations.

1 Introduction

In Chapter 16 of his second notebook [1], Ramanujan develops the theory of theta-function and is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \mid ab \mid < 1, \tag{1.1}$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

where $(a;q)_0 = 1$ and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

Following Ramanujan, we define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.4}$$

and

$$\chi(q) := (-q;q^2)_\infty. \tag{1.5}$$

Now we define a modular equation in brief. The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $|\, x \,|< 1$. Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{1.6}$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \tag{1.7}$$

where $0 < x < 1$.
Let r denote a fixed natural number and assume that the following relation holds:

$$r\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{1.8}$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (1.8). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

Using Ramanujan continued fraction [1], Nikos Bagis [5],[6] deduced the following result:

$$\frac{q^{B-A}}{1-a_1b_1} + \frac{(a_1-b_1q_1)(b_1-a_1q_1)}{(1-a_1b_1)(q_1^2+1)} + \frac{(a_1-b_1q_1^3)(b_1-a_1q_1^3)}{(1-a_1b_1)(q_1^4+1)} + \cdots$$

$$= \frac{\prod_{n=0}^{\infty}(1-q^aq^{np})(1-q^{p-a}q^{np})}{\prod_{n=0}^{\infty}(1-q^bq^{np})(1-q^{p-b}q^{np})} \tag{1.9}$$

where $a_1 = q^A$, $b_1 = q^B$, $q_1 = q^{A+B}$, $a = 2A + 3p/4$, $2B + p/4$, and $p = 4(A+B)$, $|q| < 1$,.

In this paper, we establish several new modular relations between $\dfrac{R(-q)}{R(q)}$ and $\dfrac{R(-q^n)}{R(q^n)}$ for $n = 2, 3, 4, 5, 7, 9$ and values of $\dfrac{R(-q)}{R(q)}$ for $q = e^{-\pi}, e^{-2\pi}$.

2 Preliminary results

Definition 2.1. [6]

$$[a, p; q] = (q^{p-a}; q^p)_\infty (q^a; q^p)_\infty \tag{2.1}$$

where $q = e^{-\pi\sqrt{r}}$ and $a, p, r > 0$.

Definition 2.2. [6]

$$R(a, b, p; q) := q^{-(a-b)/2+(a^2-b^2)/(2p)} \frac{[a, p; q]}{[b, p; q]}. \tag{2.2}$$

Lemma 2.3. *[1, Ch. 17, Entry 10-11, p.122-123]*

$$\psi(q) = \sqrt{\frac{1}{2}z}\{\alpha q^{-1}\}^{1/8} \tag{2.3}$$

$$\psi(-q) = \sqrt{\frac{1}{2}z}\{\alpha(1-\alpha)q^{-1}\}^{1/8} \tag{2.4}$$

where $q = e^{-y}$.

Lemma 2.4. *[4, Entry 17.3.1, p.385] If β is of degree 2 over α, then*

$$(1-\sqrt{1-\alpha})(1-\sqrt{\beta}) = 2\sqrt{\beta(1-\alpha)}. \tag{2.5}$$

Lemma 2.5. *[1, Entry 5(ii), p.230] If β has degree 3 over α, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{2.6}$$

Lemma 2.6. *[4, Entry 17.3.2, p.385] If β has degree 4 over α, then*

$$(1-\sqrt[4]{1-\alpha})(1-\sqrt[4]{\beta}) = 2\sqrt[4]{\beta(1-\alpha)}. \tag{2.7}$$

Lemma 2.7. *[1, Entry 13(i), p.280] If β has degree 5 over α, then*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{2.8}$$

Lemma 2.8. *[1, Entry 19(i), p.314] If β has degree 7 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \tag{2.9}$$

Lemma 2.9. *[1, Entry 3(x),(xi), p.352] If β has degree 9 over α, then*

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{m}. \tag{2.10}$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} = \frac{3}{\sqrt{m}}. \tag{2.11}$$

Lemma 2.10. *[1, Entry 24(i), p.39]*

$$\frac{\psi(-q)}{\psi(q)} = \sqrt{\frac{\varphi(-q)}{\varphi(q)}} \tag{2.12}$$

3 Modular relations of Ratio's of Ramanujan Quantities of $R(q)$

In this section, we obtain certain modular relations between $\frac{a}{b} := \frac{R(-q)}{R(q)}$ and $\frac{c}{d} := \frac{R(-q^n)}{R(q^n)}$.

Theorem 3.1. *If $\frac{a}{b} := \frac{R(-q)}{R(q)}$ and $\frac{c}{d} := \frac{R(-q^2)}{R(q^2)}$ then*

$$\left(\frac{ac}{bd}\right)^2 + \left(\frac{bc}{ad}\right)^2 = 2\left(\frac{d}{c}\right)^2. \tag{3.1}$$

Proof. Putting $a = 1, b = 2, p = 4$ in (0.1), we obtain

$$R(q) := R(1,2,4;q) = \frac{(q^2;q^4)_\infty (q^2;q^4)_\infty}{(q;q^4)_\infty (q^3;q^4)_\infty}. \tag{3.2}$$

Using the equations (1.1), (1.2) and (1.3), the above equation can be written as

$$R(q) = \frac{f(-q^2,-q^2)}{f(-q,-q^3)} = \frac{\varphi(-q^2)}{\psi(-q)}. \tag{3.3}$$

Replacing q by $-q$ in the above equation, we obtain

$$R(-q) = \frac{f(-q^2,-q^2)}{f(q,q^3)} = \frac{\varphi(-q^2)}{\psi(q)}. \tag{3.4}$$

Dividing (3.4) by (3.3), we obtain

$$\frac{R(-q)}{R(q)} = \frac{\psi(-q)}{\psi(q)}. \tag{3.5}$$

Employing the equations (2.3) and (2.4), we obtain

$$\frac{\psi(-q)}{\psi(q)} = (1-\alpha)^{1/8}. \tag{3.6}$$

Dividing (3.4) by (3.3) and then using (3.6), we obtain

$$(1-\alpha)^{1/8} = \frac{R(-q)}{R(q)}. \tag{3.7}$$

From the equation (3.7) and lemma (2.4), we obtain

$$(b^4c^4 + 2b^2d^4a^2 + c^4a^4)(b^4c^4 - 2b^2d^4a^2 + c^4a^4) = 0. \tag{3.8}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the second factor is zero; whereas first factor are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 3.2. *If* $\frac{a}{b} := \frac{R(-q)}{R(q)}$ *and* $\frac{c}{d} := \frac{R(-q^3)}{R(q^3)}$ *then*

$$\left(\frac{ad}{bc}\right)^2 + 2\frac{bd}{ac} = \left(\frac{bc}{ad}\right)^2 + 2\frac{ac}{bd}. \tag{3.9}$$

Proof. From the equation (3.7) and lemma (2.5), we obtain

$$(d^4a^4 + 2dc^3ba^3 - 2d^3cb^3a - c^4b^4)(d^4a^4 - 2dc^3ba^3 + 2d^3cb^3a - c^4b^4) = 0. \tag{3.10}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the second factor is zero; whereas first factor are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 3.3. *If* $\frac{a}{b} := \frac{R(-q)}{R(q)}$ *and* $\frac{c}{d} := \frac{R(-q^4)}{R(q^4)}$ *then*

$$\left(\frac{ac}{bd}\right)^4 + \left(\frac{bc}{ad}\right)^4 + \frac{c^4}{d^4}\left(4\left[\frac{a^2}{b^2} + \frac{b^2}{a^2}\right] + 6\right) = 8\frac{d^4}{c^4}\left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right) \tag{3.11}$$

Proof. From the equation (3.7) and lemma (2.6), we obtain(3.11). □

Theorem 3.4. *If* $\frac{a}{b} := \frac{R(-q)}{R(q)}$ *and* $\frac{c}{d} := \frac{R(-q^5)}{R(q^5)}$ *then*

$$\left(\frac{ad}{bc}\right)^3 + 4\left(\frac{bd}{ac}\right)^2 + 5\frac{ad}{bc} = \left(\frac{bc}{ad}\right)^3 + 4\left(\frac{ac}{bd}\right)^2 + 5\frac{bc}{ad}. \tag{3.12}$$

Proof. From the equation (3.7) and lemma (2.9), we obtain

$$\begin{aligned}
&(-5b^4d^2c^4a^2 - 4d^5b^5ca + d^6a^6 + 4bdc^5a^5 + 5d^4b^2c^2a^4 - b^6c^6)\\
&(-5b^4d^2c^4a^2 + 4d^5b^5ca + d^6a^6 - 4bdc^5a^5 + 5d^4b^2c^2a^4 - b^6c^6)\\
&(15b^4d^8c^4a^8 - 10b^{10}d^2c^{10}a^2 + 15b^8d^4c^8a^4 + 20b^6d^6c^6a^6 + b^{12}c^{12}\\
&+ d^{12}a^{12} + 16d^2b^2c^{10}a^{10} + 16b^{10}d^{10}c^2a^2 - 10d^{10}b^2c^2a^{10})(b^{24}c^{24}\\
&+ 58b^{20}d^4c^{20}a^4 - 320b^{20}d^{12}c^{12}a^4 + 256a^4c^4b^{20}d^{20} + 1423b^{16}a^8d^8c^{16}\\
&- 1408b^{16}a^8d^{16}c^8 - 320b^{12}d^4c^{20}a^{12} + 620a^{12}c^{12}b^{12}d^{12} + a^{24}d^{24}\\
&- 1408b^8a^{16}d^8c^{16} + 1423b^8a^{16}d^{16}c^8 + 256a^{20}c^{20}b^4d^4 - 320b^4d^{12}c^{12}a^{20}\\
&+ 58b^4d^{20}c^4a^{20} - 320d^{20}b^{12}c^4a^{12}) = 0.
\end{aligned} \tag{3.13}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 3.5. *If* $\frac{a}{b} := \frac{R(-q)}{R(q)}$ *and* $\frac{c}{d} := \frac{R(-q^7)}{R(q^7)}$ *then*

$$\begin{aligned}&\left(\frac{ad}{bc}\right)^4 + \left(\frac{bc}{ad}\right)^4 + 28\left[\left(\frac{ac}{bd}\right)^2 + \left(\frac{bd}{ac}\right)^2\right] + 70\\ &= 8\left[\left(\frac{ac}{bd}\right)^3 + \left(\frac{bd}{ac}\right)^3\right] + 56\left[\left(\frac{ac}{bd}\right) + \left(\frac{bd}{ac}\right)\right].\end{aligned} \tag{3.14}$$

Proof. From the equation (3.7) and lemma (2.8), we obtain(3.14). □

Theorem 3.6. *If* $\frac{a}{b} := \frac{R(-q)}{R(q)}$ *and* $\frac{c}{d} := \frac{R(-q^9)}{R(q^9)}$ *then*

$$\begin{aligned}&\left(\frac{ad}{bc}\right)^8 + \left(\frac{bc}{ad}\right)^6 + 8\left[\left(\frac{ad}{bc}\right)^5 + \left(\frac{bc}{ad}\right)^5\right] + 10\left[\left(\frac{ad}{bc}\right)^4 + \left(\frac{bc}{ad}\right)^4\right]\\ &+ 16\left[\left(\frac{ac}{bd}\right)^4 + \left(\frac{bd}{ac}\right)^4\right] + 15\left[\left(\frac{ad}{bc}\right)^2 + \left(\frac{bc}{ad}\right)^2\right] + 48\left[\left(\frac{ad}{bc}\right) + \left(\frac{bc}{ad}\right)\right]\\ &= 24\left[\left(\frac{ad}{bc}\right)^3 + \left(\frac{bc}{ad}\right)^3\right] + 16\left(\frac{bd}{ac}\right)^3\left[\frac{b^2}{a^2} + \frac{d^2}{c^2}\right] + 16\left(\frac{ac}{bd}\right)^3\left[\frac{a^2}{b^2} + \frac{c^2}{d^2}\right] + 84.\end{aligned} \tag{3.15}$$

Proof. From the equation (3.7) and lemma (2.9), we obtain (3.15). □

Remark 3.7. Similarly, we obtain the Modular relations between $\frac{a}{b} := \frac{R(-q)}{R(q)}$ and $\frac{c}{d} := \frac{R(-q^n)}{R(q^n)}$, for $n = 8, 11, 13, 15, 17, 19, 23$ and 25.

4 Explicit values of Ratio's of Ramanujan Quantities of $R(q)$

In this section, we obtain explicit values of Ratio's of Ramanujan Quantities of $R(q)$.

Theorem 4.1. *We have*

$$\frac{R(-e^{-n\pi})}{R(e^{-n\pi})} = \frac{\psi(-e^{-n\pi})}{\psi(e^{-n\pi})} \tag{4.1}$$

Proof. Put $q = e^{-n\pi}$ in equation (3.5), we get (4.1) □

In his first notebook, Ramanujan's second notebook [7] recorded many elementary values of $\varphi(q)$. In particularly, he recorded $\varphi(e^{-n\pi})$ and $\varphi(-e^{-n\pi})$ for $n = 1, 2, 4$ and etc. Noting from [[3], Entry 1, p.325], we have

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)} \quad and \quad \varphi(-e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)}2^{-1/4}. \tag{4.2}$$

Employing the equations (4.1), (4.2) and (2.12), we obtain

$$\frac{R(-e^{-\pi})}{R(e^{-\pi})} = 2^{-1/8}. \tag{4.3}$$

Using the above value in Theorem (3.1), we get

$$\frac{R(-e^{-2\pi})}{R(e^{-2\pi})} = 2^{5/16}[\sqrt{2} - 1]^{1/4}. \tag{4.4}$$

References

[1] B. C. Berndt, *Ramanujan's Notebooks*, Part III, Springer-Verlag, New York (1991).

[2] B. C. Berndt, *Ramanujan's Notebooks*, Part IV, Springer-Verlag, New York (1994).

[3] B. C. Berndt, *Ramanujan's Notebooks*, Part V, Springer-Verlag, New York (1998).

[4] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook*, Part I, Springer, New York (2005).

[5] Nikos Bagis and M. L. Glasser, Jacobian Elliptic Functions, Continued Fractions and Ramanujan Quantities, *arXiv:1001.2660v1 [math.GM]*, (15 Jan 2010).

[6] Nikos Bagis, Generalizations of Ramanujan Continued fractions, *arXiv:1107.2393v2 [math.GM]*, (7 Aug 2012).

[7] S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay (1957).

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Received: November 6, 2015.

Accepted: January 12, 2016.