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3-Difference cordial labeling of some union of graphs

R. Ponraj and M.Maria Adaickalam

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Abstract. Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \dots, k\}$ be a function where k is an integer, $2 \leq k \leq |V(G)|$. For each edge uv, assign the label $|f(u) - f(v)|$. f is called k-difference cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labelled with x, $x \in \{1, 2, \dots k\}$, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labelled with 1 and not labelled with 1. A graph with a k-difference cordial labeling is called a k-difference cordial graph. In this paper we investigate the 3-difference cordial labeling behavior some union of graphs.

1 Introduction

Graphs considered here are finite and simple. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. For a graph G, the splitting graph of G, $spl(G)$, is obtained from G by adding for each vertex v of G a new vertex v' so that v' is adjacent to every vertex that is adjacent to v. Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. If $x = uv$ is an edge of G and w is not a vertex of of G, then x is subdivided when it is replaced by the lines uw and wv. If every edges of G is subdivided, the resulting graph is the subdivision graph $S(G)$. The graph $P_n + K_1$ is called a fan F_n. The graph $P_n + 2K_1$ is called a double fan DF_n. Cahit [1], introduced the concept of cordial labeling of graphs. Recently Ponraj et al. [4], introduced k-difference cordial labeling of graphs and 3-difference cordial labeling of wheel, helms, flower graph, sunflower graph, lotus inside a circle, closed helm, double wheel, $K_{1,n} \odot K_2$, $P_n \odot 3K_1$, mC_4, $spl(K_{1,n})$, $DS(B_{n,n})$, $C_n \odot K_2$, and some more graphs have been studied in [5, 6]. In this paper we investigate the 3-difference cordial labeling behavior of some union of graphs. Terms are not defined here follows from Harary [3].

2 k-Difference cordial labeling

Definition 2.1. Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \dots, k\}$ be a map where k is an integer, $2 \leq k \leq |V(G)|$. For each edge uv, assign the label $|f(u) - f(v)|$. f is called k-difference cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labelled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labelled with 1 and not labelled with 1. A graph which admits a k-difference cordial labeling is called a k-difference cordial graph.

Theorem 2.2. If G is (p, q) 3-difference cordial graph with $p \equiv 0 \pmod 2$ and $q \equiv 0 \pmod 3$, then $G \cup G$ also 3-difference cordial.

Proof. Let f be a 3-difference cordial labeling of G. Then $v_f(1) = v_f(2) = v_f(3) = \frac{p}{3}$ and $e_f(0) = e_f(1) = \frac{q}{2}$. Let h be a map from $V(G \cup G) \to \{1, 2, 3\}$ defined by $h(u) = f(u)$ for all $u \in V(G \cup G)$. Clearly $v_h(1) = v_h(2) = v_h(3) = \frac{2p}{3}$ and $e_h(0) = e_h(1) = q$. Therefore h is a 3-difference cordial labeling of $G \cup G$. □

Notation 1. We denote the vertex set of the star $K_{1,n}$ is $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and edge set of $K_{1,n}$ is $V(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$.

First we investigate the 3-difference cordial labeling behavior of union of graphs with the star.

Theorem 2.3. $P_n \cup K_{1,n}$ is 3-difference cordial.

Proof. Let P_n be the $v_1 v_2 \dots v_n$. Note that $P_n \cup K_{1,n}$ has 2n+1 vertices and 2n-1 edges.
Case 1. $n \equiv 0 \pmod 3$.

Assign the labels 1,3,2 to the first three vertices of the path v_1, v_2, v_3 respectively. Then we assign the labels $1, 3, 2$ to the next three vertices of the path v_4, v_5, v_6 respectively. Continuing in this way, we assign the next three vertices and so on. Next we move to the graph $K_{1,n}$. First we assign the label 1 to the vertices u_i $(1 \leq i \leq \frac{n}{3})$. Next we assign the label 2 to the vertices $u_{\frac{n}{3}+i}$ $(1 \leq i \leq \frac{n}{3})$. Then we assign the label 3 to the vertices $u_{\frac{2n}{3}+i}$ $(1 \leq i \leq \frac{n}{3})$. Finally we assign the label 1 to the central vertex u.
Case 2. $n \equiv 1 \pmod 3$.

Assign the labels u_i, v_i, u $(1 \leq i \leq n-1)$ as in case 1. Then assign the labels 1,2 to the vertices v_n and u_n respectively.
Case 3. $n \equiv 2 \pmod 3$.

As in case 2, assign the labels to the vertices u_i, v_i, u $(1 \leq i \leq n-1)$. Finally assign the labels 3,3 to the vertices u_n and v_n respectively. The vertex and edge condition are given in table 1 and 2 respectively. □

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0, 2 \pmod 3$	n	$n-1$
$n \equiv 1 \pmod 3$	$n-1$	n

Table 1.

values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$
$n \equiv 0 \pmod 3$	$\frac{2n+3}{3}$	$\frac{2n}{3}$	$\frac{2n}{3}$
$n \equiv 1 \pmod 3$	$\frac{2n+1}{3}$	$\frac{2n+1}{3}$	$\frac{2n+1}{3}$
$n \equiv 2 \pmod 3$	$\frac{2n+2}{3}$	$\frac{2n-1}{3}$	$\frac{2n+2}{3}$

Table 2.

Next investigation is union of star with $K_{2,n}$.

Theorem 2.4. $K_{1,n} \cup K_{2,n}$ is 3-difference cordial.

Proof. Let $V(K_{2,n}) = \{v, w, v_i : 1 \leq i \leq n\}$ and $E(K_{2,n}) = \{vv_i, wv_i : 1 \leq i \leq n\}$.
Case 1. $n \equiv 0 \pmod 3$.
Subcase 1a. $n \equiv 0 \pmod 6$.

First we consider the graph $K_{1,n}$. Assign the labels 1,1,2 to the first three vertices u_1, u_2, u_3 respectively. Then assign the labels 2,2,1 to the next three vertices u_4, u_5, u_6 respectively. Next we assign the labels 1,1,2 to the next three vertices u_7, u_8, u_9 respectively and and assign the labels 2,2,1 to the next three vertices u_{10}, u_{11}, u_{12} respectively. Continuing this way, we assign the next three vertices and so on. Clearly in this process, the last vertex u_n received the label 2 or 1. Finally we assign the 1 to the vertex u. Now we move to the graph $K_{2,n}$. Assign the labels 3,3,2,3,3,1 to the first six vertices $v_1, v_2, v_3, v_4, v_5, v_6$ respectively. Then we assign the labels 3,3,2,3,3,1 to the next six vertices $v_7, v_8, v_9, v_{10}, v_{11}, v_{12}$ respectively. Proceeding like this, we assign the labels to the next six vertices and so on. Clearly the last vertex v_n received the label 1. Finally we assign the labels 2,3 to the vertices v and w respectively.
Subcase 1b. $n \equiv 3 \pmod 6$.

Assign the label to the vertices u, v, w, u_i $1 \leq i \leq n-3$ and v_i $1 \leq i \leq n-3$ as in subcase 1a. Finally assign the labels 1,1,2 to the vertices u_{n-2}, u_{n-1}, u_n respectively and 3,3,2 to the vertices v_{n-2}, v_{n-1}, v_n respectively.
Case 2. $n \equiv 1 \pmod 3$.

Subcase 2a. $n \equiv 4 \pmod 6$.

Fix the labels 1,1,2,2 to the vertices u_1, u_2, u_3, u_4 respectively. Then we assign the labels 2,2,1,1,1,2 to the next six vertices $u_5, u_6, \ldots, u_{10}$ respectively. Now assign the labels 2,2,1,1,1,2 to the next six vertices $u_{11}, u_{12}, \ldots, u_{15}$ respectively. Continuing in this way, we assign the next six vertices and so on. In this process, the last vertex u_n received the label 2. Next we assign the label 1 to the vertex u. Now our attention move to the vertices of the graph $K_{2,n}$. Fix the label 1 to the vertex v_1. Then assign the labels 3,3,2,3,3,1 to the next six vertices $v_2, v_3, \ldots, v_7$ respectively. Proceeding like this, we asign the next six vertices and so on. Clearly, in this process the vertex v_{n-3} received the label 1. Next we assign the labels 3,3,2 respectively to the vertices v_{n-2}, v_{n-1}, v_n. Finally we assign the labels 2,3 to the vertices v and w respectively.

Subcase 2b. $n \equiv 1 \pmod 6$.

Assign the label to the vertices u, v, w, u_i $1 \leq i \leq n-3$ and v_i $1 \leq i \leq n-3$ as in subcase 2a. Finally assign the labels 2,2,1 to the vertices u_{n-2}, u_{n-1}, u_n respectively and 3,3,2 to the vertices v_{n-2}, v_{n-1}, v_n respectively.

Case 3. $n \equiv 2 \pmod 3$.

Subcase 3a. $n \equiv 2 \pmod 6$.

Fix the labels 1,2 to the vertices u_1, u_2 respectively. Then we assign the labels 1,1,2,2,2,1 to the next six vertices $u_3, u_4, \ldots, u_8$ respectively. Now we assign the labels 1,1,2,2,2,1 to the next six vertices $u_9, u_{10}, \ldots, u_{14}$ respectively. Continuing this process until we reach the last vertex u_n. In this pattern, the last vertex u_n labeled by the integer 1. Then we assign the label 1 to the vertex u. Next we move to the graph $K_{2,n}$. Fix the labels 1,3 to the vertices v_1, v_2 respectively. Then we assign the labels 3,3,2,3,3,1 to the next six vertices $v_3, v_4, \ldots v_8$ respectively. Next we assign the labels 3,3,2,3,3,1 to the next six vertices $v_9, v_{10}, \ldots v_{14}$ respectively. Continuing in this way, we assign the next six vertices and so on. Finally we assign the labels 2,3 to the vertices v,w respectively. The vertex and edge condition are given in table 3 and 4. □

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0, 2, 4 \pmod 6$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 6$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$
$n \equiv 3, 5 \pmod 6$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$

Table 3.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$
$n \equiv 0 \pmod 3$	$\frac{2n+3}{3}$	$\frac{2n+3}{3}$	$\frac{2n+3}{3}$
$n \equiv 1 \pmod 3$	$\frac{2n+4}{3}$	$\frac{2n+4}{3}$	$\frac{2n+1}{3}$
$n \equiv 2 \pmod 3$	$\frac{2n+5}{3}$	$\frac{2n+2}{3}$	$\frac{2n+2}{3}$

Table 4.

We now investigate union of star with subdivision of star.

Theorem 2.5. $K_{1,n} \cup S(K_{1,n})$ is 3-difference cordial.

Proof. Let $V(S(K_{1,n})) = \{v, v_i, w_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{vv_i, v_iw_i : 1 \leq i \leq n\}$.

Case 1. n is even.

Assign 1 to the vertices $u, u_1, u_2, \ldots u_{\frac{n}{2}}$. Then assign the label 2 to the vertices $u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}$, $\ldots u_n$. Next we move to the graph $S(K_{1,n})$. Assign 2 to the vertex v. Then assign the label 2 to the vertices $v_1, v_2, \ldots v_{\frac{n}{2}}$ and 3 to the vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots v_n$ and $w_{\frac{n}{2}+1}, w_{\frac{n}{2}+2}, \ldots w_n$. Finally assign the label 1 to the vertices $w_1, w_2, \ldots w_{\frac{n}{2}}$.

Case 2. n is odd.

Assign the label 1 to the vertex u. Then assign the integer 3 to the vertex $u_1, u_2, \ldots u_{\frac{n+1}{2}}$. Then assign the label 2 to the remaining vertices of the star $K_{1,n}$. Then we move to the graph $S(K_{1,n})$. Now we assign the label 2 to the vertex v. Then we assign the label 2 to the vertices $v_1, v_2, \ldots v_{\frac{n+1}{2}}$ and 1 to the vertices $v_{\frac{n+1}{2}+1}, v_{\frac{n+1}{2}+2}, \ldots v_n$. Then assign the label 1 to the vertices

$w_1, w_2, \dots w_{\frac{n+1}{2}}$ and 3 to the vertices $w_{\frac{n+1}{2}+1}, w_{\frac{n+1}{2}+2}, \dots w_n$. Then f is a 3-difference cordial labeling follows from $v_f(1) = v_f(2) = n + 1$ and $v_f(3) = n$ and the table 5. □

values of n	$e_f(0)$	$e_f(1)$
n is odd	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$
n is even	$\frac{3n}{2}$	$\frac{3n}{2}$

Table 5.

Next is union of two stars.

Theorem 2.6. If $n \equiv 0, 1 \pmod 3$, then $K_{1,n} \cup K_{1,n}$ is 3-difference cordial.

Proof. Let u, v be the central vertex of the first and second star respectively. Let u_i $(1 \le i \le n)$ and v_i $(1 \le i \le n)$ be the pendent vertices of first and second copies of the star $K_{1,n}$.

Case 1. $n \equiv 0 \pmod 3$.

Assign the label 1 to the vertices u_i, v_i $(1 \le i \le \frac{n}{3})$ and assign the label 2 to the vertices $u_{\frac{n}{3}+i}$, $v_{\frac{n}{3}+i}$ $(1 \le i \le \frac{n}{3})$. Next we assign the label 3 to the vertices $u_{\frac{2n}{3}+i}$, $v_{\frac{2n}{3}+i}$ $(1 \le i \le \frac{n}{3})$. Finally we assign the labels 1,2 to the vertices u and v respectively.

Case 2. $n \equiv 1 \pmod 3$.

Assign the label 1 to the vertices u_i, $(1 \le i \le \frac{n+2}{3})$. Then assign the label 2 to the vertex $u_{\frac{n+2}{3}+i}$, $(1 \le i \le \frac{n-1}{3})$. Next we assign the label 3 to the vertices $u_{\frac{2n+1}{3}+i}$ $(1 \le i \le \frac{n-1}{3})$. Next we move to the next copy of the star $K_{1,n}$. Assign the label 3 to the vertices v_i, $(1 \le i \le \frac{n+2}{3})$. Then assign the label 2 to the vertices $v_{\frac{n+2}{3}+i}$, $(1 \le i \le \frac{n-1}{3})$. Next we assign the label 1 to the vertex $v_{\frac{2n+1}{3}+i}$ $(1 \le i \le \frac{n-1}{3})$. Finally we assign the labels 1,2 to the vertices u and v respectively. The edge condition is $e_f(0) = e_f(1) = 1$ and the vertex condition is given in table 6. □

values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$
$n \equiv 0 \pmod 3$	$\frac{2n+3}{3}$	$\frac{2n+3}{3}$	$\frac{2n}{3}$
$n \equiv 1 \pmod 3$	$\frac{2n+4}{3}$	$\frac{2n+1}{3}$	$\frac{2n+1}{3}$

Table 6.

Next investigation is about union of graphs with splitting graph of the star.

Theorem 2.7. $spl(K_{1,n}) \cup K_{1,n}$ is 3-difference cordial.

Proof. Let $V(spl(K_{1,n})) = \{v, w, v_i, w_i : 1 \le i \le n\}$ and $E(spl(K_{1,n}) = \{vv_i, vw_i, ww_i : 1 \le i \le n\}$. Note that $spl(K_{1,n}) \cup K_{1,n}$ has $3n + 3$ vertices and $4n$ edges. Assign the labels $1, 2, 3$ to the vertices u, v, w respectively. We now assign the label 3 to u_i $(1 \le i \le n)$, assign the label 1 to the vertices v_i $(1 \le i \le n)$. Finally assign the label 2 to the vertices w_i $(1 \le i \le n)$. It is easy to verify that $e_f(0) = e_f(1) = 2n$ and $v_f(1) = v_f(2) = v_f(3) = n + 1$. Hence f is a 3-difference cordial labeling. □

Now our attention is move to union of graphs with splitting graph of the star.

Theorem 2.8. $spl(K_{1,n}) \cup P_n$ is 3-difference cordial.

Proof. Let $u_1u_2 \dots u_n$ be the path. Let $V(spl(K_{1,n})) = \{v, w, v_i, w_i : 1 \le i \le n\}$ and $E(spl(K_{1,n})) = \{vv_i, vw_i, ww_i : 1 \le i \le n\}$. Clearly $spl(K_{1,n}) \cup P_n$ has $3n + 2$ vertices and 4n-1 edges. Define a function $f : V(G) \to \{1, 2, 3\}$ by $f(v) = 2$, $f(w) = 1$,

$$\begin{aligned} f(u_i) &= 1, \quad 1 \le i \le n \\ f(v_i) &= 3, \quad 1 \le i \le n \\ f(w_i) &= 2, \quad 1 \le i \le n \end{aligned}$$

Clearly $e_f(0) = 2n$, $e_f(1) = 2n - 1$ and $v_f(1) = v_f(2) = n + 1$ and $v_f(3) = n$. Hence f is 3-difference cordial labeling. □

Theorem 2.9. $K_{3,n} \cup spl(K_{1,n})$ is 3-difference cordial.

Proof. Let $V(K_{3,n}) = \{u, v, w, u_i : 1 \le i \le n\}$ and $E(K_{3,n}) = \{uu_i, vu_i, wu_i : 1 \le i \le n\}$. Let $V(spl(K_{1,n})) = \{x, y, x_i, y_i : 1 \le i \le n\}$ and $E(spl(K_{1,n})) = \{xx_i, xy_i, yy_i : 1 \le i \le n\}$. Clearly $K_{3,n} \cup spl(K_{1,n})$ has 3n+5 vertices and 6n edges. Define a map $f : V(G) \to \{1, 2, 3\}$ by $f(u) = 1$, $f(v) = 2$, $f(w) = 3$, $f(x) = 2$, $f(y) = 1$,

$$\begin{aligned} f(u_i) &= 1, \quad 1 \le i \le n \\ f(x_i) &= 3, \quad 1 \le i \le n \\ f(y_i) &= 2, \quad 1 \le i \le n \end{aligned}$$

Clearly $e_f(0) = e_f(1) = 3n$ and $v_f(1) = v_f(2) = n + 2$ and $v_f(3) = n + 1$. Hence f is 3-difference cordial labeling. □

Theorem 2.10. $DF_n \cup spl(K_{1,n})$ is 3-difference cordial.

Proof. Let $V(DF_n) = \{u, v, u_i : 1 \le i \le n\}$ and $E(DF_n) = \{uu_i, vu_i, u_iu_{i+1} : 1 \le i \le n\}$, $V(spl(K_{1,n})) = \{x, y, x_i, y_i : 1 \le i \le n\}$ and $E(spl(K_{1,n})) = \{xx_i, xy_i, yy_i : 1 \le i \le n\}$. Assign the label 1 to the vertex u. Then assign the label 2 to all the vertices v_i $(1 \le i \le n)$ and assign the label 3 to the vertex v. Now we move move to the graph $spl(K_{1,n})$. First we assign the label 1 to the vertex x. Then assign the label 3 to all the vertices x_i $(1 \le i \le n)$ and assign the label 1 to all the vertices y_i $(1 \le i \le n)$. Finally assign the label 2 to the vertex y. Clearly $v_f(1) = n + 2$ and $v_f(3) = n + 1$, $e_f(0) = 3n - 1$ and $e_f(1) = 3n$. Hence f is a 3-difference cordial labeling. □

Example 2.11. A 3-difference cordial labeling of $DF_4 \cup spl(K_{1,4})$ is displayed in the below figure.

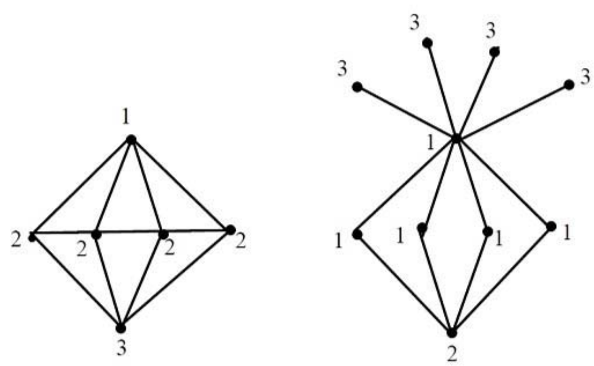

Figure 1.

We now investigate the 3-difference cordial labeling behavior of union of graphs with subdivided star.

Theorem 2.12. $S(K_{1,n}) \cup S(B_{n,n})$ is 3-difference cordial.

Proof. Let $V(S(K_{1,n})) = \{u, u_i, v_i : 1 \le i \le n\}$ and $E(S(K_{1,n})) = \{uu_i, v_iu_i : 1 \le i \le n\}$. Let $V(S(B_{n,n})) = \{w, x, y, w_i, x_i, y_i, z_i : 1 \le i \le n\}$ and $E(S(B_{n,n})) = \{wx, xy, ww_i, w_ix_i, yy_i, y_iz_i : 1 \le i \le n\}$. Assign the label 1 to the vertex u. Then assign the label 2 to the vertices u_i $(1 \leq i \leq n)$ and assign the label 3 to the vertices v_i $(1 \leq i \leq n)$. Next we move to the graph $S(B_{n,n})$. Now we assign the labels 1,2,3 to the vertices w,x,y respectively. Then assign the label 3 to the vertices w_i $(1 \leq i \leq n)$ and assign the label 1 to the vertices x_i $(1 \leq i \leq n)$ and the vertices z_i $(1 \leq i \leq n)$. Finally we assign the label 2 to the vertices y_i $(1 \leq i \leq n)$. Clearly $e_f(0) = e_f(1) = 3n + 1$ and $v_f(2) = v_f(3) = 2n + 1$, $v_f(1) = 2n + 2$. Hence f is 3-difference cordial labeling. □

Theorem 2.13. $K_{2,n} \cup S(K_{1,n})$ is 3-difference cordial.

Proof. Let $V(S(K_{1,n})) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{uu_i, u_iv_i : 1 \leq i \leq n\}$. Let $V(K_{2,n}) = \{w, z, w_i : 1 \leq i \leq n\}$ and $E(K_{2,n}) = \{ww_i, zw_i : 1 \leq i \leq n\}$. Clearly $K_{2,n} \cup S(K_{1,n})$ has 3n+3 vertices and 4n edges. Assign the label 1 to the vertex u. Then we assign the label 3 to the vertices u_i $(1 \leq i \leq n)$ and assign the label 2 to the vertices v_i $(1 \leq i \leq n)$. We now move to the graph $K_{2,n}$. Assign the labels 2,3 to the vertices w and z respectively. Finally we assign the label 1 to the vertices w_i $(1 \leq i \leq n)$. The edge and vertex condition are $e_f(0) = e_f(1) = 2n$ and $v_f(1) = v_f(2) = v_f(3) = n + 1$. Hence f is a 3-difference cordial labeling. □

Theorem 2.14. $F_n \cup S(K_{1,n})$ is 3-difference cordial.

Proof. Let $V(F_n) = \{u, u_i : 1 \leq i \leq n\}$ and $E(F_n) = \{uu_i, u_iu_{i+1} : 1 \leq i \leq n\}$. Let $V(S(K_{1,n})) = \{v, v_i, w_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{vv_i, v_iw_i : 1 \leq i \leq n\}$. Assign the label 1 to the vertex u. Then assign the label 2 to the vertices u_i $(1 \leq i \leq n)$. Next we move to the graph $S(K_{1,n})$. Now we assign the label 2 to the vertex v. Then assign the label 3 to all the vertices v_i $(1 \leq i \leq n)$. Finally assign the label 3 to all the vertices w_i $(1 \leq i \leq n)$. The edge and vertex condition of this labeling f is $e_f(0) = 2n - 1$ and $e_f(1) = 2$, $v_f(1) = v_f(2) = n + 1$ and $v_f(3) = n$. Hence f is a 3-difference cordial labeling of $F_n \cup S(K_{1,n})$. □

Theorem 2.15. $W_n \cup S(K_{1,n})$ is 3-difference cordial.

Proof. Let $V(W_n) = \{u, u_i, : 1 \leq i \leq n\}$ and $E(W_n) = \{uu_i, u_iu_{i+1}, u_nu_1 : 1 \leq i \leq n\}$. Let $V(S(K_{1,n})) = \{v, v_i, w_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{vv_i, v_iw_i : 1 \leq i \leq n\}$. Clearly $W_n \cup S(K_{1,n})$ has 3n+2 vertices and 4n edges. First we assign the label 2 to the vertex u. Then we assign the label 1 to all the vertices of u_i $(1 \leq i \leq n)$. Next we move to the graph $S(K_{1,n})$. We assign the label 1 to the vertex v. Next we assign the label 3 to the vertices v_i $(1 \leq i \leq n)$. Finally we assign the label 2 to all the vertices w_i $(1 \leq i \leq n)$. f is a 3-difference cordial labeling follows from $e_f(0) = e_f(1) = 2n$ and $v_f(1) = v_f(2) = n + 1$, $v_f(3) = n$. □

Next is union of graphs with bistar.

Theorem 2.16. $B_{n,n} \cup S(B_{n,n})$ is 3-difference cordial.

Proof. Let $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \leq i \leq n\}$. Let $V(S(B_{n,n})) = \{w, x, y, w_i, x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E(S(B_{n,n})) = \{wx, xy, ww_i, w_ix_i, yy_i, y_iz_i : 1 \leq i \leq n\}$. Note that $B_{n,n} \cup S(B_{n,n})$has 6n+5 vertices and 6n+3 edges. Assign the label 1,2 to the verticesu,v respectively. Then assign the label 2 to the vertices u_i $(1 \leq i \leq n)$ and assign the label 3 to the vertices v_i $(1 \leq i \leq n)$. Now we move to the next graph $S(B_{n,n})$. Assign the labels 1,3,2 to the vertices w,x,y respectively. Then assign the label 1 to the vertices w_i $(1 \leq i \leq n)$. Next we assign the label 3 to the vertices x_i $(1 \leq i \leq n)$ and assign the label 2 to the vertices y_i $(1 \leq i \leq n)$. Finally assign the label 1 to the vertices z_i $(1 \leq i \leq n)$. Obviously $e_f(0) = 3n + 1$ and $e_f(1) = 3n + 2$, $v_f(1) = v_f(2) = 2n + 2$ and $v_f(3) = 2n + 1$. Hence f is 3-difference cordial labeling. □

Theorem 2.17. $K_{2,n} \cup B_{n,n}$ is 3-difference cordial.

Proof. Let $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uv, vv_i, uu_i : 1 \leq i \leq n\}$. Let $V(K_{2,n}) = \{w, x, w_i : 1 \leq i \leq n\}$ and $E(K_{2,n}) = \{ww_i, xw_i : 1 \leq i \leq n\}$. Clearly $K_{2,n} \cup B_{n,n}$ has 3n+4 vertices and 4n+1 edges. Assign the labels 1,2 to the vertices u and v respectively. Then assign the label 1 to the vertices u_i $(1 \leq i \leq n)$. Next we assign the label 3 to the vertices v_i $(1 \leq i \leq n)$. Then we move to the graph $K_{2,n}$. We assign the labels 2,3 to the vertices w and x respectively. Next we assign the label 2 to the vertices w_i $(1 \leq i \leq n)$. Since $e_f(0) = 2n$, $e_f(1) = 2n + 1$, $v_f(1) = v_f(3) = n + 1$ and $v_f(2) = n + 2$, f is a 3-difference cordial labeling. □

The final investigation is about $P_n \cup P_n$, $C_n \odot K_1 \cup P_n \odot K_1$ and $F_n \cup F_n$ where $F_n = P_n + K_1$ and it is called fan.

Theorem 2.18. $(C_n \odot K_1) \cup (P_n \odot K_1)$ is 3-difference cordial.

Proof. Let $n = 3t + r$ where $0 \leq r \leq 3$. Assign the label 3 to all the cycle vertices and assign the label 2 to all the pendent vertices of $C_n \odot K_1$. Next we move to the graph $P_n \odot K_1$. Assign the labels 1,2,3 to the first three vertices of the P_n. Next we assign the labels 1,2,3 to the next three vertices of the path. Continuing in this way, assign the next three vertices and so on. If $r = 0$, we have labeled all the vertices of the path. If $r = 1$, assign the label 1 to the next non labeled vertex of the path. If $r = 2$, then we assign the labels 1,2 to the non labeled vertices of the path respectively. Finally assign the label 1 to all the pendent vertices of $P_n \odot K_1$. Clearly $e_f(0) = 2n - 1$ and $e_f(1) = 2n$ and the vertex condition is shown in the table 7.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$
$n \equiv 0 \pmod 3$	$\frac{4n}{3}$	$\frac{4n}{3}$	$\frac{4n}{3}$
$n \equiv 1 \pmod 3$	$\frac{4n+2}{3}$	$\frac{4n-1}{3}$	$\frac{4n-1}{3}$
$n \equiv 2 \pmod 3$	$\frac{4n+1}{3}$	$\frac{4n+1}{3}$	$\frac{4n-2}{3}$

Table 7.

□

Theorem 2.19. $F_n \cup F_n$ is 3-difference cordial.

Proof. Let $V(F_n \cup F_n) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(F_n \cup F_n) = \{uu_i, u_iu_{i+1}, vv_i, v_iv_{i+1} : 1 \leq i \leq n\}$. Note that $F_n \cup F_n$ has 2n+2 vertices and 4n-2 edges.
Case 1. $n \equiv 0 \pmod 3$.
Subcase 1a. $n \equiv 0 \pmod 6$.
Assign the labels 1,3,2,1,3,2 to the first six vertices $u_1, u_2, \ldots u_6$ respectively. Then assign the labels 1,3,2,1,3,2 to the next six vertices u_7, u_8, u_{12} respectively. Continuing this way, we assign the next six vertices and so on. In this process the last vertex received the label 2. Assign the label 1 to the vertex u. Next we move to the vertices v and v_i. Assign the labels 1,3,2,2,3,1 to the first six vertices $v_1, v_2, \ldots v_6$ respectively. Then we assign the labels 1,3,2,2,3,1 to the next six vertices $v_7, v_8, \ldots v_{12}$ respectively. Continuing this pattern, we reach the last vertex v_n. Clearly in this pattern the last vertex v_n received the label 3. Finally assign the label 2 to the vertex v.
Subcase 1b. $n \equiv 3 \pmod 6$.
As in subcase 1a, assign the label to the vertices u, v, u_i $1 \leq i \leq n-3$ and v_i $1 \leq i \leq n-3$. Finally assign the labels 1,3,2 respectively to the vertices u_{n-2}, u_{n-1}, u_n and 1,3,2 to the vertices v_{n-2}, v_{n-1}, v_n respectively.
Case 2. $n \equiv 1 \pmod 3$.
Subcase 2a. $n \equiv 1 \pmod 6$.
Fix the label 3 to the vertex u_1. Then assign the labels 1,2,3 to the next three vertices u_2, u_3, u_4 respectively. Now we assign the labels 1,2,3 to the next three vertices v_5, v_6, v_7 respectively. Continuing in this way, we assign the next three vertices and so on. In this process the last vertex v_n labeled by the integer 3. Then we assign the label 1 to the vertex u. Now we move to the second copy of F_n. Fix the label 2 to the vertex v_1. Assign the labels 3,1,2,2,3,1 to the next six vertices $v_2, v_3, \ldots v_7$ respectively. Then assign the labels 3,1,2,2,3,1 to the next six vertices $v_7, v_8, \ldots v_{13}$ respectively. Proceeding like this, we assign the label to the next six vertices and so on. Clearly in this pattern the last vertex v_n labeled received the label 1. Finally we assign the label 2 to the vertex v.
Subcase 2b. $n \equiv 4 \pmod 6$.
As in subcase 2a, assign the label to the vertices u, v, u_i $1 \leq i \leq n-3$ and v_i $1 \leq i \leq n-3$. Finally assign the labels 1,2,3 respectively to the vertices u_{n-2}, u_{n-1}, u_n and 3,1,2 to the vertices v_{n-2}, v_{n-1}, v_n respectively.
Case 3. $n \equiv 2 \pmod 3$.
Subcase 3a. $n \equiv 2 \pmod 6$.

Fix the labels 2,1 to the vertices u_1 and u_2 respectively. Then we assign the labels 1,3,2,1,3,2 to the six vertices $v_4, v_5, \dots v_8$ respectively. Now we assign the labels 1,3,2,1,3,2 to the next six vertices $v_9, v_{10}, \dots v_{14}$ respectively. Continuing this way, we assign the next six vertices and so on. Clearly in this process the last vertex u_n labeled by the integer 2. Assign the label 1 to the vertex. Now we move to the second copy of F_n. Fix the labels 3,2 to the vertices v_1, v_2 respectively. Assign the labels 1,3,2 to the next three vertices v_3, v_4, v_5 respectively. Then assign the labels 1,3,2 to the next three vertices v_6, v_7, v_8 respectively. Proceeding like this, we assign the label to the next three vertices and so on. In this pattern 2 is the label of the last vertex v_n. Finally assign the the label 2 to the vertex v.

Subcase 3b. $n \equiv 5 \pmod 6$.

As in subcase 3a, assign the label to the vertices u, v, u_i $1 \leq i \leq n-3$ and v_i $1 \leq i \leq n-3$. Finally assign the labels 1,3,2 respectively to the vertices u_{n-2}, u_{n-1}, u_n and 1,3,2 to the vertices v_{n-2}, v_{n-1}, v_n respectively.

Clearly all these cases the edge condition is $e_f(0) = e_f(1) = 2n-1$. The vertex condition is given in table 8. □

Values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$
$n \equiv 0 \pmod 3$	$\frac{2n+3}{3}$	$\frac{2n+3}{3}$	$\frac{2n}{3}$
$n \equiv 1 \pmod 3$	$\frac{2n+1}{3}$	$\frac{2n+4}{3}$	$\frac{2n+1}{3}$
$n \equiv 2 \pmod 3$	$\frac{2n+2}{3}$	$\frac{2n+2}{3}$	$\frac{2n+2}{3}$

Table 8.

Theorem 2.20. $P_n \cup P_n$ is 3-difference cordial.

Proof. Let $u_1 u_2 \dots u_n$ be the first copy of the path and $v_1 v_2 \dots v_n$ be the second copy of the path.

Case 1. $n \equiv 0 \pmod 3$.

Subcase 1b. $n \equiv 0 \pmod 6$.

Assign the labels 1,3,2 to the first three vertices of the path u_1, u_2, u_3 respectively. Then assign the labels 1,3,2 to the next three verties u_4, u_5, u_6 of the path P_n. Proceeding like this, we assign the next three vertices and so on. Note that in this process 2 is the label of the last vertex u_n. Now our attention turn to the second copy of the path. Assign the labels 1,3,2,2,3,1 to the first six vertices $v_1, v_2, \dots v_6$ respectively. Then assign the labels 1,3,2,2,3,1 to the next six vertices of the second copy of the path $v_7, v_8 \dots v_{12}$. Continuing this way until we reach the vertex v_n. It is obvious that the last vertex v_n is received the label 1.

Subcase 1b. $n \equiv 3 \pmod 6$.

Assign the label to the vertices u_i $1 \leq i \leq n-3$ and v_i $1 \leq i \leq n-3$ as in subcase 1a. Finally assign the labels 1,3,2 respectively to the vertices u_{n-2}, u_{n-1}, u_n and 1,3,2 to the vertices v_{n-2}, v_{n-1}, v_n respectively.

Case 2. $n \equiv 1 \pmod 3$.

Subcase 2a. $n \equiv 1 \pmod 6$.

Fix the label 1 to the vertex u_1. Next we aassign the labels 1,3,2 to the next three vertices u_2, u_3, u_4 of the path. Then assign the labels 1,3,2 to the next three vertices u_5, u_6, u_7 respectively. Proceeding like this we assign the next three vertices and so on. Now we move to the second copy of the path P_n. Fix the label 2 to the first vertex v_1. Then assign the labels 1,3,2,2,3,1 to the next six vertices $v_2, v_3, \dots v_7$ of the second path. Next we assign the labels 1,3,2,2,3,1 to the next six vertices $v_8, v_9, \dots v_{13}$ respectively. Continuing in this pattern until we reach the last vertex v_n. It is easy to verify that 1 is the label of the last vertex v_n.

Subcase 2b. $n \equiv 4 \pmod 6$.

Assign the label to the vertices u_i $1 \leq i \leq n-3$ and v_i $1 \leq i \leq n-3$ as in subcase 1a. Finally assign the labels 1,3,2 respectively to the vertices u_{n-2}, u_{n-1}, u_n and 1,3,2 to the vertices v_{n-2}, v_{n-1}, v_n respectively.

Case 3. $n \equiv 2 \pmod 3$.

Subcase 3a. $n \equiv 2 \pmod 6$.

Consider the first copy of the path P_n. Fix the labels 1,3 to the vertices u_1 and u_2 respectively. Then assign the labels 1,3,2 to the six vertices u_3, u_4, u_5 respectively. Next we assign the labels

1,3,2 to the next three vertices u_6, u_7, u_8 respectively. Proceeding like this, we assign the next three vertices and so on. It is obvious that, the last vertex u_n received the label 2. Next we turn to the second copy of P_n. Fix the labels 2,3 to the vertices v_1, v_2 respectively. Then assign the labels 2,3,1,1,3,2 to the next six vertices $v_3, v_4, \ldots v_8$ respectively. Next we assign the labels 2,3,1,1,3,2 to the next six vertices of the second copy of the path $v_9, v_{10}, \ldots v_{14}$ respectively. Continuing in this way until we reach the last vertex v_n. In this process the vertex v_n received the label 2.

Subcase 3b. $n \equiv 5 \pmod 6$.

Assign the label to the vertices u_i $1 \le i \le n-3$ and v_i $1 \le i \le n-3$ as in subcase 1a. Finally assign the labels 1,3,2 respectively to the vertices u_{n-2}, u_{n-1}, u_n and 2,3,1 to the vertices v_{n-2}, v_{n-1}, v_n respectively.

Hence f is a 3-difference cordial labeling follows from the edge condition $e_f(0) = e_f(1) = n-1$ and the vertex condition in table 9.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$
$n \equiv 0 \pmod 3$	$\frac{2n}{3}$	$\frac{2n}{3}$	$\frac{2n}{3}$
$n \equiv 1 \pmod 3$	$\frac{2n+1}{3}$	$\frac{2n+1}{3}$	$\frac{2n-2}{3}$
$n \equiv 2 \pmod 3$	$\frac{2n-1}{3}$	$\frac{2n-1}{3}$	$\frac{2n+2}{3}$

Table 9.

□

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Author information

R. Ponraj, Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, India.
E-mail: `ponrajmaths@gmail.com`

M.Maria Adaickalam, Department of Economics and Statistics, Government of Tamilnadu, Chennai-600 006, India.
E-mail: `mariaadaickalam@gmail.com`

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