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THE STUDY OF RADIAL VELOCITY OF BLOOD THROUGH STENOSED ARTERY AND SHAPE FUNCTION

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Abstract. The blood flow characteristics through an artery in the presence of overlapping stenosis is studied mathematically. The artery is assumed to have an axisymmetric shape. The arterial wall deformability is taken to be elastic (moving wall). In the present study, Radial variation of radial velocity is studied. The shape function distribution over the length of the artery is also studied. The blood is treated as Newtonian fluid and the arterial wall is treated to be elastic having stenosis in its lumen which is due to deposition of fatty material. The equations of motion are solved numerically by finite difference scheme. The present results show consistency with the available results.

Introduction

Cardiovascular disease is the most common problems related to biological fluid flow in the human body. Circulatory disorders are known to be responsible for over seventy five percent of all deaths and stenosis or atherosclerosis is one of the oftenly occurring cardiovascular diseases .Stenosis is narrowing of a body passage, tube or orifice [1]. It is an abnormal growth that develops at different locations of the cardiovascular system. It is due to deposition of fatty material inside the artery. In the region of narrowing arterial constriction, the flow accelerates and consequently velocity gradient near the wall is steeper due to increased core velocity resulting in relatively large shear stress on the wall. The properties of blood and arterial wall motion play a vital role in physiology of cardiovascular system. Thus we should understand the factor controlling blood flow in the presence of stenosis.
A large number of researchers have contributed in studying blood flow in the presence of stenosis. Haldar. K [2] has studied resistance to blood flow through an artery due to shape function. The paper investigated that the resistance to flow decreases as the shape of stenosis changes. They have considered the flow to be steady state one dimensional. The fluid dynamic variables were numerically analysed in human carotid artery bifurcation model by Perktold. K [3] et.al. They clarify the geometric factor in carotid bifurcation atherogenesis. The governing Navier-Stokes equations describing incompressible, pulsatile, three dimensional viscous flows are approximated using a pressure correction finite element procedure which has been developed for three-dimensional, time-dependent viscous flow problems. Chakravarty. S and Mandal. P. K [4] studied blood flow in overlapping arterial stenosis. The paper presented the unsteady behaviour of blood treating blood as non-Newtonian fluid. Sannigrahi. A. [5] observed the blood flow in the presence of body accelerations. Chakravarty. S, Mandal P. K [6] studied blood flow in tapered arteries in the presence of stenosis. The paper analysed the effects of tapering, arterial wall motion and the pressure gradient on the blood flow characteristic. Yakhot. A et.al [7] have studied a pulsatile flow of a viscous, incompressible fluid through a stenosed artery and explored the influence of the shape and surface roughness on the flow resistance. Ruchi Agarwal [8] studied pulsatile blood flow in carotid artery bifurcation. The paper studied the flow at various locations in common carotid artery and internal carotid artery for different frequencies 60, 90, 120 pulse/min. N. Mustapha et.al [9] simulated unsteady blood flow through multi-irregular arterial stenosis. They have used marker and cell method to study blood characteristic with surface irregularities. Daniel Riahi et.al [10] have considered the blood flow in an artery in the presence

of an overlapping stenosis and studied the effects of the hematocrit and constriction due to red blood cells-plasma combination of variable fluid viscosity and height of the stenosis on axial flow velocity, pressure gradient, pressure drop and impedance.
This present paper attempts to study radial variation of blood under the diseased conditions. The shape function distribution and effects are also studied. Finite difference scheme involving central differences is implemented to solve governing equations.

Governing Equations

We consider the stenotic blood flow to be unsteady, axisymmetric and laminar. The basic equations of motion governing such flow may be written as

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} + \frac{u}{r} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right] \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right] \tag{3}$$

Where r, z are radial and axial directions. u and w are velocity components along radial and axial directions respectively. ρ and μ are density and viscosity of blood, p is the pressure. The pressure gradient $-\frac{\partial p}{\partial z}$ is due to the pumping action of the heart which is given by $-\frac{\partial p}{\partial z} = A_0 + A_1 \cos\omega t$, Burton. A. C [11].Where A_0 is taken as the constant amplitude and $A_1$is the amplitude of the pulsatile component which gives rise to systolic and diastolic pressure. Here $w = 2f_p$, where f_p is pulse frequency.

Geometry of stenosis

The time dependent geometry of the overlapping stenosis may be described by

$$\begin{aligned} R(z,t) &= a\left\{1 - a_1(t)\left[\frac{11}{32}(z-d)l_0^3 - \frac{47}{48}(z-d)^2 l_0^2 + (z-d)^3 l_0 - \frac{1}{3}(z-d)^4\right]\right\} \\ &= 0.8 \qquad\qquad otherwise \end{aligned}$$

where $a_1(t) = \frac{32}{4*a*l_0^4}\tau_m(1 - e^{\frac{-t}{T}})$in which τ_m is the maximum height of stenosis.

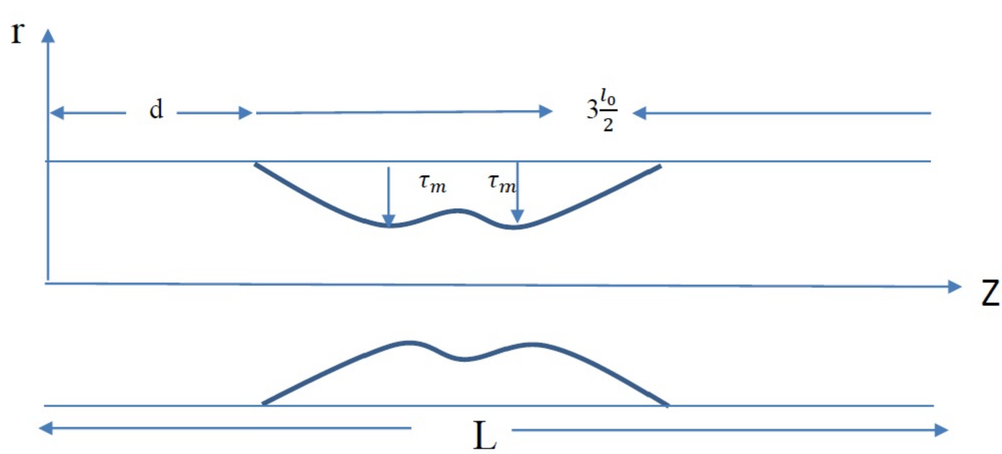

Figure 1. The geometry of the overlapping stenosis in an artery

R (z, t) is the radius of the arterial segment in the stenotic region, a is the constant radius of the normal artery in the non-stenotic region, $3l_0/2$ is the length of the stenosis and d indicates the location of the stenosis. The maximum height τ_m of the stenosis appears at two specific locations namely $z = d + (l_0/2)$and $z = d + l_0$ such that the ratio of the height of the stenosis to the radius of the normal artery is much smaller than unity. Also the height of the stenosis at a distance of $d + (3l_0/4)$ from origin is chosen to be $3\tau_m/4$. The arterial segment under consideration is taken to be of finite length L.Here T denotes a time parameter.

Boundary conditions

Axially, there is no radial flow, therefore the radial velocity is zero and the axial velocity gradient of the flowing blood along the axis may be assumed to be equal to zero. This may be stated as

$$w(r,z,t) = 0,\ u(r,z,t) = \frac{\partial R}{\partial t} \text{ at } r = R(z,t) \tag{4}$$

Along the axis of symmetry, the axial velocity gradient and the normal components of the velocity vanish. Thus

$$\frac{\partial w}{\partial r} = 0,\ u(r,z,t) = 0 \text{ at } r = 0. \tag{5}$$

It is assumed that initially radial and axial velocity both are zero.

i.e. u(r,z,0)=0,w(r,z,0)=0 (6)

The velocities at the inlet and outlet of an arterial segment of finite length are taken as Sarifuddin et al [12] $u(r,z,t) = 0$ and $w(r,z,t) = \frac{5}{3}\left[1 - \left(\frac{r}{R(z,t)}\right)^3\right]$at $z = 0$ (7)

$$\frac{\partial w(r,z,t)}{\partial z} = 0, \frac{\partial u(r,z,t)}{\partial z} = 0 \text{ at } z = L \tag{8}$$

Numerical Method and implemention

We introduce radial co-ordinate transformation

$x = \frac{r}{R(z,t)}$

Using this transformation,equations (1),(3) along with boundary conditions (4)-(8) take the following form

$$\frac{1}{R}\frac{\partial u}{\partial x} + \frac{\mu}{xR} + \frac{\partial w}{\partial z} - \frac{x}{R}\frac{\partial R}{\partial z} - \frac{x}{R}\frac{\partial R}{\partial z}\frac{\partial w}{\partial x} = 0 \tag{9}$$

$$\frac{\partial w}{\partial t} = \frac{1}{R}[x(w\frac{\partial R}{\partial z} + \frac{\partial R}{\partial t}) - u]\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} + \frac{\mu}{\rho R^2}(\frac{\partial^2 w}{\partial x^2} + \frac{1}{x}\frac{\partial w}{\partial x}) - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{10}$$

with boundary conditions

$$u(x,z,t) = \frac{\partial R}{\partial t}, w(x,z,t) = 0 \text{ at } x = 1 \tag{11}$$

$$u(x,z,t) = 0,\ \frac{\partial w}{\partial x} = 0 \text{ at } x = 0 \tag{12}$$

$$u(x,z,0) = 0, w(x,z,0) = 0 \tag{13}$$

$$u(x,z,t) = 0,\ w(x,z,t) = \frac{5}{3}\left(1 - x^3\right) \text{ at } z = 0 \tag{14}$$

$$\frac{\partial w(x,z,t)}{\partial z} = 0 = \frac{\partial u(x,z,t)}{\partial z} \text{ at } z = L \tag{15}$$

Multiplying equation (9) by xR and integrating w.r.t. x within limits 0 to x,

$$\int_0^x x\frac{\partial u}{\partial x}dx + \int_0^x u dx + \int_0^x xR\frac{\partial w}{\partial z}dx - \int_0^x x^2\frac{\partial R}{\partial z}\frac{\partial w}{\partial x}dx = 0 \tag{16}$$

Which gives us

$$u(x,z,t) = xR\frac{\partial w}{\partial z} - \frac{R}{x}\int_0^x(x\frac{\partial w}{\partial z}dx) - \int_0^x x^2\frac{\partial R}{\partial z}\frac{\partial w}{\partial x}dx \tag{17}$$

Using boundary condition (11), we will get

$$\int_0^1 x\frac{\partial w}{\partial z}dx - \int_0^1 x[\frac{2}{R}\frac{\partial w}{\partial z}w + \frac{1}{R}\frac{\partial R}{\partial t}f(x)]dx = 0 \tag{18}$$

Here, choice of the f(x) is arbitrary. Let f(x) be of the form, $f(x) = \frac{10}{3}\left(1 - x^3\right)$ satisfying

$\int_0^1 xf(x)dx = 1$
Considering equality between the integrals and integrands from (15), we will get

$\frac{\partial w}{\partial z} = -\frac{2}{R}\frac{\partial R}{\partial z}w + \frac{10}{3R}\frac{\partial R}{\partial t}(x^3 - 1)$

Using this equation in equation (17)

$$u(x, z, t) = x\frac{\partial R}{\partial z}w + 0.3333\frac{\partial R}{\partial t}(5x - 2x^4) \qquad (19)$$

Solving equation (10) using finite difference approximations in which central differences have been used.

$\frac{\partial w}{\partial x} = \frac{w_{i,j+1}^k - w_{i,j-1}^k}{2\Delta x}$, $\frac{\partial w}{\partial z} = \frac{w_{i+1,j}^k - w_{i-1,j}^k}{2\Delta z}$

Also the time derivative can be approximated as

$\frac{\partial w}{\partial t} = \frac{w_{i,j}^{k+1} - w_{i,j}^k}{\Delta t}$

$\frac{\partial^2 w}{\partial x^2} = \frac{\left(w_{i,j+1}^k - 2w_{i,j}^k + w_{i,j-1}^k\right)}{(\Delta x)^2}$

Where $x_j = (j-1)\Delta x$, $z_i = (i-1)\Delta z$, and $t_k = (k-1)\Delta t$, Δx is the increment in radial direction, Δz is the increment in the axial direction and Δt is the increment in t.
Using above differences, equation (10) can be written as

$$w_{i,j}^{k+1} = w_{i,j}^k + \Delta t \left(\frac{-1}{\rho}\left(\frac{\partial p}{\partial z}\right)^{k+1} - w_{i,j}^k\left(\frac{w_{i+1,j}^k - w_{i-1,j}^k}{2\Delta z}\right) + \left(\frac{x_j}{R_i^k}\right) w_{i,j}^k \left(\frac{\partial R}{\partial z}\right)_i^k + \frac{x_j}{R_i^k}\left(\frac{\partial R}{\partial z}\right)_i^k - \frac{u_{i,j}^k}{R_i^k}\left(\frac{w_{i,j+1}^k - w_{i,j-1}^k}{2\Delta x}\right) + \frac{\mu}{\rho (R_i^k)^2}\left(\frac{w_{i,j+1}^k - 2w_{i,j}^k + w_{i,j}^k}{(\Delta x)^2}\right) + \frac{1}{x_j}\left(\frac{w_{i,j+1}^k - w_{i,j-1}^k}{2\Delta x}\right)\right) \qquad (20)$$

We solve equation (20) for value of w by using boundary conditions $(11)-(15)$. After obtaining axial velocity, radial velocity can be obtained using equation (19). For numerical calculations following data is used. This data have been made use of Chakravarty S. et al.[4] and Mandal P.K. [13]
$a = 0.8mm, \rho = 1.024 * 10^3 kg/m^3, L = 30mm, l_0 = 10mm$, $d = 4.5mm, T = 1s$, $f_p = 1.2Hz, \mu = 0.035P, A_0 = 500kgm^{-2}s^{-2}, A_1 = 0.2A_0, \tau_m = 0.4a$

Results

Using the parameters shown above, the shape function R(z,t) is plotted over the length of the artery z taken into consideration.

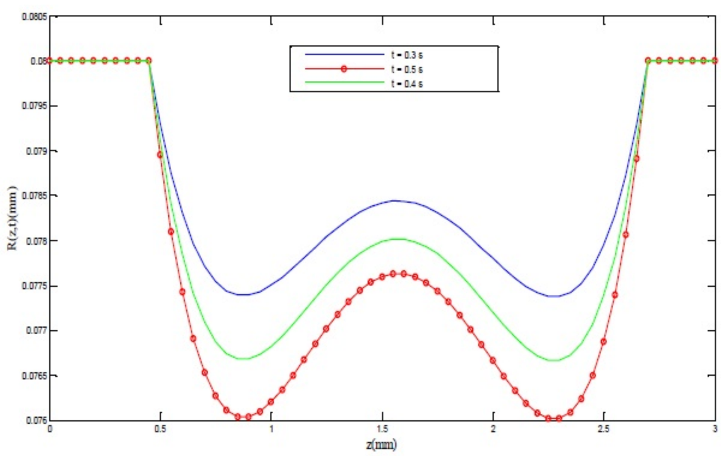

Figure 2. Shape function for varying time

Figure 2 shows distribution of time dependent stenosis over the length of the artery for different time periods. There are three different curves correspond to the distributions of the shape of the stenosis for three different times. As the time increases, we can observe concavity in the curves. Therefore, variation of stenosis with time is important factor in studying the blood flow through stenosed arterial system. Also the curves follow the shape of the stenosis over the length of artery and remain constant before and after the stenosis region.

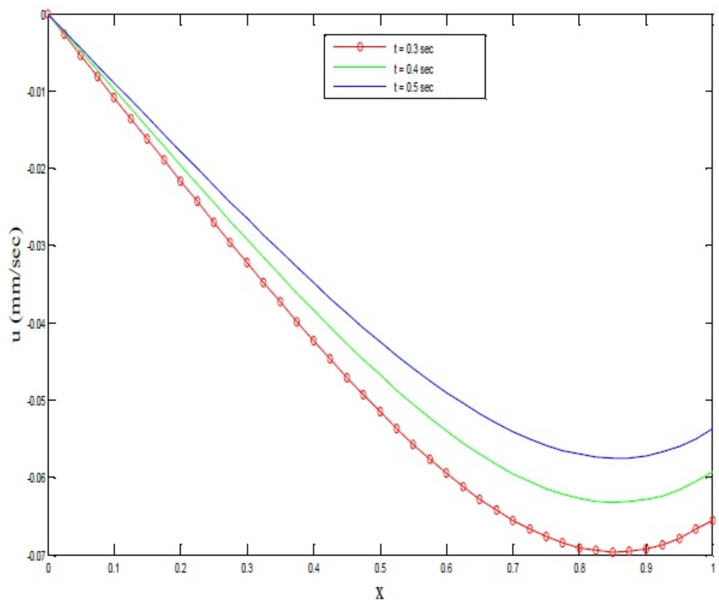

Figure 3. Radial variation of radial velocity for different times (z = 20mm)

Figure 3 shows radial variation of radial velocity for different times at $z = 20mm$ from the inlet of artery. It is observed that all the curves of radial velocity decrease from zero on the axis as one moves away from the axis and finally increase towards the wall to attend some finite value on the wall which is due to the presence of wall motion. Also, it is clear from the graph that as time increases, the radial velocity also increases in the same pattern. The present figure also displays the results for the flowing blood having Newtonian rheology and it turns out that Newtonian characteristics of the flowing blood affects the radial velocity.

Conclusion

The paper investigated that the radial velocity decreases radially in the stenosis region. The results indicating the unsteady behavior of the flowing blood over a single cardiac cycle, presented in the Fig. 3 in the stenosis region are found to be quite interesting to note. The radial velocity profile assumes negative values as we increase time from 0.3 s to 0.5 s which causes back flow.

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