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AN ANALYTICAL METHOD FOR SOLVING LINEAR AND NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. The main aim of this article is to introduce an analytical method called the Natural Homotopy Perturbation Method (NHPM) for solving linear and nonlinear Schrödinger equations. The proposed analytical method is a combination of the Natural transform method (NTM) and homotopy perturbation method (HPM). The analytical method is applied directly without using any linearization, transformation, discretization or taking some restrictive assumptions, and it reduces the computational size and avoids round-off errors.

1 Introduction

Schrödinger equations play a significant role in quantum mechanics. The Schrödinger equations are partial differential equations which arise in the study of the time evolution of the wave function. The standard linear Schrödinger equation is given by:

$$v_t = iv_{xx}, \;\; i^2 = -1, \;\; t > 0, \tag{1.1}$$

subject to the initial condition

$$v(x,0) = g(x), \tag{1.2}$$

where $g(x)$ is a continuous function and square integrable. While the nonlinear case is given by:

$$iv_t + v_{xx} + \beta|v|^2 v = 0, \tag{1.3}$$

subject to the initial condition

$$v(x,0) = g(x), \tag{1.4}$$

where β is a constant term and $v(x,t)$ is complex.
The linear and Schrödinger equation always describe the time evolution of a free particle of mass m and the nonlinear Schrödinger equation is a solitary wave equation, where the speed of propagation is independent of the amplitude of the wave function.
In the last few decades, several numerical techniques have been used to solved linear and nonlinear Schrödinger equations, such as natural decomposition method (NDM) [11, 19, 20, 23, 24, 25], Adomian decomposition method (ADM) [4, 9], Bilinear formalism (BF) [2, 3], variational iteration method (VIM) [4, 5, 17], Laplace decomposition method (LDM) [6, 22], homotopy perturbation method (HPM) [7, 33, 34, 35], inverse scattering method (ISM) [1], reduce differential transform method (RDTM) [30, 31, 32], and so on.
In this paper, we introduce an analytical method called the Natural Homotopy Perturbation Method (NHPM) for solving linear and nonlinear Schrödinger equations. The N-transform properties and its application to unsteady fluid flow over a plane wall were first introduced by Khan ZH and Khan WA [16] in the year 2008 and recently renamed as Natural transform by Belgacem FBM and Silambarasan R [14, 29] in the year 2012. The Natural transform is similar to Laplace integral transform [21] and Sumudu integral transform [26, 27, 28]. It converges to both Laplace and Sumudu integral transform by changing of variable [16]. Recently, in the year 2012,

Belgacem FBM and Silambarasan R [14, 15] successfully applied the Natural transform and obtained the solution of Bessel's differential equation with a polynomial coefficient and Maxwell's equation. In this paper, we enhance the application of the Natural transform method by using homotopy perturbation method. The proposed analytical method had a broad applicability to all sorts of linear and nonlinear Schrödinger equations. The analytical method reduces the computational size and lead to exact or approximate solution to the form of a rapidly convergence series solution. The proposed analytical method is based on coupling the Natural transform method (NTM) [14, 16, 29] and homotopy perturbation method (HPM) [33, 34, 35]. The analytical procedure is applied successfully and obtained an exact solution of linear and nonlinear Schrödinger equations, and the results are compared with the results of the existing methods. Thus, the Natural Homotopy Perturbation Method is a powerful mathematical technique for solving linear and nonlinear Schrödinger equations.

2 Natural Transform

In this section, we present some definitions and properties of the Natural transform.
Definition: The Natural transform of the function $v(t)$ for $t \in (0, \infty)$ is defined over the set of functions:
$A=\left\{ v(t) : \exists\, M,\, \tau_1,\, \tau_2 > 0,\ |v(t)| < Me^{\frac{|t|}{\tau_j}},\ if\ t \in (-1)^j \times [0, \infty) \right\}$
by the following integral

$$\mathbb{N}^+\left[v(t)\right] = V(s,u) = \frac{1}{u}\int_0^\infty e^{\frac{-st}{u}}\, v(t)\, dt; \quad s > 0,\, u > 0. \tag{2.1}$$

Where s and u are the Natural transform variables [14, 15].
Fundamental properties of the Natural transform are given below. See [14, 15, 16].

Property 1. If $V(s,u)$ is the Natural transform and $F(s)$ is the Laplace transform of the function $f(t) \in A$, then $\mathbb{N}^+\left[f(t)\right] = V(s,u) = \frac{1}{u}\int_0^\infty e^{-\frac{st}{u}}\, f(t)\, dt = \frac{1}{u}F\left(\frac{s}{u}\right)$.

Property 2. If $V(s,u)$ is the Natural transform and $G(u)$ is the Sumudu transform of the function $v(t) \in A$, then $\mathbb{N}^+\left[v(t)\right] = V(s,u) = \frac{1}{s}\int_0^\infty e^{-t}\, v\left(\frac{ut}{s}\right)\, dt = \frac{1}{s}G\left(\frac{u}{s}\right)$.

Property 3. $\mathbb{N}^+\left[v(at)\right] = \frac{1}{a}V(s,u)$.

Property 4. $\mathbb{N}^+\left[v'(t)\right] = \frac{s}{u}V(s,u) - \frac{1}{u}v(0)$.

Property 5. $\mathbb{N}^+\left[v''(t)\right] = \frac{s^2}{u^2}V(s,u) - \frac{s}{u^2}v(0) - \frac{1}{u}v'(0)$.

Property 6. The Natural transform is a linear operator. That is, if α and β are non–zero constants, then
$\mathbb{N}^+\left[\alpha\, f(t) \pm \beta\, g(t)\right] = \alpha\mathbb{N}^+\left[f(t)\right] \pm \beta\mathbb{N}^+\left[g(t)\right] = \alpha F^+(s,u) \pm \beta G^+(s,u)$.
Therefore, $F^+(s,u)$ and $G^+(s,u)$ are the Natural transforms of $f(t)$ and $g(t)$, respectively.

Table 1. List of Natural transforms of some functions.

Functional Form	Natural transform Form
1	$\frac{1}{s}$
t	$\frac{u}{s^2}$
$e^{\alpha t}$	$\frac{1}{s-\alpha u}$
$\frac{t^{n-1}}{(n-1)!}, n = 1, 2, ...$	$\frac{u^{n-1}}{s^n}$
cos $\alpha(t)$	$\frac{s}{s^2+\alpha^2 u^2}$

3 Analysis of the Natural Homotopy Perturbation Method

In this section, we demonstrate the basic idea of (NHPM) to the standard nonlinear Schrödinger of form:

$$iv_t + v_{xx} + \beta|v|^2 v = 0, \tag{3.1}$$

subject to the initial condition

$$v(x,0) = g(x), \tag{3.2}$$

where β is a constant term and $v(x,t)$ is complex.
Applying the Natural transform to eq.(3.1) subject to the given initial condition, we get:

$$V(x,s,u) = \frac{1}{s}g(x) - \frac{ui}{s}\mathbb{N}^{+}\left[v_{xx} + \beta|v|^2 v\right]. \tag{3.3}$$

Taking the inverse Natural transform of eq.(3.3), we get:

$$v(x,t) = G(x,t) + i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{xx} + \beta|v|^2 v\right]\right], \tag{3.4}$$

where $G(x,t)$ is a term arising from the source term and the prescribed initial condition.
Now we apply the homotopy perturbation method. According to homotopy perturbation method, we use the embedding parameter p as small parameter and assume that the solution of eq.(3.1) can be represented as a power series in p of the form:

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t), \tag{3.5}$$

and the nonlinear term $F(v(x,t)) = |v|^2 v = v^2\bar{v}$ can be decomposed as:

$$v^2\bar{v} = \sum_{n=0}^{\infty} p^n H_n(v), \tag{3.6}$$

where $H(v)_n$ is a He's polynomials which can be evaluated using the following formula:

$$H_n(v_1, v_2, \cdots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[F\left(\sum_{j=0}^{n} p^j v_j\right)\right]_{p=0}, n = 0,1,2,\cdots \tag{3.7}$$

Some few components of He's polynomial $(H_n(v))$ are computed below:

$$\begin{aligned}
H_0(v) &= v_0^2\bar{v}_0,\\
H_1(v) &= 2v_0v_1\bar{v}_0 + v_0^2\bar{v}_1,\\
H_2(v) &= 2v_0v_2\bar{v}_0 + v_1^2\bar{v}_0 + 2v_0v_1\bar{v}_1 + v_0^2\bar{v}_2,\\
H_3(v) &= 2v_0v_3\bar{v}_0 + 2v_1v_2\bar{v}_0 + 2v_0v_2\bar{v}_1 + v_1^2\bar{v}_1 + 2v_0v_1\bar{v}_2 + v_0^2\bar{v}_3,\\
&\vdots,
\end{aligned}$$

and so on.
Now, by substituting eq.(3.5) and eq.(3.6) into eq.(3.4), we get:

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = G(x,t) + ip\left(\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} p^n v_{nxx} + \sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right). \tag{3.8}$$

Comparing the coefficient of like powers of p in eq.(3.8), the following approximations are obtained:

$$\begin{aligned}
p^0 : v_0(x,t) &= G(x,t),\\
p^1 : v_1(x,t) &= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{0xx} + H_0(v)\right]\right],\\
p^2 : v_2(x,t) &= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{1xx} + H_1(v)\right]\right],\\
p^3 : v_3(x,t) &= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{2xx} + H_2(v)\right]\right],\\
&\vdots,
\end{aligned}$$

and so on.
Thus, the series solution of eq.(3.1) is given by:

$$v(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} v_n(x,t). \tag{3.9}$$

The series solution always converges rapidly. (ADM) [12] gave the detailed classical convergence of the series solution.

4 Applications

In this section, we demonstrate the applicability and flexibility of the Natural Homotopy Perturbation Method (NHPM) to some linear and nonlinear Schrödinger equations.

Example 1. Consider the following linear Schrödinger equation of the form:

$$v_t = i v_{xx}, \tag{4.1}$$

subject to the initial condition

$$v(x,0) = \sin(\beta x), \tag{4.2}$$

where β is a constant term.
Applying the Natural transform to eq.(4.1) subject to the given initial condition, we get:

$$V(x,s,u) = \frac{\sin(\beta x)}{s} + i\frac{u}{s}\left[\mathbb{N}^{+}\left[v_{xx}\right]\right]. \tag{4.3}$$

Taking the inverse Natural transform of eq.(4.3), we get:

$$v(x,t) = \sin(\beta x) + i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{xx}\right]\right]. \tag{4.4}$$

Now we apply the homotopy perturbation method.

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t). \tag{4.5}$$

Then eq.(4.4) will become:

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = \sin(\beta x) + ip\left(\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} p^n v_{nxx}\right]\right]\right). \tag{4.6}$$

Comparing the coefficient of like powers of p in eq.(4.6), the following approximations are obtained:

$$\begin{aligned}
p^0 : v_0(x,t) &= \sin(\beta x),\\
p^1 : v_1(x,t) &= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{0xx}\right]\right]\\
&= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[-\beta^2\sin(\beta x)\right]\right]\\
&= -i\beta^2\sin(\beta x)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[1\right]\right]\\
&= -i\beta^2\sin(\beta x)\mathbb{N}^{-1}\left[\frac{u}{s^2}\right]\\
&= -it\beta^2\sin(\beta x),
\end{aligned}$$

$$\begin{aligned}
p^2: v_2(x,t) &= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[v_{1xx}\right]\right]\\
&= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[it\beta^4\sin(\beta x)\right]\right]\\
&= -\beta^4\sin(\beta x)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[t\right]\right]\\
&= -\beta^4\sin(\beta x)\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right]\\
&= \frac{(it)^2}{2!}\beta^4\sin(\beta x),
\end{aligned}$$

$$\begin{aligned}
p^3: v_3(x,t) &= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[v_{2xx}\right]\right]\\
&= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[\frac{t^2}{2!}\beta^6\sin(\beta x)\right]\right]\\
&= \frac{i\beta^6}{2!}\sin(\beta x)\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[t^2\right]\right]\\
&= \frac{i\beta^6}{2!}\sin(\beta x)\mathbb{N}^{-1}\left[\frac{2!u^3}{s^4}\right]\\
&= -\frac{(it)^3}{3!}\beta^6\sin(\beta x),\\
&\vdots,
\end{aligned}$$

and so on.
Thus, the series solution of eq.(4.1) is given by:

$$\begin{aligned}
v(x,t) &= \lim_{N\to\infty}\sum_{n=0}^{N} v_n(x,t)\\
&= v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+\cdots\\
&= \sin(\beta x)-(it)\beta^2\sin(\beta x)+\frac{(it^2)^2}{2!}\beta^4\sin(\beta x)-\frac{(it)^3}{3!}\beta^6\sin(\beta x)+\cdots\\
&= \sin(\beta x)\left(1-(it)\beta^2+\frac{(it)^2}{2!}\beta^4-\frac{(it)^3}{3!}\beta^6+\cdots\right).
\end{aligned}$$

When $\beta = 1$, then the exact solution of the Schrodinger equation (4.1) is given by:

$$v(x,t) = \sin(x)e^{-it}. \tag{4.7}$$

The exact solution is in close agreement with the result obtained by (ADM) [4], (NDM) [11], and (VIM) [4, 5].

Example 2. Consider the following nonlinear Schrödinger equation of the form:

$$iv_t + v_{xx} + 6|v|^2 v = 0, \tag{4.8}$$

subject to the initial condition

$$v(x,0) = e^{3ix}. \tag{4.9}$$

Applying the Natural transform to eq.(4.8) subject to the given initial condition, we get:

$$V(x,s,u) = \frac{e^{3ix}}{s} + i\frac{u}{s}\mathbb{N}^+\left[v_{xx}+6|v|^2 v\right] = 0 \tag{4.10}$$

Taking the inverse Natural transform of eq.(4.10), we get:

$$v(x,t) = e^{3ix} + i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^+\left[v_{xx}+6|v|^2 v\right]\right]. \tag{4.11}$$

Now by applying the homotopy perturbation method we get:

$$v(x,t)=\sum_{n=0}^{\infty}p^n v_n(x,t). \tag{4.12}$$

Then eq.(4.11) will become:

$$\sum_{n=0}^{\infty}p^n v_n(x,t)=e^{3ix}+ip\left(\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty}p^n v_{nxx}+6\sum_{n=0}^{\infty}p^n H_n(v)\right]\right]\right), \tag{4.13}$$

where $H_n(v)$ is a He's polynomials which represent the nonlinear term $|v|^2v$
Comparing the coefficient of like powers of p in eq.(4.13), the following approximations are obtained:

$$\begin{aligned}
p^0:\ v_0(x,t) &= e^{3ix},\\
p^1:\ v_1(x,t) &= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{0xx}+H_0(v)\right]\right]\\
&= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{0xx}+6v_0^2\bar{v}\right]\right]\\
&= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[-3e^{3ix}\right]\right]\\
&= -3ie^{3ix}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[1\right]\right]\\
&= -3ie^{3ix}\mathbb{N}^{-1}\left[\frac{u}{s^2}\right]\\
&= -(3i)te^{3ix},
\end{aligned}$$

$$\begin{aligned}
p^2:\ v_2(x,t) &= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{1xx}+6H_1(v)\right]\right]\\
&= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{1xx}+6\left(2v_0v_1\bar{v_0}+v_0^2v_1\right)\right]\right]\\
&= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[9ite^{3ix}\right]\right]\\
&= -9e^{3ix}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t\right]\right]\\
&= -9e^{3ix}\mathbb{N}^{-1}\left[\frac{u^2}{s^3}\right]\\
&= \frac{(3it)^2}{2!}e^{3ix},
\end{aligned}$$

$$\begin{aligned}
p^3:\ v_3(x,t) &= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{2xx}+6H_2(v)\right]\right]\\
&= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[v_{2xx}+6(2v_0v_2\bar{v_0}+v_1^2\bar{v_0}+2v_0v_1\bar{v_1}+v_0^2\bar{v_2})\right]\right]\\
&= i\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[\frac{27t^2}{2!}e^{3ix}\right]\right]\\
&= \frac{27i}{2!}e^{3ix}\mathbb{N}^{-1}\left[\frac{u}{s}\mathbb{N}^{+}\left[t^2\right]\right]\\
&= \frac{27i}{2!}e^{3ix}\mathbb{N}^{-1}\left[\frac{2u^3}{s^4}\right]\\
&= -\frac{(3it)^3}{3!}e^{3ix},
\end{aligned}$$

$$\vdots,$$

and so on.
Thus, the series solution of eq.(4.8) is given by:

$$\begin{aligned}
v(x,t) &= \lim_{N\to\infty}\sum_{n=0}^{N} v_n(x,t) \\
&= v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+\cdots \\
&= e^{3ix}-3ite^{3ix}+\frac{(3it)^2}{2!}e^{3ix}+\cdots \\
&= e^{3ix}\left(1-(3it)+\frac{(3it)^2}{2!}-\frac{(3it)^3}{3!}\cdots\right). \\
&= e^{3i(x-t)}.
\end{aligned}$$

Hence, the exact solution of the Schrodinger equation (4.8) is given by:

$$v(x,t) = e^{3i(x-t)}. \tag{4.14}$$

The exact solution is in close agreement with the result obtained by (ADM) [4], (NDM) [11], and (VIM) [4, 5].

5 Conclusion

In this paper, an analytical method called the Natural Homotopy Perturbation Method (NHPM) is successfully applied to linear and nonlinear Schrödinger equations. The analytical method doesn't require the use of Adomian polynomials which is an advantage over the Adomian decomposition method. The flexibility and high accuracy of the analytical method is successfully illustrated. Thus, the analytical method can be use to solve many linear and nonlinear Schrödinger equations and related applications in science and engineering.

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