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THE WEAKLY SIGN SYMMETRIC Q-MATRIX COMPLETION PROBLEM

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Abstract. In this paper, some necessary and sufficient conditions for a digraph to have weakly sign symmetric Q-completion are provided. The digraphs of order at most three that have weakly sign symmetric Q-completion are singled out.

1 Introduction

A *partial matrix* is a rectangular array of numbers in which some entries are specified while others are free to be chosen. A *pattern* for $n \times n$ matrices is a subset of $\{1, \dots, n\} \times \{1, \dots, n\}$. A partial matrix *specifies a pattern* if its specified entries lie exactly in those positions listed in the pattern. For $\alpha \subseteq \{1, \dots, n\}$, the principal submatrix $B[\alpha]$ is obtained by deleting from B all rows and columns whose indices are not in α. A principal minor is the determinant of a principal submatrix.

A real $n \times n$ matrix B is a P-*matrix* (P_0-*matrix*) if every principal minor of B is positive (nonnegative). A real $n \times n$ matrix $B = [b_{ij}]$ is a Q-*matrix* if for every $k \in \{1, 2, \dots, n\}$, $S_k(B) > 0$, where $S_k(B)$ is the sum of all $k \times k$ principal minors of B. The matrix B is *weakly sign symmetric* if $b_{ij}b_{ji} \geq 0$ for each pair of $i, j \in \{1, \dots, n\}$.

For a given class Π of matrices (e.g., P, P_0 or Q-matrices) a *partial* Π-*matrix* is a partial matrix for which the specified entries satisfy the properties of a Π-matrix. Thus, a *partial* P-*matrix* (*partial* P_0-*matrix*) is one in which all fully specified principal minors are positive (nonnegative). Similarly, a partial weakly sign symmetric matrix is a matrix in which fully specified principal submatrices are weakly sign symmetric matrices.

A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A Π-*completion* of a partial Π-matrix M is a completion of M which is a Π-matrix. For a particular class Π of matrices, we say a pattern has Π-*completion* if every partial Π-matrix specifying the pattern can be completed to a Π-matrix and the Π-*matrix completion problem* studies the properties and classifications of patterns having Π-completions. Matrix completion problems for several classes of matrices including P and P_0-matrices have been studied by a number of authors (e.g., [5, 6, 7, 8, 9, 10, 11]). For a survey of matrix completion results one may see [3].

1.1 Digraphs

Graph theory has played an important role in the study of matrix completion problems. Most of the graph-theoretic terms can be found in any standard reference, for example, in [1] and [4]. For our purposes, a *directed graph* or *digraph* $D = (V_D, A_D)$ of order $n > 0$ is a finite nonempty set V_D, with $|V_D| = n$ of objects called *vertices* together with a (possibly empty) set A_D of ordered pairs of vertices, called *arcs* or *directed edges*. Sometimes, we simply write $v \in D$ (resp. $(u, v) \in D$) to mean $v \in V_D$ (resp. $(u, v) \in A_D$). If $x = (u, v)$ is an arc in D, we say that x is *incident* with u and v. If $x = (u, u)$, then x is called a *loop* at the vertex u.

A *symmetric edge* of D is a pair of arcs $\{(u, v), (v, u)\} \subseteq A_D$, usually written as $\{u, v\}$. A digraph $H = (V_H, A_H)$ is a *subdigraph of order* k of the digraph D if $|V_H| = k$ and $V_H \subseteq V_D$, $A_H \subseteq A_D$. A subdigraph H of D is an *induced subdigraph* if $A_H = (V_H \times V_H) \cap A_D$ (*induced by* V_H) and is a *spanning subdigraph* if $V_H = V_D$. By K_n we denote the digraph with vertex set $\langle n \rangle = \{1, 2, \ldots, n\}$, and arc set $\langle n \rangle \times \langle n \rangle$, i.e., one with all possible arcs including loops on the vertex set $\langle n \rangle$. The *complement of a digraph* D is the digraph $\overline{D}$, where $V_{\overline{D}} = V_D$ and $(v, w) \in A_{\overline{D}}$ if and only if $(v, w) \notin A_D$. A digraph is called asymmetric if it does not contain a symmetric edge.

A (*directed*) u-v *path* P of length $k \geq 0$ in D is an alternating sequence $(u = v_0, x_1, v_1, \ldots, x_k, v_k = v)$ of vertices and arcs, where $v_i,\ 1 \leq i \leq k$, are distinct vertices and $x_i = (v_{i-1}, v_i)$. Then, the vertices v_i and the arcs x_i are said to be on P. Further, if $k \geq 2$ and $u = v$, then a u-v path is a *cycle* of length k. We then write $C_k = \langle v_1, v_2, \ldots, v_k \rangle$ and call C_k a k-cycle in D. Naturally, paths and cycles in a digraph D are considered to be subdigraphs of D.

A cycle C is *even* (resp. *odd*) if its length is even (resp. odd). A digraph D is said to be connected (resp. strongly connected) if for every pair u, v of vertices, D contains a u-v path (resp. both a u-v path and a v-u path). The maximal connected (resp. strongly connected) subdigraphs of D are called *components* (resp. *strong components*) of D.

1.2 Digraphs and matrices

Let π be a permutation of a nonempty finite set V. The digraph $D_\pi = (V, A_\pi)$, where $A_\pi = \{(v, \pi(v)) : v \in V\}$, is called a *permutation digraph*. Clearly, each component of a permutation digraph is a loop or a cycle. The digraph D_π is said to be *positive* (resp. *negative*) if π is an even permutation (resp. an odd permutation). It is clear that D_π is negative if and only if it has odd number of even cycles.

A *permutation subdigraph* H (of order k) of a digraph D is a permutation digraph that is a subdigraph of D (of order k). A digraph D is *stratified* if D has a permutation subdigraph of order k for every $k = 2, 3, \ldots, |D|$.

Let $B = [b_{ij}]$ be an $n \times n$ matrix. We have

$$\det(B) = \sum (\operatorname{sgn} \pi) b_{1\pi(1)} \cdots b_{n\pi(n)} \tag{1.1}$$

where the sum is taken over all permutations π of $\langle n \rangle$.

A signing of a digraph is an assignment of a sign $+$ or $-$ to each arc of the digraph. The result of signing of a digraph is called a *signed digraph*. For an arc $e \in D$ by $s(e)$ we mean e has sign $s(e)$.

For a k-cycle in C_k in D, the sign $s(C_k)$ is defined to be,

$$s(C_k) = (-1)^{k+1} \prod_{e \in C_k} s(e)$$

For a permutation subdigraph k of D, the sign $s(k)$ of k is

$$s(k) = \prod_{C \in k} s(C)$$

Now if is useful to associate a partial matrix with a digraph that describes the positions of the specified entries in the partial matrix. We say that an $n \times n$ partial matrix B *specifies* a digraph D if D = $(\langle n \rangle, A_D)$, and for $1 \leq i, j \leq n$, $(i, j) \in A_D$ if and only if the entry b_{ij} of B is specified. Let $M = [m_{ij}]$ be a partial matrix specifying D. The sign of an arc (i, j) is defined as follows:

$$sgn(i,j) = \begin{cases} 1, & \text{if } m_{ij} > 0 \\ -1 & \text{if } m_{ij} < 0 \end{cases}$$

The resulting signed digraph D is the sign pattern of a partial matrix of M. In the case of a symmetrically placed pair, a_{ij} and x_{ji}, in a partial matrix, the specified entry a_{ij} shall be referred to as the specified twin. The other member of the pair, x_{ij}, shall be referred to as the unspecified twin. If any specified twin a_{ij} of a partial matrix M specifying D is zero, we can assign suitable sign (+ or -) to the arcs of D according to our choice. In that case, we can choose also suitable sign of unspecified twin x_{ij}.

The property of being a weakly sign symmetric Q-matrix is preserved under similarity and transposition, but it is not inherited by principal submatrices, as it can easily be verified. Thus the weakly sign symmetric Q-matrix completion problem is quite different from completion problems involving P-matrix classes, where principal submatrices inherit the properties of the class under consideration.

2 Partial weakly sign symmetric Q-matrices and their completion problem

A *partial Q-matrix* M is a partial matrix such that $S_k(M) > 0$ for every $k = 1, \dots, n$ for which all $k \times k$ principal submatrices of M are fully specified. A *partial weakly sign symmetric Q-matrix* $M = [a_{ij}]$ is a partial Q-matrix in which all fully specified principal submatrices are weakly sign symmetric.

Let $M = [a_{ij}]$ be a partial weakly sign symmetric matrix. If all 1×1 principal submatrices (i.e., all diagonal entries) in M are specified, then their sum $S_1(M)$ (the trace of M) must be positive. If all $k \times k$ principal submatrices are fully specified for some $k \geq 2$, then M is fully specified and, therefore, is a weakly sign symmetric Q-matrix. Thus, a partial weakly sign symmetric Q-matrix is characterized as follows.

Proposition 2.1. *Suppose $M = [a_{ij}]$ is a partial weakly sign symmetric matrix. Then M is a partial weakly sign symmetric Q-matrix if and only if exactly one of the following holds:*

(i) *At least one diagonal entry of M is not specified.*

(ii) *All diagonal entries are specified, at least one diagonal entry is positive so that $Tr(M) > 0$ and M has an off diagonal unspecified entry.*

(iii) *All entries of M are specified and M is a Q-matrix.*

A completion B of a partial weakly sign symmetric Q-matrix M is called a *weakly sign symmetric Q-completion* of M, if B is a weakly sign symmetric Q-matrix. Since any matrix which is permutation similar to a Q-matrix is a Q-matrix, it is evident that if a partial weakly sign symmetric Q-matrix has a weakly sign symmetric Q-completion, so does any partial matrix which is permutation similar to M.

It is easy to see that any partial weakly sign symmetric matrix M with all unspecified diagonal entries has weakly sign symmetric Q-completion. A completion can be obtained by choosing sufficiently large values for the unspecified diagonal entries. Let M be a partial weakly sign symmetric Q-matrix in which the diagonal entries at (i, i) positions $(i = k + 1, \dots, n)$ are unspecified. In case $M[1, \dots, k]$ is fully specified, M may not have a weakly sign symmetric Q-completion. For example, the partial matrix,

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & * \end{bmatrix},$$

where $*$ denotes an unspecified entry, does not have weakly sign symmetric Q-completion. Indeed, for any completion B of M, $S_3(B) = \det B = 0$. On the other hand, if $M[1, \dots, k]$ has an unspecified entry and has a weakly sign symmetric Q-completion, then M has a weakly sign symmetric Q-completion. A completion of M can be obtained by choosing sufficiently large values for the unspecified diagonal entries. We list these observations in the following results.

Theorem 2.2. *If a matrix M omits all diagonal entries, then M has weakly sign symmetric Q-completion.*

Proof. Let $M = [a_{ij}]$ be a partial weakly sign symmetric Q-matrix. For any $t > 1$ consider a completion $B = [b_{ij}]$ of M by setting all diagonal entries equal to t and rest of the off diagonal entries to be equal to zero. Then, any $k \times k$ principal minor will be of the form $t^k + p(t)$, where $p(t)$ is a polynomial of degree $\leq k-1$. Now by choosing t large enough we have, $S_k(B) > 0$ for all $k \times k$ principal minors of B. Since only finitely many principal minors are to be considered, thus for sufficiently large t, M has weakly sign symmetric Q-completion. ■

Theorem 2.3. *Let M be a partial weakly sign symmetric Q-matrix in which the diagonal entry at $(r+1, r+1)$ position is unspecified. If the principal submatrix $M[1,\dots,r]$ of M is not fully specified and has weakly sign symmetric Q-completion, then M has weakly sign symmetric Q-completion.*

Proof. Let $M = [a_{ij}]$ be a partial weakly sign symmetric Q-matrix which omits the diagonal entry at $(r+1, r+1)$ position. Then, M is of the form,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where, $M_{11} = M[1,\dots,r]$ and $M_{22} = M[r+1, r+1]$.

Let A_1 be the weakly sign symmetric Q-matrix completion of $M[1,\dots,r]$. Then,

$$M' = \begin{bmatrix} A_1 & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

is a partial weakly sign symmetric Q-matrix, since M_{22} has an unspecified diagonal entry. Now for $t > 0$, consider a completion $B = [b_{ij}]$ of M' obtained by choosing $b_{ii} = t, \quad i = r+1$ and $b_{ij} = 0$ against all other unspecified entries in M'. Then B is of the form,

$$B = \begin{bmatrix} A_1 & B_{12} \\ B_{21} & t \end{bmatrix}.$$

Since A_1 is a weakly sign symmetric Q-matrix, $S_i(A_1) > 0$ for $1 \le i \le r$. For $2 \le j \le r+1$,

$$S_j(B) \quad = \quad S_j(A_1) + tS_{j-1}(A_1) + s_j,$$

where s_j is a constant. Now $S_j(B) > 0$ for sufficiently large values of t and clearly B is weakly sign symmetric Q-matrix. ∎

Corollary 2.4. *Let M be a partial weakly sign symmetric Q-matrix in which the diagonal entries at (i,i) positions $(i = r+1,\dots,n)$ are unspecified. If the principal submatrix $M[1,\dots,r]$ of M is not fully specified and has weakly sign symmetric Q-completion, then M has weakly sign symmetric Q-completion.*

The converse of Corollary 2.4 is not true which can be seen from the following example.

Example 2.5. Consider the partial matrix,

$$M = \begin{bmatrix} * & * & a_{13} & * \\ a_{21} & d_2 & * & * \\ * & a_{32} & * & * \\ a_{41} & * & a_{43} & d_4 \end{bmatrix},$$

where $*$ denotes the unspecified entries. We show that for any choice of values of the specified entries M has weakly sign symmetric Q-completions, though there are occasions when $M[1,2,3]$ does not have (see Example 3.3). For $t > 0$, consider the completion $B(t)$ of M defined as follows:

$$B(t) = \begin{bmatrix} t & 0 & a_{13} & 0 \\ a_{21} & d_2 & b_{23} & 0 \\ 0 & a_{32} & t & b_{34} \\ b_{41} & b_{42} & a_{43} & d_4 \end{bmatrix},$$

where, we put $b_{23} = s(a_{32})t,\ b_{34} = s(a_{43})t,\ b_{42} = s(a_{32}a_{43})t$. Then,

$$\begin{aligned} S_1(B(t)) &= 2t + \sum d_i, \\ S_2(B(t)) &= t^2 + f_1(t), \\ S_3(B(t)) &= t^3 + f_2(t), \\ S_4(B(t)) &= t^4 + f_3(t), \end{aligned}$$

where $f_i(t)$ is a polynomial in t of degree at most $i,\ i = 1, 2, 3$. Consequently, $B(t)$ is a weakly sign symmetric Q-matrix for sufficiently large t, and therefore, M has weakly sign symmetric Q-completion. On the other hand, the partial weakly sign symmetric Q-matrix

$$M[2,4] = \begin{bmatrix} 0 & x_{24} \\ x_{42} & 1 \end{bmatrix},$$

with unspecified entries x_{24}, x_{42}, is the principal submatrix of M induced by its diagonal $\Delta = \{2, 4\}$. That $M[2, 4]$ does not have weakly sign symmetric Q-completion is evident, because $S_2(M[2, 4]) \leq 0$ for any completion of $M[2, 4]$.

3 Digraphs and weakly sign symmetric Q-completions

We have seen that an $n \times n$ partial matrix M *specifies* a digraph $D = (\langle n \rangle, A_D)$ if for $1 \leq i, j \leq n$, $(i, j) \in A_D$ if and only if the (i, j)-th entry of M is specified. For example, the partial weakly sign symmetric Q-matrix M in Example 2.5 specifies the digraph D in Figure 1.

Theorem 3.1. *Suppose M is a partial weakly sign symmetric matrix specifying the digraph D. If the partial submatrix of M induced by every strongly connected induced subgraph of D has weakly sign symmetric Q-completion, then M has weakly sign symmetric Q-completion.*

Proof. We prove the result for the case when D has two strong components D_1 and D_2. The general result will then follow by induction. By a relabeling of the vertices of D, if required, we have

$$M = \begin{bmatrix} M_{11} & M_{12} \\ X & M_{22} \end{bmatrix},$$

where M_{ii} is a partial weakly sign symmetric Q-matrix specifying $D_i,\ i = 1, 2$, and all entries in X are unspecified. By the hypothesis, M_{ii} has a weakly sign symmetric Q-completion B_{ii}. Consider the completion

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

by choosing all entries in X as well as all unspecified entries in M_{12} as 0. Then, for $2 \leq k \leq |D|$ we have

$$S_k(B) = S_k(B_{11}) + S_k(B_{22}) + \sum_{r=1}^{k-1} S_r(B_{11})S_{k-r}(B_{22}) > 0,$$

Here, we mean $S_k(B_{ii}) = 0$ whenever k exceeds the size of B_{ii}. Thus M can be completed to a weakly sign symmetric Q-matrix. ■

The proof of the following result is similar.

Theorem 3.2. *Suppose M is a partial weakly sign symmetric matrix specifying the digraph D. If the partial submatrix of M induced by each component of D has a weakly sign symmetric Q-completion, then M has a weakly sign symmetric Q-completion.*

The converse of Theorem 3.1 is not true. For example, every partial weakly sign symmetric Q-matrix specifying the digraph D in Figure 1 has weakly sign symmetric Q-completion, although the strong component D_1 induced by vertices $\{1, 2, 3\}$ does not have weakly sign symmetric Q-completion (see Example 3.3).

Example 3.3. Consider the digraph D in the Figure 1. We show that D has weakly sign symmetric Q-completion, but the strong component D_1 induced by vertices $\{1, 2, 3\}$ does not have weakly sign symmetric Q-completion. Let $M = [a_{ij}]$ be a partial weakly sign symmetric Q-

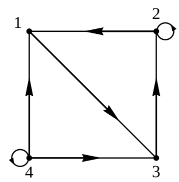

Figure 1. The Digraph D

matrix specifying D. Then for $t > 0$, M can be completed to a weakly sign symmetric Q-matrix $B(t)$ (see Example 2.5) but the principal submatrix induced by the digraph D_1 i.e. $M[1, 2, 3]$ does not have weakly sign symmetric Q-completion. To see that $M[1, 2, 3]$ does not have weakly sign symmetric Q-completion, consider the partial weakly sign symmetric Q-matrix

$$M[1,2,3] = \begin{bmatrix} x & u & -1 \\ -1 & 0 & w \\ z & -1 & y \end{bmatrix},$$

with unspecified entries x, u, w, y and z. Then for any weakly sign symmetric Q-completion B of $M[1, 2, 3]$, we have $S_3(B) \leq 0$ and hence $M[1, 2, 3]$ does not have weakly sign symmetric Q-completion.

The property of having weakly sign symmetric Q-completion is not inherited by induced subdigraphs. This can be also seen from the Example 2.5.

4 The weakly sign symmetric Q-completion problem

We say that a digraph D has *weakly sign symmetric Q-completion*, if every partial weakly sign symmetric Q-matrix specifying D can be completed to a weakly sign symmetric Q-matrix. The *weakly sign symmetric Q-matrix completion problem* aims at studying and classifying all digraphs D which have weakly sign symmetric Q-completion.

It is clear that if a digraph D has weakly sign symmetric Q-completion, then any digraph which is isomorphic to D has weakly sign symmetric Q-completion.

4.1 Necessary conditions for weakly sign symmetric Q-matrix completion

In this section we provide some necessary conditions for a digraph to have weakly sign symmetric Q-completion.

Theorem 4.1. *Let D be a digraph with at least two vertices. If D has weakly sign symmetric Q-completion, then D omits at least two loops.*

Proof. Let D be a digraph with n vertices having loops at $2, 3, \ldots, n$. Suppose D omits only one loops. Let $M = [a_{ij}]$ be a partial weakly sign symmetric Q-matrix specifying the digraph D which is defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } (i,j) = (1,1)\,, (i,j) \in D \\ 0, & \text{for all } (i,j) \neq (1,1)\,, (i,j) \in D. \end{cases}$$

Then for any weakly sign symmetric completion B of M, $S_2(B) \leq 0$, where $S_2(B)$ is the sum of all principal minors of order 2×2 in B. Hence D omits at least two loops. ∎

The next theorem shows that for a digraph D that omits at least two loops, stratification of $\overline{D}$ is necessary condition for D to have weakly sign symmetric Q-completion.

Theorem 4.2. *Suppose D be a digraph of order $n \geq 2$ such that D has weakly sign symmetric Q-completion, then $\overline{D}$ is stratified.*

Proof. Suppose D has weakly sign symmetric Q-completion. Let $k \geq 2$ and assume that $\overline{D}$ has no order k permutation digraph. If M is a partial weakly sign symmetric matrix that specifies D with all specified entries zero and B is a completion of M, then all $k \times k$ principal minors of B are zero, so B is not a Q-matrix. This implies that $\overline{D}$ must be stratified. ∎

But the condition of the Theorem 4.1 and Theorem 4.2 are not sufficient for weakly sign symmetric Q-completion which can be seen from the following Example 4.3.

Example 4.3. Let D_2 be the digraph shown in the Figure 2 in which $\overline{D_2}$ is stratified. Consider a

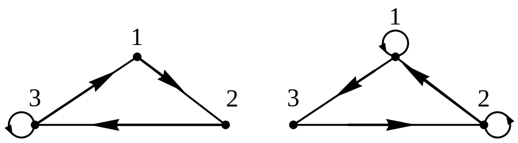

Figure 2. $D_2, \overline{D_2}$

partial weakly sign symmetric Q-matrix,

$$M = \begin{bmatrix} 0 & 1 & x \\ y & z & 1 \\ -1 & w & u \end{bmatrix},$$

where x, y, z, w, u are unspecified entries. It is clear that M specifies D_2. Now, from sign symmetric conditions of M, $x \leq 0$, $y, w, z, u \geq 0$ and at least one of z and u is > 0. But M cannot be completed to a weakly sign symmetric Q-matrix because for any completion B of M, $S_3(B) \leq 0$.

Corollary 4.4. *Suppose D be a digraph of order n that omits two loops and such that $|A_D| > n(n-1)+2$. Then D does not have weakly sign symmetric Q-completion.*

Proof. If D has more than $n(n-1)+2$ arcs (including loops), then $\overline{D}$ has fewer than $n^2 - n(n-1) - 2 = n-2$ arcs. Thus $\overline{D}$ does not contain an order n permutation digraph. Therefore by Theorem 4.2, D is not stratified and hence D does not have weakly sign symmetric Q-completion. ∎

Theorem 4.5. *Suppose D be a digraph of order $n > 2$ which omits only 2 loops. If $\overline{D}$ is stratified and $\overline{D}$ has no symmetric edge, then D does not have weakly sign symmetric Q-completion.*

Proof. Suppose $M = [a_{ij}]$ be a partial weakly sign symmetric Q-matrix specifying the digraph D which is defined as follows:

$$a_{ij} = \begin{cases} 0, & \text{if } i = j\ , (i,j) \in D \\ -1, & otherwise \end{cases}$$

Then for completion B of M, $S_3(B) \leq 0$, where $S_3(B)$ is the sum of all principal minors of order 3×3 in B. Hence the result follows. ∎

Example 4.6. Consider the digraph D_2 in 4.3. It is easy to see that D_2 satisfies the hypothesis of the Theorem 4.5. Thus the digraph D_2 does not have weakly sign symmetric Q-completion.

4.2 Sufficient conditions for weakly sign symmetric Q-matrix completion

Theorem 4.7. *If a digraph $D \neq K_n$ of order n has weakly sign symmetric Q-completion, then any spanning subdigraph of D has weakly sign symmetric Q-completion.*

Proof. Suppose H be a spanning subdigraph of D and M_H be a partial weakly sign symmetric Q-matrix specifying the digraph H. Consider a partial matrix M_D obtained from M_H by specifying the entries corresponding to $(i,j) \in A_D \setminus A_H$ as 0. Since $D \neq K_n$, by Proposition 2.1, M_D is a partial weakly sign symmetric Q-matrix specifying D. Let B be a weakly sign symmetric Q-completion of M_D. Clearly, B is a weakly sign symmetric Q-completion of M_H. ■

Definition 4.8. A digraph D_1 is said to be weakly sign symmetric compatible digraph with a digraph D if the sign of the arcs of every 2-cycle $\langle u, v\rangle$ in $D \cup D_1$ satisfies the relation $s(u,v)s(v,u) \geq 0$.

Theorem 4.9. *Suppose D be a digraph such that D omits at least two loops and $\overline{D}$ is stratified. If for any signing of the arcs of D, $\overline{D}$ is weakly sign symmetric compatible with D and every cycle of length ≥ 3 of $\overline{D}$ is of positive sign, then D has weakly sign symmetric Q-completion.*

Proof. Let $M = [a_{ij}]$ be a partial weakly sign symmetric Q-matrix specifying the digraph D. Now for any signing of D, $\overline{D}$ is weakly sign symmetric compatible with D. Then for $t > 0$, consider a completion $B(t) = [b_{ij}]$ of M as follows:

$$b_{ij} = \begin{cases} t^2, & \text{if } i = j \text{ and } (i,j) \in \overline{D} \\ sgn(i,j)t, & \text{if } (i,j) \in \overline{D} \\ a_{ij}, & \text{if } (i,j) \in D \end{cases}$$

Then for $k = 2, \ldots, n$, we have,

$$S_2(B(t)) = \alpha t^4 + p(t)$$
$$S_k(B(t)) = \gamma t^k + q(t); k > 2,$$

where $p(t), q(t)$ is a polynomial of degree at most 3 and $k-1$ and $\alpha, \gamma > 0$. Hence the result follows. ■

Example 4.10. Consider a digraph D_2 and its complement $\overline{D}$ in the following Figure 3. Now,

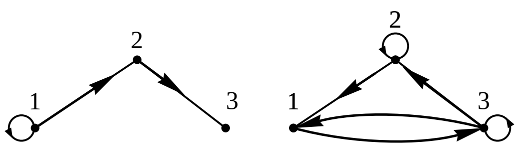

Figure 3. $D, \overline{D}$

for any signing of D, it is possible to sign the arcs of $\overline{D}$ so that $\overline{D}$ is weakly sign symmetric compatible with D. Now, with the sign of the arcs of D, it can be possible to sign the arcs of $\overline{D}$ so that every cycle of length$\geq$ 3 is of positive sign. Hence, by Theorem 4.9, D has weakly sign symmetric Q-completion.

5 Classification of small digraphs as to weakly sign symmetric Q-completion

We have examined the digraphs of order at most three to weakly sign symmetric Q-completion. Clearly, any digraph of order 1 (with or without a loop) has weakly sign symmetric Q-completion. Any digraph of order 2 and without a loop has weakly sign symmetric Q-completion.

There are only three non-isomorphic digraphs of order 3 without loops for which the digraphs obtained by attaching a loop at any of the vertices have weakly sign symmetric Q-completion (by Theorem 4.9). These digraphs are precisely are the following:

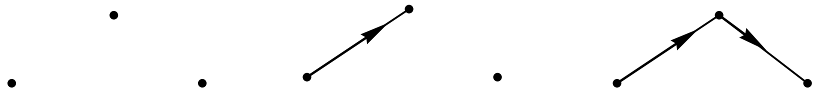

Figure 4. The digraphs of order 3 having weakly sign symmetric Q-completion

6 Comparison between Q-completion and weakly sign symmetric Q-completion

We know that every weakly sign symmetric Q-matrix is a Q-matrix, and every partial weakly sign symmetric Q-matrix is a partial Q-matrix, but the following examples show that the completion problems for these two classes are quite different.

Example 6.1. Consider the digraphs $D_i,\ i = 3, 4, 5$ in Figure 5.

(i) The digraph D_3 has both Q-completion and weakly sign symmetric Q-completion.

(ii) The digraph D_4 has Q-completion, but does not have weakly sign symmetric Q-completion.

(iii) The digraph D_5 has neither Q-completion nor weakly sign symmetric Q-completion.

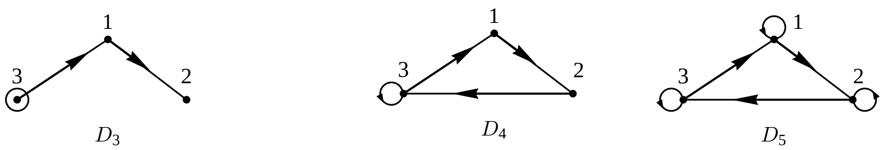

Figure 5. Q-completion vs. weakly sign symmetric Q-completion

Since the digraph D_3 and D_4 satisfies the Theorem 2.3 of [2], thus D_3 or D_4 has Q-completion. On the other hand, $\overline{D_5}$ is not stratified, hence by Theorem 2.8 of [2], D_5 does not have Q-completion. Again D_3 satisfies the Theorem 4.9, therefore it has weakly sign symmetric Q-completion. But by example 3.3, D_4 does not have weakly sign symmetric Q-completion. Also $\overline{D_5}$ is not stratified, hence by Theorem 4.1, D_5 does not have weakly sign symmetric Q-completion.

Suppose D is a digraph having weakly sign symmetric Q-completion. Then, $\overline{D}$ is stratified and omits at least two loops. For all small digraphs (including all digraph of order 4) having these properties are seen to have Q-completion. Whether a stratified digraph omitting a loop necessarily have Q-completion is not known (see Question 2.9 in [2]). We do not know whether there is a digraph having weakly sign symmetric Q completion, but not Q-completion.

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