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Pointwise estimate for the Bergman Kernel of holomorphic line bundles

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Abstract. Under curvatures conditions, we prove upper pointwise estimates for the Bergman kernel of the L^2-space of holomorphic sections of a holomorphic hermitian line bundle over a Stein Kähler manifold with bounded geometry.

1 Introduction and statment of the main result

Let L be a holomorphic hermitian line bundle over a complex manifold M, and let (U_j) be a covering of the manifold by open sets over which L is locally trivial. A section s of L is represented by a collection of complex valued functions f_j on U_j that are related by the holomorphic transition functions (g_{jk}) of the bundle

$$f_j = g_{jk} f_k \quad \text{on } U_j \cap U_k$$

We say that s is holomorphic if each f_i is holomorphic on U_i. A metric h on L is given by a collection of real valued functions Φ_j on U_j , related so that

$$|s|_h^2 := |f_j|^2 e^{-\Phi_j} \text{ on } U_j$$

is globally well defined. We will write h for the collection (Φ_j) , and refer to h as the metric on L. We say that L is positive, $L > 0$, if h can be chosen smooth with curvature

$$c(L) := i\partial\bar{\partial}\Phi_j$$

strictly positive, and that L is semipositive, $L \geq 0$, if it has a smooth metric of semipositive curvature. We say that $(L, h) \longrightarrow (M, g)$ has bounded curvature if $-M\omega_g \leq c(L) \leq M\omega_g$ for some positive constant M. Let $\mathcal{F}^2(M, L)$ the Hilbert space of holomorphic sections $s : M \longrightarrow L$ such that

$$\|s\|_2 := \Big(\int_M |s|_h^2 dv_g\Big)^{\frac{1}{2}} < \infty$$

Let P the orthogonal projection from the Hilbert space of $L^2(M, L)$ onto its closed subspace $\mathcal{F}^2(M, L)$.Let $K \in C^\infty(M \times M, L \otimes \overline{L})$ the reproducing (or Bergman) kernel of P, that is

$$K(z, w) = \sum_{j=1}^{d} s_j(z) \otimes \overline{s_j(w)} \;\in\; L_z \otimes \overline{L_w}$$

where $\overline{L}$ is the conjugate bundle of L which is the hermitian anti-holomorphic line bundle $\overline{L}$ whose transition functions are $(\overline{g}_{jk})$, (s_j) is an orthonormal basis for $\mathcal{F}^2(M, L)$ and $0 \leq d = dim\mathcal{F}^2(M, L) \leq \infty$. The distribution kernel K is called the Bergman Kernel of $(L, h) \longrightarrow (M, g)$. For all $s \in L^2(M, L)$

$$(Ps)(z) = \int_M K(z, w) \bullet s(w) dv_g(w)$$

where

$$K(z,w) \bullet s(w) = \sum_{j=1}^{d} < s(w), s_j(w) > s_j(z)$$

Since

$$|K(z,w)|^2 = \sum_j \sum_k < s_j(z), s_k(z) > \overline{< s_j(w), s_k(w) >}$$

then $K(z,w)$ is Hermitian : $|K(z,w)| = |K(w,z)|$. The function $|K(z,z)|$ is called the Bergman function of $\mathcal{F}^2(M,L)$. It satisfies

$$|K(z,z)| = \int_M |K(z,w)|^2 dv_g(w)$$

The main result of this paper is an estimate for the Bergman kernel of L similar to those obtained in [4,10] for weighted trivial line bundles with bounded curvature.

Theorem 1.1. *Let (M,g) be a Stein Kähler manifold with bounded geometry. Let $(L,h) \longrightarrow (M,g)$ be a hermitian holomorphic line bundle with bounded curvature such that*

$$c(L) + Ricci(g) \geq a\omega_g$$

for some positive constant a. There are constants $\alpha,\ C > 0$ such that for all $z, w \in M$,

$$|K(z,w)| \leq Ce^{-\alpha d_g(z,w)}$$

where d_g is the geodesic distance associated to the metric g.

From the above estimate for the Bergman kernel, we obtain the boundedness of the Bergman projection from $L^p(M,L)$ to $\mathcal{F}^p(M,L)$.

Proposition 1.2. *Let (M,g) be a Stein Kähler manifold with bounded geometry such that for all $\epsilon > 0$*

$$\sup_{z\in M} \int_M e^{-\epsilon d_g(z,w)} dV_g(w) < \infty$$

Let $(L,h) \longrightarrow (M,g)$ be a hermitian holomorphic line bundle with bounded curvature such that

$$c(L) + Ricci(g) \geq a\omega_g$$

for some positive constant a. Let $p \in [1,+\infty]$. Then the Bergman projection is bounded as a map from $L^p(M,L)$ to $\mathcal{F}^p(M,L)$.

2 Background

For the proof of Theorem 1.1, we need some notation and background.

Definition 2.1. A Hermitian manifold (M,g) is said to have bounded geometry if there exists positive numbers R and c such that for all $z \in M$ there exists a biholomorphic mapping $F_z : (U,0) \subset \mathbb{C}^n \longrightarrow (V,z) \subset M$ such that
(i) $F_z(0) = z$,
(ii) $B_g(z,R) \subset F_z(U)$ and
(iii) $\frac{1}{c} g_e \leq F_z^* g \leq c g_e$ on $F_z^{-1}(B_g(z,R))$ where g_e is the euclidean metric.

By (iii)

$$\forall\, w \in B_g(z,R)\ :\ \frac{1}{c}\|F_z^{-1}(w)\|_e \leq d_g(w,z) \leq c\|F_z^{-1}(w)\|_e$$

Remark 2.2. If an Hermitian manifold (M,g) has bounded geometry then the geodesic exponential map $\exp_z : T_z^{\mathbb{R}}M \to M$ is defined on a ball $B(0,r) \subset T_z^{\mathbb{R}}M$ for any $r < R$ and provide a diffeomorphism of this ball onto the ball $B_g(z,r) \subset M$. It follows that the manifold (M,g) is complete.

Remark 2.3. It is well known that if (M,g) has bounded geometry and $Ric(g) \geq Kg$ then (M,g) satisfy the uniform ball size condition ([3] Prop. 14), i.e. for every $r \in \mathbb{R}^+$

$$\inf_{z\in M} vol(B_g(z,r)) > 0 \quad \text{and} \quad \sup_{z\in M} vol(B_g(z,r)) < \infty$$

Also by volume comparison theorem [2], there are nonnegative constants C, α, β such that

$$vol_g(B_g(z,r)) \leq Cr^{\alpha}e^{\beta r}, \quad \forall\, r \geq 1,\ z \in M$$

In particular if (M,g) has polynomial volume growth, i.e $\beta = 0$, then

$$\sup_{z\in M}\int_M e^{-\epsilon d_g(z,w)}dV_g(w) = \sup_{z\in M}\int_0^\infty \mathrm{vol}_g(\partial B(z,r))e^{-\epsilon r}dr \leq C(\epsilon)$$

Bounded geometry allows one to produce an exhausion function which behaves like the distance function and whose gradient and hessian are bounded on M [9].

Lemma 2.4. *Let (M,g) be a Hermitian manifold with bounded geometry. For every $z \in M$ there exists a smooth function $\Psi_z : M \longrightarrow \mathbb{R}$ such that*
(i) $C_1 d_g(.,z) \leq \Psi_z \leq C_2(d_g(.,z)+1)$,
(ii) $|\partial\Psi_z|_g \leq C_3$, and
(iii) $-C_4\omega_g \leq i\partial\bar{\partial}\Psi_z \leq C_5\omega_g$.
Furthermore, the constants in (i), (ii) and (iii) depend only on the constants associated with the bounded geometry of (M,g).

We recall Demailly's theorem [5], which generalizes Hörmander's L^2 estimates [6] (Theorem 2.2.1, p. 104) for forms with values in a line bundle.

Theorem 2.5. *Let (X,ω) be a complete Kähler manifold, (L,h) a holomorphic hermitian line bundle over X, and let ϕ be a locally integrable function over X. If the curvature $c(L)$ is such that*

$$c(L) + Ric(\omega) + i\partial\bar{\partial}\phi \geq \gamma\omega$$

for some positive and continuous function γ on X, then for all $v \in L^2_{(0,1)}(X,L,loc)$, $\bar{\partial}$-closed and such that

$$\int_X \gamma^{-1}|v|^2 e^{-\phi}dv_\omega < \infty$$

there exists $u \in L^2(X,L)$ such that

$$\bar{\partial}u = v \quad \text{and} \quad \int_X |u|_h^2 e^{-\phi}dv_\omega \leq \int_X \gamma^{-1}|v|^2_{\omega,h}e^{-\phi}dv_\omega$$

Also, we recall J.McNeal-D.Varolin's theorem [8](Theorem 2.2.1, p. 104), which generalizes Berndtsson-Delin's improved L^2-estimate of $\bar{\partial}$-equation having minimal L^2-norm [1],[4] for forms with values in a line bundle.

Theorem 2.6. *Let (M,g) be a Stein Kähler manifold, and $(L,h) \longrightarrow (M,g)$ a holomorphic hermitian line bundle with Hermitian metric h. Suppose there exists a smooth function $\eta : M \to \mathbb{R}$ and a positive, a.e. strictly positive Hermitian $(1,1)$-form Θ on M such that*

$$c(L) + Ric(g) + i\partial\bar{\partial}\eta - i\partial\eta \wedge \bar{\partial}\eta \geq \Theta$$

Let v be an L-valued $(0,1)$-form such that $v = \bar{\partial}u$ for some L-valued section u satisfying

$$\int_M |u|_h^2 dv_g < \infty$$

Then the solution u_0 of $\bar{\partial}u = v$ having minimal L^2-norm i.e

$$\int_M < u_0, \sigma > dv_g = 0 \text{ for all } \sigma \in \mathcal{F}^2(M,L)$$

satisfies the estimate

$$\int_M |u_0|_h^2 e^{\eta}dv_g \leq \int_M |v|^2_{\Theta,h}e^{\eta}dv_g.$$

3 Preliminary results

3.1 Weighted Bergman Inequalities

Proposition 3.1. *Let (M, g) be a complete noncompact Kähler manifold with bounded geometry and lower Ricci curvature bound. Let $(L, h) \longrightarrow (M, g)$ be a hermitian holomorphic line bundle with bounded curvature. Fix $p \in]0, \infty[$. Then for each $r > 0$ there exists a constant C_r such that if $s \in \mathcal{F}^2(M, L)$ then*

$$|s(z)|^p \leq C_r^p \int_{B_g(z,r)} |s|^p dv_g \tag{3.1}$$

in particular $\mathcal{F}^p(M, L) \subset \mathcal{F}^\infty(M, L)$ and

$$|\nabla |s(z)|^p|_g(z) \leq C_r^p \int_{B_g(z,r)} |s|^p dv_g \tag{3.2}$$

Proof. Since (M, g) has bounded geometry there exists positive numbers R and c such that for all $z \in M$ there exists a biholomorphic mapping $\Psi_z : (U, 0) \subset \mathbb{C}^n \longrightarrow (V, z) \subset M$ such that
(i) $\Psi_z(0) = z$,
(ii) $B_g(z, R) \subset \Psi_z(U)$ and
(iii) $\frac{1}{c} g_e \leq \Psi_z^* g \leq c g_e$ on $\Psi_z^{-1}(B_g(z, R))$ where g_e is the euclidean metric.
Consider the $(1, 1)$-form defined on $B_e(0, \delta(R)) \subset\subset \Psi_z^{-1}(B_g(z, R)) \subset \mathbb{C}^n$ by

$$\Theta := \Psi_z^* c(L)$$

Since $-K\omega_g \leq c(L) \leq K\omega_g$, by [11] Lemma 4.1 there exists a function $\phi \in C^2(B_e(0, \delta))$ such that

$$i\partial\bar{\partial}\phi = \Theta \quad \text{and} \quad \sup_{B_e(0,\delta)} (|\phi| + |d\phi|_{g_e}) \leq M$$

On $B_g(z, \eta) \subset\subset \Psi_z(B_e(0, \delta(R))$, consider the C^2-function

$$\psi := \phi \circ \Psi_z^{-1}$$

By (iii) we have

$$i\partial\bar{\partial}\psi = c(L) \quad \text{and} \quad \sup_{B_g(z,\eta)} (|\psi| + |\nabla\psi|_g) \leq M^{'}$$

where $M^{'}$ and η depend only on R and c.
Let e be a frame of L arround $z \in B_g(z, \eta)$ and $\Phi(w) = -\log |e(w)|^2$. Then $i\partial\bar{\partial}\psi = i\partial\bar{\partial}\Phi$ on $B_g(z, \eta)$. Hence the function

$$\rho(w) = \Phi(w) - \Phi(z) + \psi(z) - \psi(w)$$

is pluriharmonic. Then $\rho = \Re(F)$ for some holomorphic function F with $\Im(F)(z) = 0$ and

$$\sup_{B_g(z,\eta)} |\Phi - \Phi(z) - \Re(F)| = \sup_{B_g(z,\eta)} |\psi - \psi(z)| \leq C \tag{3.3}$$

$$\sup_{B_g(z,\eta)} |\nabla(\Phi - \Phi(z) - \Re(F))|_g = \sup_{B_g(z,\eta)} |\nabla\psi|_g \leq C \tag{3.4}$$

We can suppose $0 < r \leq \eta$. According to [7] , for all $z \in M$ and all holomorphic functions f on $B_g(z, \eta)$ and all $\zeta \in B_g(z, \eta/2)$

$$|f(\zeta)|^p \leq \frac{C}{\text{Vol}(B_g(\zeta, \eta/2))} \int_{B_g(\zeta,\eta)} |f(w)|^p dv_g$$

where C depend only in K, n, η. Since g has sbounded geometry $\text{Vol}(B_g(z, \eta/2)) \succeq 1$ uniformly in z. Hence

$$|f(\zeta)|^p \leq C \int_{B_g(\zeta,\eta)} |f(w)|^p dv_g$$

Let $s \in \mathcal{F}^p(M,L)$ and $s = fe$ on $B_g(z,\eta)$. We have

$$\begin{aligned} |s|_h^p &= |fe^{-\frac{F}{2}}|^p e^{-\frac{p}{2}\Phi(z)} e^{-\frac{p}{2}(\Phi-\Phi(z)-\Re(F))} \\ &\leq C^p |fe^{-\frac{F}{2}}|^p e^{-\frac{p}{2}\Phi(z)} \end{aligned}$$

By mean value inequality

$$\begin{aligned} |f(z)e^{-\frac{F(z)}{2}}|^p e^{-\frac{p}{2}\Phi(z)} &\leq c_r^p \int_{B_g(z,r)} |fe^{-\frac{F}{2}}|^p e^{-\frac{p}{2}\Phi(z)} dv_g \\ &\leq C_r^p \int_{B_g(z,r)} |fe^{-\Phi(w)}|^p dv_g \end{aligned}$$

Hence

$$|s(z)|_h^p \leq C_r^p \int_{B_g(z,r)} |s|^p dv_g$$

By (2.3) and (2.4)

$$\begin{aligned} |\nabla |s|_h^p|_g &\leq e^{-\frac{p}{2}\Phi(z)} e^{-\frac{p}{2}(\Phi-\Phi(z)-\Re(F)))} |\nabla |fe^{-\frac{F}{2}}|^p| \\ &+ \frac{p}{2} |fe^{-\frac{F}{2}}|^p e^{-\frac{p}{2}\Phi(z)} e^{-\frac{p}{2}(\Phi-\Phi(z)-\Re(F))} |\nabla(\Phi - \Phi(z) - \Re(F))|_g \\ &\leq e^{-\frac{p}{2}\Phi(z)} e^{-\frac{p}{2}(\Phi-\Phi(z)-\Re(F))} |\nabla |fe^{-\frac{F}{2}}|^p| \\ &+ \frac{p}{2} |s|_h^p e^{-\frac{p}{2}(\Phi-\Phi(z)-\Re(F))} |\nabla(\Phi - \Phi(z) - \Re(F))|_g \\ &\leq C^p \big(e^{-\frac{p}{2}\Phi(z)} |\nabla |fe^{-\frac{F}{2}}|^p| + \frac{p}{2} |s|_h^p \big) \end{aligned}$$

By mean value inequality (Cauchy formula for partial derivates), there exists $c_r > 0$ such that

$$\begin{aligned} |\nabla |fe^{-\frac{F}{2}}|^p|(z) &\leq c_r^p \int_{B_g(z,r)} |fe^{-\frac{F}{2}}|^p dv_g \\ &\leq C_r^p \int_{B_g(z,r)} |s|^p dv_g \end{aligned}$$

From this it follows

$$|\nabla |fe^{-\frac{F}{2}}|^p|(z) e^{-\frac{p}{2}\Phi(z)} \leq c_r^p \int_{B_g(z,r)} |fe^{-\frac{F}{2}}|^p e^{-\frac{p}{2}\Phi(z)} dv_g \leq C_r^p \int_{B_g(z,r)} |s|^p dv_g$$

Thus we get (2.2).

3.2 Slow Growth of Bergman Sections

Lemma 3.2. *Let (M,g) be a Kähler manifold with bounded geometry and lower Ricci curvature bound. Let $(L,h) \longrightarrow (M,g)$ be a hermitian holomorphic line bundle with bounded curvature. Then there exists $\delta > 0$ with the following properties : if $z \in M,\ s \in \mathcal{F}^p(M,L),\ \|s\|_p \leq 1$ then*

$$|s(z)|_h \geq a \Longrightarrow |s(w)|_h \geq \frac{a}{2},\ \forall\, w \in B_g(z,\delta).$$

Proof. Le $R > \delta > 0$. By (3.2) of proposition 3.1 and mean value theorem for all $w \in B_g(z, R/2)$

$$\begin{aligned} \big| |s(w)|_h^p - |s(z)|_h^p \big| &\leq C_r^p d_g(w,z) \Big(\int_{B_g(z,R)} |s(\zeta)|^p dv_g \Big) \\ &\leq \delta C_R^p \|s\|_p^p \end{aligned}$$

Hence if δ is small enough

$$\forall w \in B_g(z,\delta) \ :\ |s(w)|_h^p \geq a^p - \delta C_R^p \geq \frac{a^p}{2^p}$$

3.3 Diagonal Bounds for the Bergman Kernel

As a consequence of (3.1) proposition 3.1, we obtain the following proposition.

Proposition 3.3. *Let (M, g) be a complete noncompact Kähler manifold with bounded geometry and lower Ricci curvature bound. Let $(L, h) \longrightarrow (M, g)$ be a hermitian holomorphic line bundle with bounded curvature. There is a constant $C > 0$ such that for all $z \in M \; : \; |K(z, z)| \preceq C$. Therefore $|K(z, w)| \leq C$ for all $z, w \in M$.*

Proof. Proof. Let (s_j) be a orhonormal basis of $\mathcal{F}^2(M, L)$. By definition of the Bergman Kernel

$$K(z, w) = \sum_j s_j(z) \otimes \overline{s_j(w)}$$

By (3.1) of Proposition 3.1 the evaluation

$$\begin{aligned} ev_z \quad : \quad & \mathcal{F}^2(M, L) \longrightarrow L_z \\ & s \longrightarrow s(z) \end{aligned}$$

is continuous and

$$\|ev_z\| = |K(z, z)| \preceq 1$$

uniformly in $z \in M$. Hence

$$\begin{aligned} |K(z, w)| \quad & \leq \quad \sum_j |s_j(z)||s_j(w)| \\ & \leq \quad \sqrt{|K(z, z)|}\sqrt{|K(w, w)|} \preceq 1 \end{aligned}$$

□

The following result gives bounds for the Bergman kernel in a small but uniform neighborhood of the diagonal

Proposition 3.4. *Let (M, g) be a complete noncompact Kähler manifold with bounded geometry and lower Ricci curvature bound. Let $(L, h) \longrightarrow (M, g)$ be a hermitian holomorphic line bundle with bounded curvature. There are constants $\delta, C_1, C_2 > 0$ such that for all $z \in M$ and $w \in B_g(z, \delta)$*

$$C_1|K(z, z)| \leq |K(z, w)| \leq C_2|K(z, z)|$$

Proof. Let $z \in M$. Fix a frame e in a neighborhood U of the point z and consider an orhonormal basis $(s_j)_{j=1}^d$ of $\mathcal{F}^2(X, L)$ (where $1 \leq d \leq \infty$). In U each s_i is represented by a holomorphic function f_i such that $s_i(x) = f_i(x)e(x)$. Let

$$s_z(w) := |e(z)| \sum_{i=1}^d \overline{f_i(z)} s_i(w)$$

Then

$$\begin{aligned} |s_z(w)| \quad & = \quad \Big|\Big(\sum_{i=1}^d \overline{f_i(z)} s_i(w)\Big) \otimes \overline{e(z)}\Big| \\ & = \quad \Big|\sum_{i=1}^d s_i(w) \otimes \overline{s_i(z)}\Big| \\ & = \quad |K(w, z)| \end{aligned}$$

and

$$\begin{aligned} \int_M |s_z|^2 dv_g(w) \quad & = \quad \int_M |K(w, z)|^2 dv_g(w) \\ & = \quad |K(z, z)| \preceq 1 \end{aligned}$$

Hence, by lemma 3.2, there exists $C, \delta > 0$ independant of z such that

$$|K(w, z)| = |s_z(w)| \geq C|s_z(z)| = C|K(z, z)|$$

for all $w \in B_g(z, \delta)$. □

4 Proofs of Theorem 1.1 and Proposition 1.2

4.1 Proof of Theorem 1.1

Let $z, w \in M$ such that $d_g(z,w) \geq \delta$ where $\delta > 0$ as in Proposition 3.4. Fix a smooth function $\chi \in C_0^\infty(B_g(w, \delta/2))$ such that
(i) $0 \leq \chi \leq 1$,
(ii) $\chi = 1$ in $B_g(w, \delta/4)$,
(iii) $|\bar\partial \chi|_g \preceq \chi$.
Let $s_z \in \mathcal{F}^2(M, L)$ defined by

$$s_z(w) := |e(z)| \sum_{i=1}^{d} \overline{f_i(z)} s_i(w)$$

where $(s_i)_{1 \leq i \leq d}$ is an orthonormal basis of $\mathcal{F}^2(M, L)$ and e is a local frame of L around z. Then $|s_z(w)| = |K(w,z)|$ and $\|s_z\|_2 = |K(z,z)| \preceq 1$. Also

$$s_z(w) \otimes \overline{\frac{e(z)}{|e(z)|}} = K(w,z)$$

By (3.1) of Proposition 3.1

$$|s_z(w)|^2 \preceq \int_{B(w,\delta/2)} \chi(\zeta)|s_z(\zeta)|^2 dv_g \preceq \|s_z\|^2_{L^2(\chi dv_g)}$$

We have

$$\|s_z\|_{L^2(\chi dV_g)} = \sup_\sigma | < \sigma, s_z >_{L^2(\chi dv_g)} |$$

where $\sigma \in \mathcal{F}^2(B_g(z,\delta), L)$ such that $\|\sigma\|_{L^2(\chi dv_g)} = 1$. Since

$$\begin{aligned}
\Big| < \sigma, s_z >_{L^2(\chi dv_g)} \Big|_{\mathbb{C}} &= \Big| \int_M < \chi(w)\sigma(w), s_z(w) > dv_g(w) \Big|_{\mathbb{C}} \\
&= \Big| \sum_{i=1}^{d} \int_M < \chi(w)\sigma(w), |e(z)|\overline{f_i(z)} s_i(w) > dv_g(w) \Big|_{\mathbb{C}} \\
&= \Big| \sum_{i=1}^{d} \int_M < \chi(w)\sigma(w), s_i(w) > f_i(z)|e(z)| dv_g(w) \Big|_{\mathbb{C}} \\
&= \Big| \sum_{i=1}^{d} \int_M < \chi(w)\sigma(w), s_i(w) > f_i(z)e(z) dv_g(w) \Big|_{L_z} \\
&= \Big| \sum_{i=1}^{d} \int_M < \chi(w)\sigma(w), s_i(w) > s_i(z) dv_g(w) \Big|_{L_z} \\
&= \Big| \int_M K(z,w) \bullet \chi(w)\sigma(w) dv_g(w) \Big|_{L_z} \\
&= |P(\chi\sigma)(z)|_{L_z}
\end{aligned}$$

then

$$\|s_z\|_{L^2(\chi dV_g)} = \sup_\sigma |P(\chi\sigma)(z)|$$

Since $c(L) + Ricci(g) \geq ag$, by Theorem 2.5 there exists a solution u of $\bar\partial u = \bar\partial\chi.\sigma$ such that

$$\int_M |u|^2 dv_g \preceq \int_M |\bar\partial\chi|_g^2 |\sigma|^2 dv_g < \infty$$

Let $u_\sigma = \chi\sigma - P(\chi\sigma)$ be the solution having minimal L^2-norm of

$$\bar\partial u = \bar\partial\chi.\sigma$$

Since $\chi(z)=0$

$$\Big| < \sigma, s_z >_{L^2(\chi dv_g)} \Big|_{\mathbb{C}} = |P(\chi\sigma)(z)|_{L_z} = |u_\sigma(z)|_{L_z}$$

Since $B(z,\delta/2)\cap B(w,\delta/2)=\emptyset$, the section u_σ is holomorphic in $B_g(z,\delta/2)$. Let $\epsilon\in]0,2/\delta]$, By (3.1) Proposition 3.1

$$|u_\sigma(z)|^2_{L_z} \preceq \int_{B_g(z,\delta/2)} |u_\sigma(\zeta)|^2_{L_\zeta} dv_g \preceq \int_{B_g(z,\delta/2)} e^{-\epsilon d(\zeta,z)} |u_\sigma(\zeta)|^2_{L_\zeta} dv_g \qquad (4.1)$$

Let $\eta := -\epsilon\Phi_z$ where Φ_z is as in lemma 2.4 and $\Theta=\epsilon\omega_g$. Choose ϵ small enough such that

$$c(L) + Ricci(g) - i\epsilon\partial\bar\partial\Phi_z - i\epsilon^2\partial\Phi_z\wedge\bar\partial\Phi_z - \epsilon\omega_g \geq 0$$

By Theorem 2.6

$$\int_M e^{-\epsilon\Phi_z}|u_\sigma|^2 dv_g \preceq \int_M e^{-\epsilon\Phi_z}|\bar\partial\chi|^2_g|\sigma|^2 dv_g$$

Since $C_1 d_g(.,z)\leq \Phi_z \leq C_2(d_g(.,z)+1)$, by (4.1)

$$|u_\sigma(z)|^2_{L_z} \preceq \int_M e^{-\epsilon C_1 d_g(\zeta,z)}\chi(\zeta)|\sigma(\zeta)|^2 dv_g$$

Since $\zeta\in B_g(w,\delta)$ we have

$$\begin{aligned} d_g(\zeta,z) &\geq d_g(z,w) - d_g(w,\zeta) \\ &\succeq d_g(z,w) - \delta \succeq d_g(z,w) \end{aligned}$$

Finally

$$|K(z,w)| \preceq \sup_\sigma |u_\sigma(z)|_{L_z} \preceq e^{-\alpha d_g(z,w)}.$$

4.2 Proof of Proposition 1.2

If $p=\infty$, we have

$$\begin{aligned} \|Ps\|_\infty &= \Big\| \int_M K(z,w).s(w)dv_g(w)\Big\|_\infty \\ &\leq \|s\|_\infty \sup_{z\in M}\int_M |K(z,w)|dv_g(w) \\ &\preceq \|s\|_\infty \sup_{z\in M}\int_M e^{-\alpha d_g(z,w)}dv_g(w) \\ &\preceq \|s\|_\infty \end{aligned}$$

and then P is bounded from $L^\infty(M,L)$ to $\mathcal{F}^\infty(M,L)$.
If $p\in[1,\infty[$, we have

$$\begin{aligned} \int_M |Ps(z)|^p dv_g(w) &= \int_M \Big|\int_M K(z,w).s(w)dv_g(w)\Big|^p dv_g(z) \\ &\leq \int_M \Big|\int_M |s(w)||K(z,w)|dv_g(w)\Big|^p dv_g(z) \\ &\leq \int_M \Big(\Big(\int_M |K(z,w)|dv_g(w)\Big)^{p-1} \\ &\quad \times \int_M |s(w)|^p|K(z,w)|dv_g(w)\Big) dv_g(z) (\text{ Jensen inequality}) \\ &\preceq \int_M \Big(\int_M e^{-\alpha d_g(w,z)}dv_g(w)\Big)^{p-1} \\ &\quad \times \int_M |s(w)|^p|K(z,w)|dv_g(w)\Big) dv_g(z) \end{aligned}$$

Thus

$$\int_M |Ps(z)|^p dv_g(w) \preceq \int_M \int_M |s(w)|^p e^{-\alpha d_g(w,z)} dv_g(w) dv_g(z)$$
$$\preceq \int_M |s(w)|^p dv_g(w)$$

and then P is bounded from $L^p(M,L)$ to $\mathcal{F}^p(M,L)$.

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