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ON TRACES OF PERMUTING n-DERIVATIONS AND PERMUTING GENERALIZED n-DERIVATIONS ON JORDAN IDEALS

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Abstract. In this paper we investigate some proprieties of permuting n-derivations acting on a Jordan Ideal of prime-rings. Some related results for left $n-$multipliers and generalized $n-$derivations are also discussed.

1 Introduction

In this paper, R will represent an associative ring. For any $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. R is 2-torsion free if whenever $2x = 0$, with $x \in R$ implies $x = 0$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$, for all $u \in J$ and $r \in R$. A mapping $f : R \longrightarrow R$ is said to be centralizing (resp. commuting) on a subset S of R if $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$) for all $x \in S$. A derivation on R is an additive mapping $d : R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xF(y)$ for all $x, y \in R$. In this case, F is called the generalized derivation associated with d. For a fixed positive integer n, a map $\Delta : R^n \longrightarrow R$ is n-additive if it satisfies $\Delta(x_1, x_2, ..., x_i + x_i', ..., x_n) = \Delta(x_1, x_2, ..., x_i, ..., x_n) + \Delta(x_1, x_2, ..., x_i', ..., x_n)$ for all $x_i, x_i' \in R$, $i = 1, 2, ..., n$. A map $\Delta : R^n \longrightarrow R$ is said to be permuting if $\Delta(x_1, x_2, x_3, ..., x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, ..., x_{\pi(n)})$ for all $x_i \in R$ and for every permutation $\pi(1), \pi(2), \pi(3), ..., \pi(n)$. A map $\delta : R \longrightarrow R$ is called the trace of Δ if $\delta(x) = \Delta(x, x, x, ..., x)$ for all $x \in R$. It is obvious to verify that if $\Delta : R \longrightarrow R$ is a permuting and n-additive mapping, then the trace δ of Δ satisfies the relation

$$\delta(x+y) = \delta(x) + \delta(y) + \sum_{i=1}^{n-1} \binom{n}{i} \Delta(x, x, ..., x, y, y, ..., y)$$

where x appears $(n-i)$-times and y appears i-times.
Park [9] introduced the notion of permuting n-derivation as follows: a permuting map $\Delta : R^n \longrightarrow R$ is said to be a permuting n-derivation if Δ is n-additive and $\Delta(x_1, x_2, ..., x_i x_i^{'}, ..., x_n) = x_i \Delta(x_1, x_2, ..., x_i^{'}, ..., x_n) + \Delta(x_1, x_2, ..., x_i, ..., x_n) x_i^{'}$ for all $x_i, x_i^{'} \in R$. Clearly, a 1-derivation is a usual derivation and a 2-derivation is a symmetric bi-derivation. However, in the case of $n = 3$ we get the concept of tri-derivation.
An n-additive mapping $\Omega : R^n \longrightarrow R$ is called a generalized n-derivation of R with associated n-derivation Δ if

$$\Omega(x_1, x_2, ..., x_i x_i^{'}, ..., x_n) = \Omega(x_1, x_2, ..., x_i, ..., x_n) x_i^{'} + x_i \Omega(x_1, x_2, ..., x_i', ..., x_n)$$

for all $x_i, x_i^{'} \in R$.
An additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$, for all $u \in J$ and $r \in R$. We shall use without explicit mention the fact that if J is a nonzero Jordan ideal of a ring R, then $2[R, R]J \subseteq J$ and $2J[R, R] \subseteq J$ ([8], Lemma 2.4). Moreover, from ([1], proof of Lemma 3) we have $4j^2R \subset J$ and $4Rj^2 \subset J$ for all $j \in J$. Since $4jrj = 2\{j(jr + rj) + (jr +$

$rj)j\} - \{2j^2 \cdot r + r \cdot 2j^2\}$, it follows that $4jRj \subset J$ for all $j \in J$ (see [1], proof of Theorem 3). Many results in literature indicate how the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R. A well known result due to Posner [7] states that a prime ring R which admits a nonzero centralizing derivation is commutative. Since then, several authors have done a great deal of work concerning commutativity of prime and semi-prime rings admitting different kinds of maps which are centralizing on some appropriate subsets of R ([3] and [6] for a further references). More recently several authors have studied various identities involving trace of permuting n-derivations (generalized n-derivation) and have obtained interesting theorems. In this paper we establish analogous results for the traces of permuting n-derivations (generalized n-derivation) acting on Jordan ideals.

2 Main results

The following lemmas are essential for developing the proofs of our results.

Lemma 2.1. ([6], Lemma 2.4) *Let n be a fixed positive integer and let R be a $n!-$torsion free prime ring. Suppose that $y_1, y_2, ..., y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + ... + \lambda^n y_n = 0$, (or $\in Z(R)$) for $\lambda = 1, 2, 3, ..., n$. Then $y_i = 0$ (or $y_i \in Z(R)$) for all i.*

Lemma 2.2. ([4])] *Let R be a prime ring. Let $d : R \to R$ be a derivation and $a \in R$. If $ad(x) = 0$ holds for all $x \in J$, then we have either $a = 0$ or $d = 0$.*

Proof. Assume that: $ad(x)$ for all $\in J$. Replacing x by $4xj^2$, where $j \in J$, we get $axd(j^2) = 0$, so that $aJd(j^2) = 0$. Applying ([8], Lemma 2.6) we have $a = 0$ or $d(j^2) = 0$ for all $j \in J$. In the last case,([5], Lemma 3) implies that $d = 0$. □

Theorem 2.3. *Let $n \geq 2$ be a fixed positive integer and R be a $(n+1)!$-torsion free prime ring and J a nonzero Jordan ideal of R. If R admits a permuting n-derivation Δ such that the trace δ satisfies $[[\delta(x), x], x] \in Z(R)$ for all $x \in J$. Then δ is commuting on J.*

Proof. We are given that

$$[[\delta(x), x], x] \in Z(R) \quad \text{for all} \quad x \in J. \tag{2.1}$$

An easier computation shows that the trace δ of Δ satisfies the relation:

$$\delta(x + y) = \delta(x) + \delta(y) + \sum_{r=1}^{n-1} \binom{n}{r} h_r(x, y) \quad \text{for all} \quad x, y \in J \tag{2.2}$$

where $h_r(x, y) = \Delta(x, x, x, ..., x, y, y, ..., y)$; y appears r times and x appears $n - r$ times. Consider a positive integer k, $1 \leq k \leq n + 1$. Replacing x by $x + ky$ in (2.1), we obtain

$$kQ_1(x, y) + k^2 Q_2(x, y) + ... + k^{n+1} Q_{n+1}(x, y) = 0 \quad \text{for all} \quad x, y \in J \tag{2.3}$$

where $Q_i(x, y)$ denotes the sum of the terms in which y appears i times. Using (2.3) together with Lemma 2.1, we have

$$[[\delta(x), x], y] + [[\delta(x), y], x] + n[[\Delta(x, x, x, ..., y), x], x] \in Z(R) \tag{2.4}$$

for all $x, y \in J$. Replacing y by $4y^2 z$ in (2.4), where $z \in J$, we get

$$[[\delta(x), x], y^2 z] + [[\delta(x), y^2 z], x] + n[[\Delta(x, x, x, ..., y^2 z), x], x] \in Z(R). \tag{2.5}$$

for all $x, y, z \in J$. Substituting $8xy^2 z$ for y in (2.4), we get

$$x([[\delta(x), x], y^2 z] + [[\delta(x), y^2 z], x] + n[[\Delta(x, x, x, ..., y^2 z), x], x]) + (n + 2)[[\delta(x), x], x]y^2 z$$
$$+(2n + 1)[\delta(x), x][y^2 z, x] + n\delta(x)[[y^2 z, x], x] \in Z(R) \tag{2.6}$$

which leads to

$$(3n + 3)[[\delta(x), x], x][y^2 z, x] + (3n + 1)[\delta(x), x][[y^2 z, x], x] + n\delta(x)[[[y^2 z, x], x], x] = 0 \tag{2.7}$$

Replacing z by $2z[\delta(x), x]$ in (2.7) and comparing with (2.4), we find that
$$(3n+3)[[\delta(x),x],x]^2y^2z + (6n+2)[\delta(x),x][y^2z,x][[\delta(x),x],x]$$
$$+3n\delta(x)[[[y^2z,x],x],x][[\delta(x),x],x] = 0 \tag{2.8}$$

and thus
$$(9n+5)[[\delta(x),x],x]^2[y^2z,x] + (9n+2)[\delta(x),x][[y^2z,x],x][[\delta(x),x],x]$$
$$+3n\delta(x)[[[y^2z,x],x],x][[\delta(x),x],x] = 0 \tag{2.9}$$

for all $x, y, z \in J$. It now follows, from (2.7) and (2.9), that
$$4[[\delta(x),x],x]^2[y^2z,x] + [\delta(x),x][[y^2z,x],x][[\delta(x),x],x] = 0. \tag{2.10}$$

Once again replacing z by $2z[\delta(x), x]$ and using (2.10), we obtain
$$[[\delta(x),x],x]^2(4[[\delta(x),x],x]y^2z + 2[\delta(x),x][y^2z,x]) = 0 \quad \text{for all} \;\; x,y,z \in J. \tag{2.11}$$

Writing $2z[r,t]$ instead of z in (2.11), where $r, t \in R$, we get
$$[[\delta(x),x],x]^2[\delta(x),x]y^2z[[r,t]x] = 0 \quad \text{for all} \;\; x,y,z \in J; \quad r,t \in R. \tag{2.12}$$

Since N is prime, then (2.12) shows that
$$[[\delta(x),x],x] = 0 \text{ for all } \; x \in J. \tag{2.13}$$

Replacing x by $x + ky$ in (2.13), we obtain
$$kQ_1(x,y) + k^2Q_2(x,y) + \ldots\ldots.. + k^{n+1}Q_{n+1}(x,y) = 0 \quad \text{for all} \;\; x,y \in J. \tag{2.14}$$

Once again applying Lemma 2.1, we obtain
$$[[\delta(x),x],y] + [[\delta(x),y],x] + n[[\Delta(x,x,x,....,y),x],x] = 0 \quad \text{for all} \;\; x,y \in J. \tag{2.15}$$

Replacing y by $8xzy^2uv$ in (2.15), where $u, v, z \in J$, we get
$$(2n+1)[\delta(x),x][zy^2uv,x] + n\delta(x)[[zy^2uv,x],x] = 0 \quad \text{for all} \;\; u,v,x,y,z \in J. \tag{2.16}$$

Replacing y by $8xrzy^2uv$ in (2.15), where $u, v, z \in J$ and $r \in R$, we get
$$(2n+1)[\delta(x),x][rzy^2uv,x] + n\delta(x)[[rzy^2uv,x],x] = 0 \quad \text{for all} \;\; u,v,x,y,z \in J; \; r \in R \tag{2.17}$$

in such a way that
$$(2n+1)[\delta(x),x](r[zy^2uv,x] + [r,x]zy^2uv) + n\delta(x)(2[r,x][zy^2uv,x]$$
$$+r[[zy^2uv,x],x] + [[r,x],x]zy^2uv) = 0 \tag{2.18}$$

for all $u, v, x, y, z \in J$, $r \in R$. Taking $r = \delta(x)$ in the last expression we get
$$(2n+1)[\delta(x),x](\delta(x)[zy^2uv,x] + [\delta(x),x]zy^2uv) + n\delta(x)(2[\delta(x),x][zy^2uv,x]$$
$$+\delta(x)[[zy^2uv,x],x]) = 0. \tag{2.19}$$

Invoking (2.17), we get
$$(2n+1)[\delta(x),x](\delta(x)[zy^2uv,x] + [\delta(x),x]zy^2uv) + 2n\delta(x)[\delta(x),x][zy^2uv,x]$$
$$-(2n+1)\delta(x)[\delta(x),x][zy^2uv,x] = 0 \tag{2.20}$$

and thus
$$(2n+1)\Big([\delta(x),x]\delta(x) - \delta(x)[\delta(x),x]\Big)[zy^2uv,x] + (2n+1)[\delta(x),x]^2zy^2uv = 0 \tag{2.21}$$

If we replace y by $8x\delta(x)zy^2$ in (2.15), then we get

$$\Big((2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]\Big)[zy^2,x]+(2n+1)[\delta(x),x]^2zy^2=0 \tag{2.22}$$

Comparing (2.21) and (2.22) we conclude that

$$\Big((2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]\Big)zy^2[uv,x]=0 \tag{2.23}$$

so that

$$((2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x])Jy^2[uv,x]=0 \quad \text{for all} \;\; u,v,x,y\in J. \tag{2.24}$$

Since R is prime, then either $(2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]=0$ or $y^2[uv,x]=0$, in which case we obtain $x\in Z(R)$. Hence in both cases we have

$$(2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]=0 \quad \text{for all} \;\; x\in J. \tag{2.25}$$

Similarly replacing y by $8uvy^2r\delta(x)x$ in (2.15), where $u,v\in J$ and $r\in R$ we get

$$(2n+1)[uvy^2r\delta(x),x][\delta(x),x]+n[[uvy^2r\delta(x),x],x]\delta(x)=0 \tag{2.26}$$

which leads to

$$(2n+1)[uvy^2r,x]\delta(x)[\delta(x),x]+(2n+1)uvy^2r[\delta(x),x]^2+2n[uvy^2r,x][\delta(x),x]\delta(x)$$
$$n[[uvy^2r,x],x]\delta(x)^2=0 \quad \text{for all} \;\; u,v,x,y\in J, \;\; r\in R. \tag{2.27}$$

If we replace y by $8uvy^2rx$ in (2.15), where $u,v,y\in J$ and $r\in R$, we find that

$$(2n+1)[uvy^2r,x][\delta(x),x]+n[[uvy^2r,x],x]\delta(x)=0 \quad \text{for all} \;\; u,v,x,y\in J; \;\; r\in R. \tag{2.28}$$

Using (2.28) together with (2.27) we see that

$$(2n+1)[uvy^2r,x]\delta(x)[\delta(x),x]+(2n+1)uvy^2r[\delta(x),x]^2+2n[uvy^2r,x][\delta(x),x]\delta(x)$$
$$-(2n+1)[uvy^2r,x][\delta(x),x]\delta(x)=0 \quad \text{for all} \;\; u,v,x,y\in J; \;\; r\in R \tag{2.29}$$

so that

$$[uvy^2r,x]((2n+1)\delta(x)[\delta(x),x]-[\delta(x),x]\delta(x))+(2n+1)uvy^2r[\delta(x),x]^2=0 \tag{2.30}$$

Similarly if we replace y by $8y^2r\delta(x)x$ in (2.15), where $r\in R$, we get

$$[y^2r,x]((2n+1)\delta(x)[\delta(x),x]-[\delta(x),x]\delta(x))+(2n+1)y^2r[\delta(x),x]^2 \tag{2.31}$$

By virtue of (2.30), equation (2.31) forces

$$[uv,x]y^2r\Big((2n+1)\delta(x)[\delta(x),x]-[\delta(x),x]\delta(x)\Big)=0 \tag{2.32}$$

and so

$$[uv,x]y^2R\Big((2n+1)\delta(x)[\delta(x),x]-[\delta(x),x]\delta(x)\Big)=0 \quad \text{for all} \;\; u,v,x,y\in J. \tag{2.33}$$

In light of the primeness of R, we conclude that either $(2n+1)\delta(x)[\delta(x),x]-[\delta(x),x]\delta(x)=0$ or $[uv,x]y^2=0$. Arguing as above, in both the cases we have

$$(2n+1)\delta(x)[\delta(x),x]-[\delta(x),x]\delta(x)=0 \quad \text{for all} \;\; x\in J. \tag{2.34}$$

Adding (2.25) and (2.34), because of $(n+1)!$ torsion freeness, we find that

$$\delta(x)[\delta(x),x]+[\delta(x),x]\delta(x)=0 \quad \text{for all} \;\; x\in J. \tag{2.35}$$

Analogously, adding (2.25) and (2.35) we see that

$$[\delta(x), x]\delta(x) = 0 \quad \text{for all} \quad x \in J. \tag{2.36}$$

Accordingly, equation (2.25) reduces to

$$\delta(x)[\delta(x), x] = 0 \quad \text{for all} \quad x \in J. \tag{2.37}$$

Replacing x by $x + ky$ in (2.37), we obtain

$$kB_1(x, y) + k^2 B_2(x, y) + ... + k^{n+1} B_{n+1}(x, y) = 0 \quad \text{for all} \quad x, y \in J \tag{2.38}$$

where $B_i(x, y)$ denotes the sum of the terms in which y appears i times. Application of Lemma 2.1 and (2.3) gives that

$$\delta(x)[\delta(x), y] + n\delta(x)[\Delta(x, x, x, ..., y), x] + n\Delta(x, x, x, ..., y)[\delta(x), x] = 0 \tag{2.39}$$

Replacing y by $8ury^2vx$ in (2.39), where $u, v, x, y \in J$ and $r \in R$, we obtain

$$(n + 1)\delta(x)ury^2v[\delta(x), x] + n\delta(x)[ury^2v, x]\delta(x) = 0 \tag{2.40}$$

Substituting $8xury^2vx$ for y in (2.39), where $u, v, x, y \in J$ and $r \in R$, we get

$$(n + 1)\delta(x)xury^2v[\delta(x), x] + n\delta(x)x[ury^2v, x]\delta(x) = 0 \tag{2.41}$$

Left multiplying (2.40) by x and using (2.41), it is obvious to see that

$$(n + 1)[\delta(x), x]ury^2v[\delta(x), x] + n[\delta(x), x][ury^2v, x]\delta(x) = 0 \tag{2.42}$$

Replacing y by $8\delta(x)sz^2ry^2vx$ in (2.4), where $s \in R$ we get

$$(2n + 1)[\delta(x), x]sz^2ry^2v[\delta(x), x] + 2n[\delta(x), x][sz^2ry^2v, x]\delta(x) = 0 \tag{2.43}$$

Writing sz^2 instead of u in (2.42), where $z \in J$ and $s \in R$ we get

$$(n + 1)[\delta(x), x]sz^2ry^2v[\delta(x), x] + n[\delta(x), x][sz^2ry^2v, x]\delta(x) = 0 \tag{2.44}$$

Combining (2.43) and (2.44), because of the torsion restriction, we find that

$$[\delta(x), x]sz^2ry^2v[\delta(x), x] + [\delta(x), x][sz^2ry^2v, x]\delta(x) = 0 \tag{2.45}$$

for all $v, x, y, z \in J, \ and \ r, s \in R$. Comparing (2.44) and (2.45) we conclude that

$$[\delta(x), x]sz^2ry^2v[\delta(x), x] = 0 \quad \text{for all} \quad v, x, y, z \in J, \ and \ r, s \in R \tag{2.46}$$

so that

$$[\delta(x), x]sz^2Ry^2v[\delta(x), x] = 0 \quad \text{for all} \quad v, x, y, z \in J, \ and \ s \in R. \tag{2.47}$$

Since R is prime, the we can conclude that $[\delta(x), x]sz^2 = 0$ or $y^2v[\delta(x), x] = 0$. But in both the cases one can see that for each $x \in J$, $[\delta(x), x] = 0$ and our proof is complete. □

Theorem 2.4. *Let $n \geq 2$ be a fixed positive integer and R be a noncommutative $(n+1)!$-torsion free prime ring and J a nonzero Jordan ideal of R. If R admits a permuting generalized n-derivation Ω with associated n-derivation Δ such that the trace ω of Ω is commuting on R. Then Ω is a left n-multiplier on R.*

Proof. Assume that

$$[\omega(x), x] = 0 \quad \text{for all} \quad x \in J. \tag{2.48}$$

It is obvious to verify that

$$\omega(x + y) = \omega(x) + \omega(y) + \sum \binom{n}{r} H_r(x, y) \quad \text{for all} \quad x, y \in J \tag{2.49}$$

where $H_i(x,y) = \Omega(x,x,x,....,x,y,y,....,y)$; y appears i times.
Replacing x by $x+ky$ in (2.48), where $1 \leq k \leq n$ is a positive integer, we get

$$[\omega(x+ky), x+ky] = 0 \quad \text{for all} \quad x,y \in J \tag{2.50}$$

so that

$$[\omega(x) + \omega(ky) + \sum_{i=1}^{n-1} \binom{n}{i} H_i(x,y), x+ky] = 0 \quad \text{for all} \quad x,y \in J \tag{2.51}$$

and therefore

$$k([\omega(x),y] + \tbinom{n}{1}[H_1(x,y),x]) + k^2((\tbinom{n}{1}[H_1(x,y),y] + \tbinom{n}{2}[H_2(x,y),x])$$
$$+ ..... + k^n([\omega(y),x] + \binom{n}{n-1}[H_{n-1}(x,y),x]) = 0 \ \text{for all} \ x,y \in J \tag{2.52}$$

Application of Lemma 2.1 yields

$$[\omega(x),y] + \binom{n}{1}[H_1(x,y),x] \quad = [\omega(x),y] + n[\Omega(x,x,...,y),x] = 0 \ \text{ for all } \ x,y \in J. \tag{2.53}$$

Replacing y by $8zuy^2vx$, where $u,v,x,y,z \in J$, we get

$$\begin{aligned} 0 &= zuy^2v[\omega(x),x] + [\omega(x),zuy^2v]x + n[\Omega(x,x,....,zuy^2v)x + zuy^2v\Delta(x,x,...,x),x] \\ &= [\omega(x),zuy^2v]x + nzuy^2v[\delta(x),x] + n[zuy^2v,x]\delta(x) + n[\Omega(x,x,...,zuy^2v),x]x \\ &= n[zuy^2v,x]\delta(x) + nzuy^2v[\delta(x),x] \end{aligned}$$

Since R is $(n+1)!$-torsion free, it then follows that

$$[zuy^2v,x]\delta(x) + zuy^2v[\delta(x),x] = 0 \quad \text{for all} \quad x,y \in J. \tag{2.54}$$

Replacing z by $2[r,t]z$,where $r,t \in R$, and invoking (2.54), we have

$$[[r,t],x]zy^2v\delta(x) = 0 \quad \text{for all} \quad v,x,y \in J, \quad and \quad r,t \in R. \tag{2.55}$$

Using the primeness of R, we get $\delta(x) = 0$ or $x \in Z(R)$ for all $x \in J$. Hence in all the cases we have

$$[\delta(x),x] = 0 \quad \text{for all} \quad x \in J. \tag{2.56}$$

Consider a positive integer k, $1 \leq k \leq n+1$. Replacing x by $x+\lambda y$ in (2.56), where $y \in J$, we get

$$\begin{aligned} 0 \quad &= \quad k[\delta(x),y] + k[\delta(x),y] + \binom{n}{r}[h_1(x,y),x] + k^2\binom{n}{1}[h_1(x,y),y] + \binom{n}{2}[h_2(x,y),x] \\ &+ \quad ... + k^n\binom{n}{1}[h_1(x,y),y] + \binom{n}{n-1}[h_{n-1}(x,y),y] \end{aligned}$$

In view of Lemma 2.1, (2.56) assures that

$$[\delta(x),y] + n[h_1(x,y),x] = 0 \quad \text{for all} \quad x,y \in J. \tag{2.57}$$

Replacing y by $128ry^2vj^2t$ in (2.57), where $j,v \in J$ and $r,t \in R$, we get

$$[\delta(x), 128ry^2vj^2t] + n[h_1(x, 128ry^2vj^2t),x] = 0 \quad \text{for all} \quad x \in J. \tag{2.58}$$

Writing xr instead of r in (2.58), one can easily see that

$$[\delta(x), 128xry^2vj^2t] + n[h_1(x, 128xry^2vj^2t),x] = 0 \quad \text{for all} \quad x \in J \tag{2.59}$$

which implies that

$$x[\delta(x), 128ry^2vj^2t] + n[h_1(x, 128ry^2vj^2t),x] + n\delta(x)[128ry^2vj^2t,x] = 0 \tag{2.60}$$

and thus

$$\delta(x)[128ry^2vj^2t, x] = 0 \quad \text{for all} \quad x \in J. \tag{2.61}$$

Substituting t by ts in (2.61), where $s \in R$, and using the primeness of R, we conclude that $\delta(x) = 0$ or $x \in Z(R)$ for all $x \in J$.
Let $x \in J$ $(x \in Z(R)$ and $y \in J$ $(y \notin Z(R)$. Then $y + kx \notin Z(R)$ and thus

$$0 = \delta(y + kx) = \delta(y) + k^n\delta(x) + \sum_{i=1}^{n-1} k^i \binom{n}{i} h_i(x, y)$$

accordingly,

$$\sum_{r=1}^{n-1} k^r \binom{n}{r} h_r(x, y) + k^n\delta(x) = 0 \quad \text{for all} \quad x, y \in J. \tag{2.62}$$

Application of Lemma 2.1 implies that

$$\delta(x) = 0 \quad \text{for all} \quad x \in J. \tag{2.63}$$

For $k = 1, 2, 3, ..., n$, Let $P_k(x) = \Delta(x, x, .., x, x_{k+1}, x_{k+2}, ..., x_n)$, where x appears k times and $x, x_i \in R$, $i = k + 1, k + 2, ..., n$. Let μ $(1 \leq \mu \leq n - 1)$ be any integer. By view of (2.63),

$$\begin{aligned} 0 &= \delta(\mu x + x_n) = P_n(\mu x + x_n) \\ &= \mu^n\delta(x) + \delta(x_n) + \sum_{r=1}^{n-1} \mu^r \binom{n}{r} P_r(x) \\ &= \sum_{r=1}^{n-1} \mu^r \binom{n}{r} P_r(x) \end{aligned}$$

for all $x, x_n \in J$, that is

$$\sum_{r=1}^{n-1} \mu^r \binom{n}{r} P_r(x) = 0 \quad \text{for all} \quad x \in J. \tag{2.64}$$

Using Lemma 2.2 together with (2.64), we obtain

$$P_{n-1}(x) = 0 \quad \text{for all} \quad x \in J. \tag{2.65}$$

Let ν $(1 \leq \nu \leq n - 2)$ be any integer. By virtue of (2.65) we have

$$0 = P_{n-1}(\nu x + x_{n-1}) = \nu^{n-1}P_{n-1}(x) + P_{n-1}(x_{n-1}) + \sum_{i=1}^{n-2} \nu^i \binom{n}{i} P_i(x)$$

for all $x, x_{n-1} \in J$ in such a way that

$$\sum_{r=1}^{n-2} \nu^r \binom{n}{r} P_r(x) = 0 \quad \text{for all} \quad x \in J. \tag{2.66}$$

Once again using Lemma 2.2, (2.66) yields

$$P_{n-2}(x) = 0 \quad \text{for all} \quad x \in J. \tag{2.67}$$

If we continue to carry out the same method as above, we arrive at

$$P_1(x) = 0 \quad \text{for all} \quad x \in J \tag{2.68}$$

for all $x \in J$; that is

$$\Delta(x_1, x_2, ..., x_n) = 0 \quad \text{for all} \quad x_i \in J \tag{2.69}$$

so we get the required result. □

Our aim in the following theorem is to extend ([2], Theorem 2.8) to Jordan ideals with the restriction that R is assumed to be prime.

Theorem 2.5. *Let n be a fixed positive integer and R be a $(n+1)!$-torsion free prime ring and J a nonzero Jordan ideal of R. If R admits a nonzero permuting generalized n-derivation Ω with associated n-derivation Δ such that the trace ω of Ω is centralizing on J. Then R is commutative.*

Proof. We are given that $[\omega(x), x] \in Z(R)$ for all $x \in R$. Using similar arguments as used in the proof of Theorem 2.4, we obtain

$$[\omega(x), y] + n[\Omega(x, x, ..., y), x] = 0 \quad \text{for all} \quad x, y \in J. \tag{2.70}$$

Replacing y by $128ry^2uz^2x$, where $u, z \in J$, *and* $r \in R$, we have
$128ry^2uz^2[\omega(x), x] + [\omega(x), 128ry^2uz^2]x + n[\Omega(x, x, ..., 128ry^2uz^2), x]x$

$$+128nry^2uz^2[\delta(x), x] + n[128ry^2uz^2, x]\delta(x) \in Z(R) \tag{2.71}$$

for all $x, y \in J$. Once again replacing y by $128ry^2uz^2$ in (2.70), where $u \in J$ and $r \in R$, it is straightforward to see that

$$[\omega(x), 128ry^2uz^2] + n[\Omega(x, x, ..., 128ry^2uz^2), x] = 0 \quad \text{for all} \quad x, y \in J. \tag{2.72}$$

Combining (2.71) with (2.72), we get
$[128ry^2uz^2, x][\omega(x), x] + n[128ry^2uz^2, x][\delta(x), x] + 128nry^2uz^2[[\delta(x), x], x]$

$$+n[128ry^2uz^2, x][\delta(x), x] + n[[128ry^2uz^2, x], x]\delta(x) = 0 \tag{2.73}$$

for all $u, x, y \in J$ and $r \in R$. Replacing r by $\omega(x)r$ in (2.73) and invoking (2.73), we obtain

$$128[\omega(x), x]ry^2uz^2[\omega(x), x]+128\times 2n[\omega(x), x]ry^2uz^2[\delta(x), x]+2n[\omega(x), x][128ry^2uz^2, x]\delta(x) = 0 \tag{2.74}$$

for all $u, x, y \in J$ and $r \in R$. Writing rs instead of r, where $s \in R$, we obtain

$$[\omega(x), x](128rsy^2uz^2([\omega(x), x]+2n[\delta(x), x])+2n(r[128sy^2uz^2, x]+[r, x]128sy^2uz^2)\delta(x)) = 0 \tag{2.75}$$

Using (2.74) together with (2.75), we get

$$2n[\omega(x), x][r, x]128sy^2uz^2\delta(x) = 0 \quad \text{for all} \quad u, x, y \in J, \quad \textit{and} \quad r \in R. \tag{2.76}$$

Replacing r by $\omega(x)$, we conclude that either $[\omega(x), x]^2 = 0$ or $\delta(x) = 0$.
If $[\omega(x), x]^2 = 0$, then $[\omega(x), x]R[\omega(x), x] = 0$ so that $[\omega(x), x] = 0$.
Suppose that $\delta(x) = 0$, then (2.74) gives

$$ry^2uz^2[\omega(x), x]^2 = 0 \quad \text{for all} \quad u, x, y \in J, \quad \textit{and} \quad r \in R \tag{2.77}$$

which, because of the primeness of R, leads to

$$[\omega(x), x] = 0 \quad \text{for all} \quad x \in J \tag{2.78}$$

so that ω is commuting on J.
For $x, y \in J$, replacing x by $x+ky$ for $k = 1, 2, ..., n$ in $[\omega(x), x] = 0$, we find that $[x+ky, \delta(x)+\delta(ky) + \sum_{r=1}^{n-1} \binom{n}{r} h_r(x, y)] = 0$ for all $x, y \in J$. Using the last equation together with Lemma 2.1, we get

$$[y, \omega(x)] + n[x, \Omega(x, x, x, ..., y)] = 0 \; \textit{for all} \; x, y \in J. \tag{2.79}$$

Replacing y by $8zy^2uvx$ in (2.79), where $u, v, z \in J$, and using the given condition, we get

$$[zy^2uv, \omega(x)]x + n[x, \Omega(x, x, x, ..., zy^2uv)]x + n[x, zy^2uv]\omega(x) = 0 \; \forall \; x, y \in J. \tag{2.80}$$

Using (2.79) we find that $[x, zy^2uv]\omega(x) = 0$. Replacing x by $x + kw$ for $k = 1, 2, ..., n$ and use (2.79) we obtain

$$n[x, zy^2uv]\Omega(x, x, x, ..., w) + [w, zy^2uv]\omega(x) = 0 \; \forall \; u, v, w, x, y, z \in J. \tag{2.81}$$

Then we obtain

$$n[x,z]y^2uv\mathbf{\Omega}(x,x,x,....,w)+[w,z]y^2uv\omega(x)=0\ \forall\ j,u,v,w,x,y,z\in J. \tag{2.82}$$

Replacing w by z we get

$$n[x,z]y^2uv\mathbf{\Omega}(x,x,x,....,z)=0\ \forall\ u,v,x,y,z\in J. \tag{2.83}$$

Replacing z by $z+w$ we get

$$n[x,z]y^2uv\mathbf{\Omega}(x,x,x,....,w)+[x,w]y^2uv\mathbf{\Omega}(x,x,x,....,z)=0\ \forall\ u,v,w,x,y,z\in J. \tag{2.84}$$

Using (2.83) and (2.84) we obtain

$$[x,w]y^2uv\mathbf{\Omega}(x,x,x,....,z)R[x,w]y^2uv\mathbf{\Omega}(x,x,x,....,z)=0\ \forall\ u,v,w,x,y,z\in J. \tag{2.85}$$

Since R is prime we conclude that

$$[x,w]y^2uv\mathbf{\Omega}(x,x,x,....,z)=0\ \forall\ u,v,w,x,y,z\in J. \tag{2.86}$$

Let $[x,y]z\mathbf{\Omega}(x,...,x,x_{i-1},x_{i-2},....,x_2,x_1)=0$ holds for all $x,x_{i-1},x_{i-2},...,x_1\in J$, and $2\leq i\leq n$. Replacing x by $x+kx_i$ in last equation to obtain

$$[x+kx_i,y]z\mathbf{\Omega}(x+kx_i,...,x+kx_i,x_{i-1},x_{i-2},....,x_2,x_1)=0\ \forall\ j,u,v,w,x,y,z\in J. \tag{2.87}$$

Using Lemma 2.1, we obtain

$$[x_i,y]z\mathbf{\Omega}(x,...,x,x_{i-1},...,x_1)+(n-i+1)[x,y]z\mathbf{\Omega}(x,...,x,x_i,...,x_1)=0 \tag{2.88}$$

for all $j,u,v,w,x,y,z\in J$. Then we have
$0=(n-i+1)[x,y]z\mathbf{\Omega}(x,...,x,x_i,...,x_1)R[x,y]z\mathbf{\Omega}(x,...,x,x_i,...,x_1)$
$=\ -[x_i,y]z\mathbf{\Omega}(x,...,x,x_i,...,x_1)R[x,y]z\mathbf{\Omega}(x,...,x,x_{i-1},...,x_1)$.
Since R is prime we get

$$[x,y]z\mathbf{\Omega}(x,...,x,x_i,...,x_1)=0\ \forall\ x_i,...,x_1,x,y,z\in J. \tag{2.89}$$

For $i=n-1$, we obtain

$$[x,y]z\mathbf{\Omega}(x,x_{n-1},...,x_1)=0\ \forall\ x_{n-1},...,x_1,x,y,z\in J. \tag{2.90}$$

As R is prime we obtain $[x,y]=0$ or $\mathbf{\Omega}(x,x_{n-1},...,x_1)\ \forall\, x_{n-1},...,x_1,x,y\in J$.
Let us consider

$$J_1=\{x\in J|\,[x,z]=0\ \forall\ z\in J\},\ \ J_2=\{x\in J|\,\mathbf{\Omega}(x,y_{n-1},...,y_1)=0\ \forall\ y_{n-1},...,y_1\in J\}.$$

It is clear that J_1 and J_2 are two additive subgroups of J such that $J=J_1\cup J_2$ and therefore either $J=J_1$ or $J=J_2$.
If $J=J_1$ then we get $[J,J]=0$ which proves that R is commutative.
If $J=J_2$ then $\mathbf{\Omega}(x,y_{n-1},...,y_1)=0$ for all $y_{n-1},...,y_1,x\in J$ and thus $\mathbf{\Omega}(J,J,...,J)=0$. Accordingly $\omega(J)=0$ in such a way that $\omega=0$, a contradiction. Therefore R is commutative.□

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