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$\mathbb{O}/\mathbb{Z}_p$ OCTONION ALGEBRA AND ITS MATRIX REPRESENTATIONS

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Abstract. Since $\mathbb{O}$ is a non-associative algebra over $\mathbb{R}$, this real division algebra can not be algebraically isomorphic to any matrix algebras over the real number field $\mathbb{R}$. In this study using $\mathbb{H}$ with Cayley-Dickson process we obtain octonion algebra. Firstly, We investigate octonion algebra over $\mathbb{Z}_p$. Then, we use the left and right matrix representations of $\mathbb{H}$ to construct representation for octonion algebra. Furthermore, we get the matrix representations of $\mathbb{O}/\mathbb{Z}_p$ with the help of Cayley-Dickson process.

1 Introduction

Let us summarize the notations needed to understand octonionic algebra. There are only four normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ [2], [6]. The octonion algebra is a non-commutative, non-associative but alternative algebra which discovered in 1843 by John T. Graves. Octonions have many applications in quantum logic, special relativity, supersymmetry, etc. Due to the non-associativity, representing octonions by matrices seems impossible. Nevertheless, one can overcome these problems by introducing left (or right) octonionic operators and fixing the direction of action.
In this study we investigate matrix representations of octonion division algebra $\mathbb{O}/\mathbb{Z}_p$. Now, lets start with the quaternion algebra $\mathbb{H}$ to construct the octonion algebra $\mathbb{O}$. It is well known that any octonion **a** can be written by the Cayley-Dickson process as follows [7].

$$\mathbf{a} = a' + a''e;\ a', a'' \in \mathbb{H};\ \mathbb{H} = \{a = a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

Here, $\mathbb{H}$ is the real quaternion division algebra and $\{1, i, j, k\}$ is the base for this algebra. Where $i^2 = j^2 = k^2 = -1, ijk = -1$ and e is imaginary unit like i and $e^2 = -1$. For any elements **a, b** of $\mathbb{O}$ the addition and multiplication operations are as follows. If $\mathbf{a} = a' + a''e$ and $\mathbf{b} = b' + b''e \in \mathbb{O}$ then

$$\mathbf{a+b} = (a' + a''e) + (b' + b''e) = (a' + b') + (a'' + b'')e$$

and

$$\mathbf{ab} = (a' + a''e)(b' + b''e) = (a'b' - b''\overline{a''}) + (\overline{a'}b'' + b'a'')e$$

respectively. Where $\overline{a'}$ and $\overline{a''}$ denote the conjugates of the quaternions a' and a''. $\mathbb{O}$ is a non-associative but alternative division algebra with an eight-dimension over its center field $\mathbb{R}$ and we note that the cannonical basis of $\mathbb{O}$ is

$$e_0 = 1, e_1 = i, e_2 = j, e_3 = k, e_4 = e, e_5 = ie, e_6 = je, e_7 = ke.$$

2 $\mathbb{O}/\mathbb{Z}_p$ Alternative Octonion Algebra

In this section we use a prime number $p,\ p \neq 2$ and $e_0 = 1, {e_1}^2 = {e_2}^2 = ... = {e_7}^2 = -1$. Then, the set of $\mathbb{O}/\mathbb{Z}_p$ can be written as

$$\mathbb{O}/\mathbb{Z}_p = \{k = c_0e_0 + c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6 + c_7e_7 | c_i \in \mathbb{Z}_p,\ 0 \leq i \leq 7\}.$$

Firstly, we show that $\mathbb{O}/\mathbb{Z}_p$ is a vector space. Then, we define multiplication operation on $\mathbb{O}/\mathbb{Z}_p$ and we obtain that $\mathbb{O}/\mathbb{Z}_p$ forms an alternative division algebra with respect to the defined multiplication.

Theorem 2.1. *The $\mathbb{O}/\mathbb{Z}_p$ is a vector space over the field $\mathbb{Z}_p$.*

Proof. For all $k_1, k_2 \in \mathbb{O}/\mathbb{Z}_p$ and $c_i, d_i \in \mathbb{Z}_p,\ 0 \le i \le 7$,

$$k_1 = c_0e_0 + c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6 + c_7e_7,$$

$$k_2 = d_0e_0 + d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5 + d_6e_6 + d_7e_7.$$

The addition operation over the $\mathbb{O}/\mathbb{Z}_p$ is defined as

$$\oplus : \mathbb{O}/\mathbb{Z}_p \times \mathbb{O}/\mathbb{Z}_p \longrightarrow \mathbb{O}/\mathbb{Z}_p,\ k_1 \oplus k_2 = (c_i + d_i)e_i,\ 0 \le i \le 7.$$

So, we can easily see that $(\mathbb{O}/\mathbb{Z}_p, \oplus)$ is an abelian group. For all $d \in \mathbb{Z}_p$ and $k_1 \in \mathbb{O}/\mathbb{Z}_p$ the multiplication operation over the $\mathbb{O}/\mathbb{Z}_p$ is defined as

$$\odot : \mathbb{Z}_p \times \mathbb{O}/\mathbb{Z}_p \longrightarrow \mathbb{O}/\mathbb{Z}_p$$
$$d \odot k_1 = d \odot (c_0e_0 + c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6 + c_7e_7)$$
$$d \odot k_1 = (dc_i)e_i,\ 0 \le i \le 7.$$

And, we can show the multiplication operation satisfies the following properties.

v_1) For all $d \in \mathbb{Z}_p$ and $k_1, k_2 \in \mathbb{O}/\mathbb{Z}_p$,

$$d \odot (k_1 \oplus k_2) = (d \odot k_1) \oplus (d \odot k_2),$$

v_2) For all $c, d \in \mathbb{Z}_p$ and $k \in \mathbb{O}/\mathbb{Z}_p$,

$$(c + d) \odot k = (c \odot k) \oplus (d \odot k),$$

v_3) For all $c, d \in \mathbb{Z}_p$ and $k \in \mathbb{O}/\mathbb{Z}_p$,

$$(cd) \odot k = c \odot (d \odot k),$$

v_4) For all $k \in \mathbb{O}/\mathbb{Z}_p$,

$$1 \odot k = k.$$

Thus, $\{\mathbb{O}/\mathbb{Z}_p, \oplus, \mathbb{Z}_p, ., \odot\}$ is a vector space. □

From now on, this vector space will be denoted by $\mathbb{O}/\mathbb{Z}_p$, shortly. In the following theorem, we can give an alternative algebra on this vector space.

Theorem 2.2. *$\mathbb{O}/\mathbb{Z}_p$ is an alternative algebra over the field $\mathbb{Z}_p$.*

Proof. Since $\mathbb{O}/\mathbb{Z}_p$ is a vector space, the multiplication of two vectors on this vector space is

$$\times : \mathbb{O}/\mathbb{Z}_p \times \mathbb{O}/\mathbb{Z}_p \longrightarrow \mathbb{O}/\mathbb{Z}_p \text{ for all } k_1, k_2 \in \mathbb{O}/\mathbb{Z}_p$$

$$k_1 \times k_2 = (c_0e_0 + c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6 + c_7e_7)$$
$$\times\ (d_0e_0 + d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5 + d_6e_6 + d_7e_7),$$

$$k_1 \times k_2 = (c_0d_0 - c_1d_1 - c_2d_2 - c_3d_3 - c_4d_4 - c_5d_5 - c_6d_6 - c_7d_7)e_0$$
$$+ (c_0d_1 + c_1d_0 + c_2d_3 - c_3d_2 + c_4d_5 - c_5d_4 + c_7d_6 - c_6d_7)e_1$$
$$+ (c_0d_2 + c_2d_0 + c_3d_1 - c_1d_3 + c_4d_6 - c_6d_4 + c_5d_7 - c_7d_5)e_2$$

$$
\begin{aligned}
&+ (c_0d_3 + c_3d_0 + c_1d_2 - c_2d_1 + c_4d_7 - c_7d_4 + c_6d_5 - c_5d_6)e_3 \\
&+ (c_0d_4 + c_4d_0 + c_5d_1 - c_1d_5 + c_7d_3 - c_3d_7 + c_6d_2 - c_2d_6)e_4 \\
&+ (c_0d_5 + c_5d_0 + c_1d_4 - c_4d_1 + c_7d_2 - c_2d_7 + c_3d_6 - c_6d_3)e_5 \\
&+ (c_0d_6 + c_6d_0 + c_1d_7 - c_7d_1 + c_2d_4 - c_4d_2 + c_5d_3 - c_3d_5)e_6 \\
&+ (c_0d_7 + c_7d_0 + c_6d_1 - c_1d_6 + c_2d_5 - c_5d_2 + c_3d_4 - c_4d_3)e_7.
\end{aligned}
$$

We remark that this multiplication is called as octonion multiplication. The octonion multiplication over $\mathbb{O}/\mathbb{Z}_p$ satisfies the following properties.

$i)$ For all $k_1, k_2 \in \mathbb{O}/\mathbb{Z}_p$ and $d \in \mathbb{Z}_p$

$$(d \odot k_1) \times k_2 = d \odot (k_1 \times k_2).$$

$ii)$ For all $k_1, k_2 \in \mathbb{O}/\mathbb{Z}_p$

$$
\begin{aligned}
k_1 \times (k_1 \times k_2) &= (k_1 \times k_1) \times k_2, \\
(k_2 \times k_1) \times k_1 &= k_2 \times (k_1 \times k_1).
\end{aligned}
$$

$iii)$ For all $k_1, k_2, k_3 \in \mathbb{O}/\mathbb{Z}_p$

$$
\begin{aligned}
k_1 \times (k_2 \oplus k_3) &= (k_1 \times k_2) \oplus (k_1 \times k_3), \\
(k_2 \oplus k_3) \times k_1 &= (k_2 \times k_1) \oplus (k_3 \times k_1).
\end{aligned}
$$

With these properties, we obtain that $\mathbb{O}/\mathbb{Z}_p$ forms an alternative algebra. It is well known that the property $ii)$ in the above equations is called the alternative property.

$\square$

For example, let $p = 13, d = 11$ and if we choose k_1, k_2 as follows, then we have

$$k_1 = 3e_0 + 2e_1 + e_2 + 7e_3 + 7e_4 + 11e_5 + 8e_6 + 5e_7 \in \mathbb{O}/\mathbb{Z}_p,$$

$$k_2 = 1e_0 + 4e_1 + 12e_2 + 2e_3 + 7e_4 + 11e_5 + 6e_6 + 1e_7 \in \mathbb{O}/\mathbb{Z}_p,$$

$$k_1 \oplus k_2 = 4e_0 + 6e_1 + 0e_2 + 9e_3 + 1e_4 + 9e_5 + 1e_6 + 6e_7 \in \mathbb{O}/\mathbb{Z}_p,$$

$$k_1 \times k_2 = 6e_0 + 6e_1 + 3e_2 + 1e_3 + 0e_4 + 11e_5 + 6e_6 + 7e_7 \in \mathbb{O}/\mathbb{Z}_p,$$

$$d \odot k_1 = 7e_0 + 9e_1 + 11e_2 + 12e_3 + 12e_4 + 4e_5 + 10e_6 + 3e_7 \in \mathbb{O}/\mathbb{Z}_p.$$

3 Matrix Representations and $\mathbb{O}/\mathbb{Z}_p$

It is well known that every finite dimensional associative algebra over an arbitrary field $\mathbb{F}$ is algebraically isomorphic to a subalgebra of a total matrix algebra over the field $\mathbb{F}$. For the real quaternion algebra $\mathbb{H}$, there are many studies related with matrix representations[1], [3] and [4]. For the properties of left and right matrix representations, reader who interested in these matrices may find useful these papers [3] and [5]. In [1], for all $a \in \mathbb{H}$ the matrix form of right multiplication given as

$$\varphi : \mathbb{H} \longrightarrow M, \varphi(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix},$$

where M is

$$M = \left\{ \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Thus, $\mathbb{H}$ is algebraically isomorphic to the matrix algebra M. Similarly the matrix form of left multiplication is

$$\tau : \mathbb{H} \longrightarrow M, \tau(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}.$$

Now, based on the results related with the real matrix representations of quaternions, one of real matrix representation of real octonions as follows.

Definition 3.1. Let $\mathbf{a} = a' + a''e \in \mathbb{O};\ a' = a_0 + a_1 i + a_2 j + a_3 k,\ a'' = a_4 + a_5 i + a_6 j + a_7 k$. The following 8×8 real matrix is called as left matrix representation of **a** over $\mathbb{R}$ [1].

$$\omega(\mathbf{a}) = \begin{bmatrix} \varphi(a') & -\tau(a'')K_4 \\ \varphi(a'')K_4 & \tau(a') \end{bmatrix},$$

where K_4 is

$$K_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Thus, the explicit form of $\omega(\mathbf{a})$ is

$$\omega(\mathbf{a}) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}.$$

Definition 3.2. Let $\mathbf{a} = a' + a''e \in \mathbb{O};\ a' = a_0 + a_1 i + a_2 j + a_3 k,\ a'' = a_4 + a_5 i + a_6 j + a_7 k$. In the following 8×8 real matrix $\pi(\mathbf{a})$,

$$\pi(\mathbf{a}) = \begin{bmatrix} \tau(a') & -\varphi(\overline{a''}) \\ \varphi(a'') & \tau(\overline{a'}) \end{bmatrix},$$

is called as the right matrix representation of **a** over $\mathbb{R}$ [1]. The explicit form of $\pi(\mathbf{a})$ is shown as

$$\pi(\mathbf{a}) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & a_3 & -a_2 & a_5 & -a_4 & -a_7 & a_6 \\ a_2 & -a_3 & a_0 & a_1 & a_6 & a_7 & -a_4 & -a_5 \\ a_3 & a_2 & -a_1 & a_0 & a_7 & -a_6 & a_5 & -a_4 \\ a_4 & -a_5 & -a_6 & -a_7 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & -a_7 & a_6 & -a_1 & a_0 & -a_3 & a_2 \\ a_6 & a_7 & a_4 & -a_5 & -a_2 & a_3 & a_0 & -a_1 \\ a_7 & -a_6 & a_5 & a_4 & -a_3 & -a_2 & a_1 & a_0 \end{bmatrix}.$$

Using the above definitions for matrix representations over $\mathbb{O}/\mathbb{Z}_p$, we will give the following theorems without proofs.

Theorem 3.3. *For $k = q' + q''e = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 + q_4e_4 + q_5e_5 + q_6e_6 + q_7e_7 \in \mathbb{O}/\mathbb{Z}_p$ the explicit form of the right matrix representation, $\pi(k)$ can be written as follows.*

$$\pi(k) = \begin{bmatrix} q_0 & (p-1)q_1 & (p-1)q_2 & (p-1)q_3 & (p-1)q_4 & (p-1)q_5 & (p-1)q_6 & (p-1)q_7 \\ q_1 & q_0 & q_3 & (p-1)q_2 & q_5 & (p-1)q_4 & (p-1)q_7 & q_6 \\ q_2 & (p-1)q_3 & q_0 & q_1 & q_6 & q_7 & (p-1)q_4 & (p-1)q_5 \\ q_3 & q_2 & (p-1)q_1 & q_0 & q_7 & (p-1)q_6 & q_5 & (p-1)q_4 \\ q_4 & (p-1)q_5 & (p-1)q_6 & (p-1)q_7 & q_0 & q_1 & q_2 & q_3 \\ q_5 & q_4 & (p-1)q_7 & q_6 & (p-1)q_1 & q_0 & (p-1)q_3 & q_2 \\ q_6 & q_7 & q_4 & (p-1)q_5 & (p-1)q_2 & q_3 & q_0 & (p-1)q_1 \\ q_7 & (p-1)q_6 & q_5 & q_4 & (p-1)q_3 & (p-1)q_2 & q_1 & q_0 \end{bmatrix}.$$

Theorem 3.4. *For $k = q' + q''e = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 + q_4e_4 + q_5e_5 + q_6e_6 + q_7e_7 \in \mathbb{O}/\mathbb{Z}_p$ the explicit form of the left matrix representation $\omega(k)$ can be written as follows.*

$$\omega(k) = \begin{bmatrix} q_0 & (p-1)q_1 & (p-1)q_2 & (p-1)q_3 & (p-1)q_4 & (p-1)q_5 & (p-1)q_6 & (p-1)q_7 \\ q_1 & q_0 & (p-1)q_3 & q_2 & (p-1)q_5 & q_4 & q_7 & (p-1)q_6 \\ q_2 & q_3 & q_0 & (p-1)q_1 & (p-1)q_6 & (p-1)q_7 & q_4 & q_5 \\ q_3 & (p-1)q_2 & q_1 & q_0 & (p-1)q_7 & q_6 & (p-1)q_5 & q_4 \\ q_4 & q_5 & q_6 & q_7 & q_0 & (p-1)q_1 & (p-1)q_2 & (p-1)q_3 \\ q_5 & (p-1)q_4 & q_7 & (p-1)q_6 & q_1 & q_0 & q_3 & (p-1)q_2 \\ q_6 & (p-1)q_7 & (p-1)q_4 & q_5 & q_2 & (p-1)q_3 & q_0 & q_1 \\ q_7 & q_6 & (p-1)q_5 & (p-1)q_4 & q_3 & q_2 & (p-1)q_1 & q_0 \end{bmatrix}.$$

For example, for $p = 17$ and $k \in \mathbb{O}/\mathbb{Z}_{17}$ we take k as $k = 5e_0 + 2e_1 + 2e_2 + 15e_3 + 13e_4 + 10e_5 + 4e_6 + 1e_7 \in \mathbb{O}/\mathbb{Z}_{17}$. Then, it can be calculated that $q' = 5 + 2i + 2j + 15k$ and $q'' = 13 + 10i + 4j + k$. So, using q' and q'' we can obtain the matrices below;

$$\varphi(q') = \begin{bmatrix} 5 & 15 & 15 & 2 \\ 2 & 5 & 2 & 2 \\ 2 & 15 & 5 & 15 \\ 15 & 15 & 2 & 5 \end{bmatrix}, \varphi(q'') = \begin{bmatrix} 13 & 7 & 13 & 16 \\ 10 & 13 & 16 & 4 \\ 4 & 1 & 13 & 7 \\ 1 & 13 & 10 & 13 \end{bmatrix}.$$

And

$$\tau(q') = \begin{bmatrix} 5 & 15 & 15 & 2 \\ 2 & 5 & 15 & 15 \\ 2 & 2 & 5 & 2 \\ 15 & 2 & 15 & 5 \end{bmatrix}, \tau(q'') = \begin{bmatrix} 13 & 7 & 13 & 16 \\ 10 & 13 & 1 & 13 \\ 4 & 16 & 13 & 10 \\ 1 & 4 & 7 & 13 \end{bmatrix}.$$

Moreover, the left matrix representation of $k \in \mathbb{O}/\mathbb{Z}_{17}$ is

$$\omega(k) = \begin{bmatrix} 5 & 15 & 15 & 2 & 4 & 7 & 13 & 16 \\ 2 & 5 & 2 & 2 & 7 & 13 & 1 & 13 \\ 2 & 15 & 5 & 15 & 13 & 16 & 13 & 10 \\ 15 & 15 & 2 & 5 & 16 & 4 & 7 & 13 \\ 13 & 10 & 4 & 1 & 5 & 15 & 15 & 2 \\ 10 & 7 & 1 & 13 & 2 & 5 & 15 & 15 \\ 4 & 16 & 7 & 10 & 2 & 2 & 5 & 2 \\ 1 & 4 & 7 & 4 & 15 & 2 & 15 & 5 \end{bmatrix}.$$

The right matrix representation of $k \in \mathbb{O}/\mathbb{Z}_{17}$ is

$$\pi(k) = \begin{bmatrix} 5 & 15 & 15 & 2 & 4 & 7 & 13 & 16 \\ 2 & 5 & 15 & 15 & 10 & 4 & 16 & 4 \\ 2 & 2 & 5 & 2 & 4 & 1 & 4 & 7 \\ 15 & 2 & 15 & 5 & 1 & 13 & 10 & 4 \\ 13 & 7 & 13 & 16 & 5 & 2 & 2 & 15 \\ 10 & 13 & 16 & 4 & 15 & 5 & 2 & 2 \\ 4 & 1 & 13 & 7 & 15 & 15 & 5 & 15 \\ 1 & 13 & 10 & 13 & 2 & 15 & 2 & 5 \end{bmatrix}.$$

In conclusion, we study octonion algebra over $\mathbb{Z}_p$ and give their left and right matrix representations according to defined real matrix representation [1].

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