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A CHARACTERIZATION OF NONLINEAR ξ-LIE $*$-DERIVATIONS ON VON NEUMANN ALGEBRAS

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Abstract Let $\mathcal{M}$ be a von Neumann algebra without central abelian projections. In this paper, it is proved that under some mild conditions, every nonlinear ξ-Lie $*$-derivation ($\xi \neq 0, 1$) $L : \mathcal{M} \to \mathcal{M}$ is an additive $*$-derivation.

1 Introduction and preliminaries

Let $\mathcal{A}$ be an algebra over the complex field $\mathbb{C}$. Recall that an additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$, and an additive Lie derivation if $\delta([A, B]) = [\delta(A), B]+[A, \delta(B)]$ for all $A, B \in \mathcal{A}$, where $[A, B] = AB - BA$ is the usual Lie product of A and B. The problem of how to characterize the Lie derivations and reveal the relationship between Lie derivations and derivations has received many mathematicians' attention for many years (for example, see [4], [5], [7], [9]). Let $\delta : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity or linearity assumption) and ξ be a non-zero scalar. We say that δ is a nonlinear ξ-Lie derivation if $\delta([A, B]_\xi) = [\delta(A), B]_\xi + [A, \delta(B)]_\xi$ for all $A, B \in \mathcal{A}$, where $[A, B]_\xi = AB - \xi BA$ is the ξ-Lie product of A and B. It is clear that if $\xi = 1$, a nonlinear ξ-Lie derivation is a nonlinear Lie derivation. Recently, Yu and Zhang [10] described nonlinear Lie derivation on triangular algebras. Bai and Du [1] investigated nonlinear Lie derivations on von Neumann algebras. Bai, Du and Guo [2] proved that every nonlinear ξ-Lie derivation ($\xi \neq 1$) on a von Neumann algebra with no central abelian projections is an additive derivation.

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$ and ξ be a non-zero scalar. We say that a mapping $\delta : \mathcal{A} \to \mathcal{A}$ is an additive $*$-derivation if δ is an additive derivation and satisfies $\delta(A^*) = \delta(A)^*$ for all $A \in \mathcal{A}$. A mapping (without the additivity or linearity assumption) $L : \mathcal{A} \to \mathcal{A}$ is called a nonlinear ξ-Lie $*$-derivation if $L([A^*, B]_\xi) = [L(A)^*, B]_\xi + [A^*, L(B)]_\xi$ for all $A, B \in \mathcal{A}$. If $L(A^*) = L(A)^*$ for all $A \in \mathcal{A}$, then L is a nonlinear ξ-Lie derivation if and only if L is a nonlinear ξ-Lie $*$-derivation. But, in general, a nonlinear ξ-Lie $*$-derivation does not satisfy $L(A^*) = L(A)^*$ for all $A \in \mathcal{A}$. It is clear that if $\xi = 1$, a nonlinear ξ-Lie $*$-derivation is a nonlinear $*$-Lie derivation. In [6], the authors studied the structure of nonlinear $*$-Lie derivations and proved that a nonlinear $*$-Lie derivation on a von Neumann algebra with no central abelian projections can be expressed as the sum of an additive $*$-derivation and a mapping with image in the center vanishing at commutators.

In this paper, we will give a characterization of nonlinear ξ-Lie $*$-derivations on von Neumann algebras without central abelian projections for all scalars $\xi \neq 1$.

Before giving our main result, we need some notations and preliminaries. Throughout this paper, let $\mathcal{H}$ be a complex Hilbert space, and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A von Neumann algebra $\mathcal{M}$ is a weakly closed, self-adjoint algebra of operators on $\mathcal{H}$ containing the identity operator I. The set $\mathcal{Z}_\mathcal{M} = \{S \in \mathcal{M} \mid ST = TS \text{ for all } T \in \mathcal{M}\}$ is called the center of $\mathcal{M}$. A projection P is called a central abelian projection if $P \in \mathcal{Z}_\mathcal{M}$ and $P\mathcal{M}P$ is abelian. For $A \in \mathcal{M}$, the central carrier of A, denoted by $\overline{A}$, is the intersection of all central projections P such that $PA = A$. It is well known that the central carrier of A is the projection onto the closed subspace spanned by $\{BA(x) \mid B \in \mathcal{M},\ x \in \mathcal{H}\}$. For each self-adjoint operator $A \in \mathcal{M}$, we define the core of A, denoted by $\underline{A}$, to be $\sup\{S \in \mathcal{Z}_\mathcal{M} \mid S = S^*,\ S \leq A\}$. If P is a projection, it is clear that $\underline{P}$ is the largest central projection Q satisfying $Q \leq P$. We call a projection core-free if $\underline{P} = 0$. It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$, here $\overline{I - P}$ denotes the central carrier of $I - P$.

First, we give the following lemma which will be used frequently.

Lemma 1.1. *Let $\mathcal{M}$ be a von Neumann algebra.*

(i) [8, Lemma 4] If $\mathcal{M}$ has no central abelian projections, then each nonzero central projection in $\mathcal{M}$ is the central carrier of a core-free projection in $\mathcal{M}$.

(ii) [3, Lemma 2.6] If $\mathcal{M}$ has no central abelian projections, then $\mathcal{M}$ equals the ideal of $\mathcal{M}$ generated by all commutators in $\mathcal{M}$.

(iii) [6, Lemma 2.1] Let $P \in \mathcal{M}$ be a projection with $\overline{P} = I$ and $A \in B(\mathcal{H})$. If $AMP = 0$ for all $M \in \mathcal{M}$, then $A = 0$. Consequently, if $Z \in \mathcal{Z}_{\mathcal{M}}$, then $ZP = 0$ implies $Z = 0$.

By Lemma 1.1(i), there exists a projection P such that $\underline{P} = 0$ and $\overline{P} = I$. Throughout this paper, $P_1 = P$ is fixed. Write $P_2 = I - P_1$. By the definition of central core and central carrier, P_2 is also core-free and $\overline{P_2} = I$. According to the two-side Pierce decomposition of $\mathcal{M}$ relative P_1, denote $\mathcal{M}_{ij} = P_i\mathcal{M}P_j$, $i, j = 1, 2$, then $\mathcal{M} = \sum_{i,j=1}^{2} \mathcal{M}_{ij}$. For every $A \in \mathcal{M}$, we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follows, when we write A_{ij}, it indicates that it is contained in $\mathcal{M}_{ij}$.

2 The Results

In order to prove our main theorem, we need the following result.

Lemma 2.1. *Let $\mathcal{M}$ be a von Neumann algebra with no central abelian projections, $\xi \neq 0, 1$ be a scalar and $A_{ii} \in \mathcal{M}_{ii}$, $B_{jj} \in \mathcal{M}_{jj}$, $1 \leq i \neq j \leq 2$. If $A_{ii}C_{ij} = \xi C_{ij}B_{jj}$ for all $C_{ij} \in \mathcal{M}_{ij}$, then $A_{ii} + B_{jj} \in (\xi P_i + P_j)\mathcal{Z}_{\mathcal{M}}$.*

Proof. For any $D_{ii} \in \mathcal{M}_{ii}$, we have $A_{ii}D_{ii}C_{ij} = \xi D_{ii}C_{ij}B_{jj} = D_{ii}A_{ii}C_{ij}$. Hence we get $(A_{ii}D_{ii} - D_{ii}A_{ii})P_iCP_j = 0$ for all $C \in \mathcal{M}$. It follows from Lemma 1.1(iii) that $A_{ii}D_{ii} = D_{ii}A_{ii}$, that is $A_{ii} = Z_iP_i$ for some $Z_i \in \mathcal{Z}_{\mathcal{M}}$. For any $D_{jj} \in \mathcal{M}_{jj}$, we get

$$\xi C_{ij}D_{jj}B_{jj} = A_{ii}C_{ij}D_{jj} = \xi C_{ij}B_{jj}D_{jj}.$$

Then we have $\xi C_{ij}(D_{jj}B_{jj} - B_{jj}D_{jj}) = 0$. Since $\xi \neq 0$, we obtain that

$$(D_{jj}B_{jj} - B_{jj}D_{jj})^* P_jCP_i = 0$$

for all $C \in \mathcal{M}$. By Lemma 1.1(iii), we get $D_{jj}B_{jj} = B_{jj}D_{jj}$, that is $B_{jj} = Z_jP_j$ for some $Z_j \in \mathcal{Z}_{\mathcal{M}}$. Hence we have that $Z_iP_iC_{ij} = \xi C_{ij}Z_jP_j$, i.e. $Z_iC_{ij} = \xi C_{ij}Z_j$ for all $C_{ij} \in \mathcal{M}_{ij}$. It means that $(Z_i - \xi Z_j)C_{ij} = 0$ for all $C_{ij} \in \mathcal{M}_{ij}$. By Lemma 1.1(iii), we get $Z_i = \xi Z_j$, and then $A_{ii} + B_{jj} \in Z_iP_i + Z_jP_j = (\xi P_i + P_j)Z_j \in (\xi P_i + P_j)\mathcal{Z}_{\mathcal{M}}$. □

Our main result reads as follows:

Theorem 2.2. *Let $\mathcal{M}$ be a von Neumann algebra with no central abelian projections. If $\xi \neq 0, 1$ is a scalar and $L : \mathcal{M} \to \mathcal{M}$ is a nonlinear ξ-Lie $*$-derivation satisfying $L(I) \in Z_{\mathcal{M}}$ where I is the identity operator of $\mathcal{M}$, then L is an additive $*$-derivation and $L(\xi A) = \xi L(A)$ for all $A \in \mathcal{M}$.*

Proof. We will divide the proof of the theorem into several claims.

Claim 1. $L(0) = 0$.

Indeed, $L(0) = L([0^*, 0]_\xi) = [L(0)^*, 0]_\xi + [0^*, L(0)]_\xi = 0$.

First, we will show that L is additive.

Claim 2. For every $A_{ii} \in \mathcal{M}_{ii}$, $B_{ij} \in \mathcal{M}_{ij}$ and $B_{ji} \in \mathcal{M}_{ji}$, $1 \leq i \neq j \leq 2$, we have

$$\begin{aligned} L(A_{ii} + B_{ij}) &= L(A_{ii}) + L(B_{ij}), \\ L(A_{ii} + B_{ji}) &= L(A_{ii}) + L(B_{ji}). \end{aligned}$$

Let $T := L(A_{ii} + B_{ij}) - L(A_{ii}) - L(B_{ij}) \in \mathcal{M}$. Then we have

$$\begin{aligned} L(-\xi B_{ij}) &= L([P_j^*, A_{ii} + B_{ij}]_\xi) \\ &= [L(P_j)^*, A_{ii} + B_{ij}]_\xi + [P_j^*, L(A_{ii} + B_{ij})]_\xi. \end{aligned}$$

On the other hand, by Claim 1, we have

$$\begin{aligned} L(-\xi B_{ij}) &= L([P_j^*, A_{ii}]_\xi) + L([P_j^*, B_{ij}]_\xi) \\ &= [L(P_j)^*, A_{ii}]_\xi + [P_j^*, L(A_{ii})]_\xi + [L(P_j)^*, B_{ij}]_\xi + [P_j^*, L(B_{ij})]_\xi \\ &= [L(P_j)^*, A_{ii} + B_{ij}]_\xi + [P_j^*, L(A_{ii}) + L(B_{ij})]_\xi. \end{aligned}$$

Hence $[P_j, T]_\xi = 0$, that is $P_j T - \xi T P_j = 0$. Since $\xi \neq 1$, we get

$$T_{jj} = \frac{-T_{ji} + \xi T_{ij}}{1 - \xi}.$$

Then we obtain that

$$\Big[P_j, T_{ii} + T_{ij} + T_{ji} + \frac{-T_{ji} + \xi T_{ij}}{1 - \xi}\Big]_\xi = 0.$$

With easy calculations we have $\frac{-\xi}{1-\xi}(T_{ij} + T_{ji}) = 0$. Since $\xi \neq 0, 1$, we get $T_{ij} + T_{ji} = 0$. Then we have $T_{jj} - \xi T_{jj} = 0$. Since $\xi \neq 1$, we get $T_{jj} = 0$. Thus $T_{jj} = T_{ij} + T_{ji} = 0$. Similarly,

$$\begin{aligned} L((\xi - \xi^2) A_{ii}) &= L([(\overline{\xi} P_i + P_j)^*, A_{ii} + B_{ij}]_\xi) \\ &= [L(\overline{\xi} P_i + P_j)^*, A_{ii} + B_{ij}]_\xi + [(\overline{\xi} P_i + P_j)^*, L(A_{ii} + B_{ij})]_\xi. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} L((\xi - \xi^2) A_{ii}) &= L([(\overline{\xi} P_i + P_j)^*, A_{ii}]_\xi) + L([(\overline{\xi} P_i + P_j)^*, B_{ij}]_\xi) \\ &= [L(\overline{\xi} P_i + P_j)^*, A_{ii}]_\xi + [(\overline{\xi} P_i + P_j)^*, L(A_{ii})]_\xi + [L(\overline{\xi} P_i + P_j)^*, B_{ij}]_\xi \\ &\quad + [(\overline{\xi} P_i + P_j)^*, L(B_{ij})]_\xi \\ &= [L(\overline{\xi} P_i + P_j)^*, A_{ii} + B_{ij}]_\xi + [(\overline{\xi} P_i + P_j)^*, L(A_{ii}) + L(B_{ij})]_\xi. \end{aligned}$$

Hence $[\xi P_i + P_j, T]_\xi = 0$, that is $(\xi P_i + P_j)T - \xi T(\xi P_i + P_j) = 0$. Then we get $\xi(1-\xi)T_{ii} = 0$. Since $\xi \neq 0, 1$, we have $T_{ii} = 0$. Hence $T = 0$ and thus $L(A_{ii} + B_{ij}) = L(A_{ii}) + L(B_{ij})$. Similarly, one can prove $L(A_{ii} + B_{ji}) = L(A_{ii}) + L(B_{ji})$.

Claim 3. For every $A_{ij}, B_{ij} \in \mathcal{M}_{ij}$, $1 \leq i \neq j \leq 2$, we have

$$L(A_{ij} + B_{ij}) = L(A_{ij}) + L(B_{ij}).$$

Since $A_{ij} + B_{ij} = [(A_{ij}^* + P_i)^*, B_{ij} + P_j]_\xi$, by using Claim 2, we have that

$$\begin{aligned} L(A_{ij} + B_{ij}) &= [L(A_{ij}^* + P_i)^*, B_{ij} + P_j]_\xi + [(A_{ij}^* + P_i)^*, L(B_{ij} + P_j)]_\xi \\ &= [L(A_{ij}^*)^* + L(P_i)^*, B_{ij} + P_j]_\xi + [(A_{ij}^* + P_i)^*, L(B_{ij}) + L(P_j)]_\xi \\ &= [L(A_{ij}^*)^*, B_{ij}]_\xi + [L(A_{ij}^*)^*, P_j]_\xi + [L(P_i)^*, B_{ij}]_\xi + [L(P_i)^*, P_j]_\xi \\ &\quad + [A_{ij}, L(B_{ij})]_\xi + [A_{ij}, L(P_j)]_\xi + [P_i, L(B_{ij})]_\xi + [P_i, L(P_j)]_\xi \\ &= L([(A_{ij}^*)^*, B_{ij}]_\xi) + L([(A_{ij}^*)^*, P_j]_\xi) + L([P_i^*, B_{ij}]_\xi) + L([P_i^*, P_j]_\xi) \\ &= L([A_{ij}, P_j]_\xi) + L([P_i, B_{ij}]_\xi) \\ &= L(A_{ij} P_j - \xi P_j A_{ij}) + L(P_i B_{ij} - \xi B_{ij} P_i) \\ &= L(A_{ij}) + L(B_{ij}). \end{aligned}$$

Claim 4. For any $A_{ii} \in \mathcal{M}_{ii}$, $B_{jj} \in \mathcal{M}_{jj}$, $1 \leq i \neq j \leq 2$, we have

$$L(A_{ii} + B_{jj}) = L(A_{ii}) + L(B_{jj}).$$

Let $T := L(A_{ii} + B_{jj}) - L(A_{ii}) - L(B_{jj}) \in \mathcal{M}$. We have

$$\begin{aligned} L((1-\xi)A_{ii}) &= L([P_i^*, A_{ii} + B_{jj}]_\xi) \\ &= [L(P_i)^*, A_{ii} + B_{jj}]_\xi + [P_i^*, L(A_{ii} + B_{jj})]_\xi. \end{aligned}$$

On the other hand, by Claim 1,

$$\begin{aligned} L((1-\xi)A_{ii}) &= L([P_i^*, A_{ii}]_\xi) + L([P_i^*, B_{jj}]_\xi) \\ &= [L(P_i)^*, A_{ii}]_\xi + [P_i^*, L(A_{ii})]_\xi + [L(P_i)^*, B_{jj}]_\xi + [P_i^*, L(B_{jj})]_\xi \\ &= [L(P_i)^*, A_{ii} + B_{jj}]_\xi + [P_i^*, L(A_{ii}) + L(B_{jj})]_\xi. \end{aligned}$$

Hence $[P_i^*, T]_\xi = 0$, that is $P_iT - \xi TP_i = 0$. Since $\xi \neq 1$, we get $T_{ii} = T_{ij} + T_{ji} = 0$.

Similarly,

$$\begin{aligned} L((1-\xi)B_{jj}) &= L([P_j^*, A_{ii} + B_{jj}]_\xi) \\ &= [L(P_j)^*, A_{ii} + B_{jj}]_\xi + [P_j^*, L(A_{ii} + B_{jj})]_\xi. \end{aligned}$$

On the other hand,

$$\begin{aligned} L((1-\xi)B_{jj}) &= L([P_j^*, A_{ii}]_\xi) + L([P_j^*, B_{jj}]_\xi) \\ &= [L(P_j)^*, A_{ii}]_\xi + [P_j^*, L(A_{ii})]_\xi + [L(P_j)^*, B_{jj}]_\xi + [P_j^*, L(B_{jj})]_\xi \\ &= [L(P_j)^*, A_{ii} + B_{jj}]_\xi + [P_j^*, L(A_{ii}) + L(B_{jj})]_\xi. \end{aligned}$$

Thus we have $[P_j^*, T]_\xi = 0$, that is $P_jT - \xi TP_j = 0$. Since $\xi \neq 1$, we get $T_{jj} = 0$. Then we obtain that $T = 0$, hence $L(A_{ii} + B_{jj}) = L(A_{ii}) + L(B_{jj})$.

Claim 5. For any $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$, $i = 1, 2$, we have

$$L(A_{ii} + B_{ii}) = L(A_{ii}) + L(B_{ii}).$$

Let $T := L(A_{ii} + B_{ii}) - L(A_{ii}) - L(B_{ii}) \in \mathcal{M}$. We only need to prove $T = 0$. For $i \neq j$, we have

$$\begin{aligned} 0 &= L([P_j^*, A_{ii} + B_{ii}]_\xi) \\ &= [L(P_j)^*, A_{ii} + B_{ii}]_\xi + [P_j^*, L(A_{ii} + B_{ii})]_\xi. \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 &= L([P_j^*, A_{ii}]_\xi) + L([P_j^*, B_{ii}]_\xi) \\ &= [L(P_j)^*, A_{ii}]_\xi + [P_j^*, L(A_{ii})]_\xi + [L(P_j)^*, B_{ii}]_\xi + [P_j^*, L(B_{ii})]_\xi \\ &= [L(P_j)^*, A_{ii} + B_{ii}]_\xi + [P_j^*, L(A_{ii}) + L(B_{ii})]_\xi. \end{aligned}$$

Hence $[P_j^*, T]_\xi = 0$, that is $P_jT - \xi TP_j = 0$. Since $\xi \neq 1$, we get $T_{jj} = T_{ij} + T_{ji} = 0$.

For any $C_{ij} \in \mathcal{M}_{ij}$ $(i \neq j)$, by Claim 3, we have

$$\begin{aligned} & [L(C_{ij})^*, A_{ii} + B_{ii}]_\xi + [C_{ij}^*, L(A_{ii} + B_{ii})]_\xi \\ = \; & L([C_{ij}^*, A_{ii} + B_{ii}]_\xi) \\ = \; & L(C_{ij}^*A_{ii} + C_{ij}^*B_{ii}) = L(C_{ij}^*A_{ii}) + L(C_{ij}^*B_{ii}) \\ = \; & L([C_{ij}^*, A_{ii}]_\xi) + L([C_{ij}^*, B_{ii}]_\xi) \\ = \; & [L(C_{ij})^*, A_{ii}]_\xi + [C_{ij}^*, L(A_{ii})]_\xi + [L(C_{ij})^*, B_{ii}]_\xi + [C_{ij}^*, L(B_{ii})]_\xi \\ = \; & [L(C_{ij})^*, A_{ii} + B_{ii}]_\xi + [C_{ij}^*, L(A_{ii}) + L(B_{ii})]_\xi. \end{aligned}$$

Thus we have $[C_{ij}^*, T]_\xi = 0$. That is, $C_{ij}^* T_{ii} = 0$ for all $C_{ij} \in \mathcal{M}_{ij}$. Hence $T_{ii}^* P_i C P_j = 0$ for all $C \in \mathcal{M}$. By Lemma 1.1(iii), we get $T_{ii} = 0$. Consequently, we have $T = 0$. Hence $L(A_{ii} + B_{ii}) = L(A_{ii}) + L(B_{ii})$.

Claim 6. For any $A_{ij} \in \mathcal{M}_{ij}$, $B_{ji} \in \mathcal{M}_{ji}$, we have

$$L(A_{ij} + B_{ji}) = L(A_{ij}) + L(B_{ji}).$$

Let $T := L(A_{ij} + B_{ji}) - L(A_{ij}) - L(B_{ji}) \in \mathcal{M}$. For every $C_{ij} \in \mathcal{M}_{ij}$,

$$\begin{aligned} & [L(C_{ij})^*, A_{ij} + B_{ji}]_\xi + [C_{ij}^*, L(A_{ij} + B_{ji})]_\xi \\ = \; & L([C_{ij}^*, A_{ij} + B_{ji}]_\xi) \\ = \; & L([C_{ij}^*, A_{ij}]_\xi) + L([C_{ij}^*, B_{ji}]_\xi) \\ = \; & [L(C_{ij})^*, A_{ij}]_\xi + [C_{ij}^*, L(A_{ij})]_\xi + [L(C_{ij})^*, B_{ji}]_\xi + [C_{ij}^*, L(B_{ji})]_\xi \\ = \; & [L(C_{ij})^*, A_{ij} + B_{ji}]_\xi + [C_{ij}^*, L(A_{ij}) + L(B_{ji})]_\xi. \end{aligned}$$

Hence $[C_{ij}^*, T]_\xi = 0$. That is, $C_{ij}^* T - \xi T C_{ij}^* = 0$. Thus we have $C_{ij}^* T P_j = 0$, i.e. $C_{ij}^* T_{ij} P_j = 0$ for all $C_{ij} \in \mathcal{M}_{ij}$. Hence $T_{ij}^* P_i C P_j = 0$ for all $C \in \mathcal{M}$. By Lemma 1.1(iii), we have $T_{ij} = 0$. Similarly, $T_{ji} = 0$.

On the other hand,

$$\begin{aligned} & [L(\bar{\xi} P_i + P_j)^*, A_{ij} + B_{ji}]_\xi + [(\bar{\xi} P_i + P_j)^*, L(A_{ij} + B_{ji})]_\xi \\ = \; & L([(\bar{\xi} P_i + P_j)^*, A_{ij} + B_{ji}]_\xi) \\ = \; & L([(\bar{\xi} P_i + P_j)^*, A_{ij}]_\xi) + L([(\bar{\xi} P_i + P_j)^*, B_{ji}]_\xi) \\ = \; & [L(\bar{\xi} P_i + P_j)^*, A_{ij}]_\xi + [(\bar{\xi} P_i + P_j)^*, L(A_{ij})]_\xi + [L(\bar{\xi} P_i + P_j)^*, B_{ji}]_\xi \\ & + [(\bar{\xi} P_i + P_j)^*, L(B_{ji})]_\xi \\ = \; & [L(\bar{\xi} P_i + P_j)^*, A_{ij} + B_{ji}]_\xi + [(\bar{\xi} P_i + P_j)^*, L(A_{ij}) + L(B_{ji})]_\xi. \end{aligned}$$

Hence $[\xi P_i + P_j, T]_\xi = 0$, that is $(\xi P_i + P_j) T - \xi T (\xi P_i + P_j) = 0$. Thus we have $\xi T_{ii} + T_{jj} = 0$. Similarly, we get $T_{ii} + \xi T_{jj} = 0$. Comparing these equations, we obtain that $T_{ii} = T_{jj}$. Hence $[\xi P_i + P_j, 2T_{ii}]_\xi = 0$, that is $T_{ii} = 0$. So we have $T_{ii} = T_{jj} = 0$. Then we get $T = 0$, proving the claim.

Claim 7. For any $A_{ii} \in \mathcal{M}_{ii}$, $B_{ij} \in \mathcal{M}_{ij}$, $C_{ji} \in \mathcal{M}_{ji}$, $1 \le i \ne j \le 2$, we have

$$L(A_{ii} + B_{ij} + C_{ji}) = L(A_{ii}) + L(B_{ij}) + L(C_{ji}).$$

Let $T := L(A_{ii} + B_{ij} + C_{ji}) - L(A_{ii}) - L(B_{ij}) - L(C_{ji}) \in \mathcal{M}$. It follows from Claim 6 that

$$\begin{aligned} & [L(P_j)^*, A_{ii} + B_{ij} + C_{ji}]_\xi + [P_j^*, L(A_{ii} + B_{ij} + C_{ji})]_\xi \\ = \; & L([P_j^*, A_{ii} + B_{ij} + C_{ji}]_\xi) \\ = \; & L([P_j^*, A_{ii}]_\xi) + L([P_j^*, B_{ij} + C_{ji}]_\xi) \\ = \; & [L(P_j)^*, A_{ii}]_\xi + [P_j^*, L(A_{ii})]_\xi + [L(P_j)^*, B_{ij} + C_{ji}]_\xi + [P_j^*, L(B_{ij} + C_{ji})]_\xi \\ = \; & [L(P_j)^*, A_{ii} + B_{ij} + C_{ji}]_\xi + [P_j^*, L(A_{ii}) + L(B_{ij}) + L(C_{ji})]_\xi. \end{aligned}$$

Hence $[P_j^*, T]_\xi = 0$, that is $P_j T - \xi T P_j = 0$. Since $\xi \ne 1$, we have $T_{jj} = T_{ij} + T_{ji} = 0$.

By using Claim 2, we have that

$$\begin{aligned} & [L(\bar{\xi} P_i + P_j)^*, A_{ii} + B_{ij} + C_{ji}]_\xi + [(\bar{\xi} P_i + P_j)^*, L(A_{ii} + B_{ij} + C_{ji})]_\xi \\ = \; & L([(\bar{\xi} P_i + P_j)^*, A_{ii} + B_{ij} + C_{ji}]_\xi) \\ = \; & L([(\bar{\xi} P_i + P_j)^*, A_{ii} + C_{ji}]_\xi) + L([(\bar{\xi} P_i + P_j)^*, B_{ij}]_\xi) \\ = \; & [L(\bar{\xi} P_i + P_j)^*, A_{ii} + C_{ji}]_\xi + [(\bar{\xi} P_i + P_j)^*, L(A_{ii} + C_{ji})]_\xi + [L(\bar{\xi} P_i + P_j)^*, B_{ij}]_\xi \\ & + [(\bar{\xi} P_i + P_j)^*, L(B_{ij})]_\xi \\ = \; & [L(\bar{\xi} P_i + P_j)^*, A_{ii} + B_{ij} + C_{ji}]_\xi + [(\bar{\xi} P_i + P_j)^*, L(A_{ii}) + L(B_{ij}) + L(C_{ji})]_\xi. \end{aligned}$$

Thus we get $[\xi P_i + P_j, T]_\xi = 0$, that is $(\xi P_i + P_j)T - \xi T(\xi P_i + P_j) = 0$. Hence $T_{ii} = 0$. So we have $T = 0$, this proves the claim.

Claim 8. For any $A_{11} \in \mathcal{M}_{11}$, $B_{12} \in \mathcal{M}_{12}$, $C_{21} \in \mathcal{M}_{21}$, $D_{22} \in \mathcal{M}_{22}$, we have

$$L(A_{11} + B_{12} + C_{21} + D_{22}) = L(A_{11}) + L(B_{12}) + L(C_{21}) + L(D_{22}).$$

Let $T := L(A_{11} + B_{12} + C_{21} + D_{22}) - L(A_{11}) - L(B_{12}) - L(C_{21}) - L(D_{22}) \in \mathcal{M}$. It follows from Claim 7 that

$$\begin{aligned}
& [L(P_1)^*, A_{11} + B_{12} + C_{21} + D_{22}]_\xi + [P_1^*, L(A_{11} + B_{12} + C_{21} + D_{22})]_\xi \\
= \; & L([P_1^*, A_{11} + B_{12} + C_{21} + D_{22}]_\xi) \\
= \; & L([P_1^*, A_{11} + B_{12} + C_{21}]_\xi) + L([P_1^*, D_{22}]_\xi) \\
= \; & [L(P_1)^*, A_{11} + B_{12} + C_{21}]_\xi + [P_1^*, L(A_{11} + B_{12} + C_{21})]_\xi + [L(P_1)^*, D_{22}]_\xi \\
& + [P_1^*, L(D_{22})]_\xi \\
= \; & [L(P_1)^*, A_{11} + B_{12} + C_{21} + D_{22}]_\xi + [P_1^*, L(A_{11}) + L(B_{12}) + L(C_{21}) + L(D_{22})]_\xi.
\end{aligned}$$

Hence $[P_1, T]_\xi = 0$. That is, $P_1 T - \xi T P_1 = 0$. Then we have $T_{11} = T_{12} + T_{21} = 0$. Similarly, we can obtain that $T_{22} = 0$. Hence $T = 0$.

Claim 9. L is additive.
By Claims 3, 5 and 8, we can prove that L is additive.

Since L is additive and $L(I) \in \mathcal{Z}_\mathcal{M}$, we get

$$L(A) - L(\xi A) = L((1-\xi)A) = L([I^*, A]_\xi) = [I^*, L(A)]_\xi = L(A) - \xi L(A)$$

for any $A \in \mathcal{M}$. Hence $L(\xi A) = \xi L(A)$ for all $A \in \mathcal{M}$.

Now we need to prove that L is an additive derivation and $L(A^*) = L(A)^*$ for all $A \in \mathcal{M}$.

Claim 10. $P_1 L(P_i) P_1 + P_2 L(P_i) P_2 = 0, \;\; i = 1, 2.$

For any $A_{12} \in \mathcal{M}_{12}$,

$$\begin{aligned}
L(A_{12}) &= L([P_1^*, A_{12}]_\xi) \\
&= [L(P_1)^*, A_{12}]_\xi + [P_1^*, L(A_{12})]_\xi \\
&= L(P_1)^* A_{12} - \xi A_{12} L(P_1)^* + P_1 L(A_{12}) - \xi L(A_{12}) P_1.
\end{aligned}$$

Multiplying both sides of the above equation by P_1 and P_2 from the left and right, respectively, we have

$$P_1 L(P_1)^* P_1 A_{12} = \xi A_{12} P_2 L(P_1)^* P_2.$$

By using Lemma 2.1, we get

$$P_1 L(P_1)^* P_1 + P_2 L(P_1)^* P_2 \in (\xi P_1 + P_2)\mathcal{Z}_\mathcal{M}. \tag{2.1}$$

Similarly, for any $A_{21} \in \mathcal{M}_{21}$,

$$\begin{aligned}
L(A_{21}) &= L([P_2^*, A_{21}]_\xi) \\
&= [L(P_2)^*, A_{21}]_\xi + [P_2^*, L(A_{21})]_\xi \\
&= L(P_2)^* A_{21} - \xi A_{21} L(P_2)^* + P_2 L(A_{21}) - \xi L(A_{21}) P_2.
\end{aligned}$$

Multiplying both sides of the above equation by P_2 and P_1 from the left and right, respectively, we get

$$P_2 L(P_2)^* P_2 A_{21} = \xi A_{21} P_1 L(P_2)^* P_1.$$

It follows from Lemma 2.1 that

$$P_2L(P_2)^*P_2 + P_1L(P_2)^*P_1 \in (P_1 + \xi P_2)\mathcal{Z}_{\mathcal{M}}. \tag{2.2}$$

Assume that $P_1L(P_1)^*P_1 + P_2L(P_1)^*P_2 = (\xi P_1 + P_2)Z_1$ and $P_2L(P_2)^*P_2 + P_1L(P_2)^*P_1 = (P_1 + \xi P_2)Z_2$ for some $Z_1, Z_2 \in \mathcal{Z}_{\mathcal{M}}$. We also have that

$$\begin{aligned} 0 &= L([P_2^*, P_1]_\xi) \\ &= [L(P_2)^*, P_1]_\xi + [P_2^*, L(P_1)]_\xi \\ &= L(P_2)^*P_1 - \xi P_1L(P_2)^* + P_2L(P_1) - \xi L(P_1)P_2. \end{aligned}$$

If we multiply both sides of the above equation by P_2 from the left and right, respectively, then we get $(1-\xi)P_2L(P_1)P_2 = 0$. Since $\xi \neq 1$, we obtain that $P_2L(P_1)P_2 = 0$. Similarly, since $\xi \neq 1$, we can get $P_1L(P_2)^*P_1 = 0$. Since $P_2L(P_1)P_2 = 0$, we also have $P_2L(P_1)^*P_2 = 0$, and it follows from that

$$[(\xi P_1 + P_2)Z_1, P_2]_\xi = [P_1L(P_1)^*P_1 + P_2L(P_1)^*P_2, P_2]_\xi = 0.$$

Hence $(1-\xi)P_2Z_1 = 0$. Since $\xi \neq 1$, we get $Z_1P_2 = 0$. By Lemma 1.1(iii), we have that $Z_1 = 0$. Thus

$$P_1L(P_1)^*P_1 + P_2L(P_1)^*P_2 = 0. \tag{2.3}$$

Similarly, we can obtain that

$$P_2L(P_2)^*P_2 + P_1L(P_2)^*P_1 = 0. \tag{2.4}$$

From the equations (2.3) and (2.4), we easily reach the desired result.

Define a mapping $\Delta : \mathcal{M} \to \mathcal{M}$ by $\Delta(A) = L(A) - [A, T_0]$ for all $A \in \mathcal{M}$, where $T_0 := P_1L(P_1)P_2 - P_2L(P_1)P_1$.

Claim 11. $T_0^* = -T_0$.

Since L is additive and $L(\xi A) = \xi L(A)$ for all $A \in \mathcal{M}$, we have

$$\begin{aligned} L(P_1) - \xi L(P_1) &= L([P_1^*, P_1]_\xi) \\ &= [L(P_1)^*, P_1]_\xi + [P_1^*, L(P_1)]_\xi \\ &= L(P_1)^*P_1 - \xi P_1L(P_1)^* + P_1L(P_1) - \xi L(P_1)P_1. \end{aligned} \tag{2.5}$$

Multiplying both sides of the above equation by P_1 and P_2 from the left and right, respectively, we get

$$-\xi P_1L(P_1)P_2 = -\xi P_1L(P_1)^*P_2.$$

Since $\xi \neq 0$, we have

$$P_1L(P_1)P_2 = P_1L(P_1)^*P_2. \tag{2.6}$$

On the other hand, if we multiply both sides of the equation (2.5) by P_2 and P_1 from the left and right, respectively, we get

$$P_2L(P_1)P_1 = P_2L(P_1)^*P_1. \tag{2.7}$$

Then by using the equations (2.6) and (2.7), we have that $T_0^* = -T_0$.

Since $T_0^* = -T_0$, we have $\Delta([A^*, B]_\xi) = [\Delta(A)^*, B]_\xi + [A^*, \Delta(B)]_\xi$ for all $A, B \in \mathcal{M}$.

Claim 12. $\Delta(P_i) = 0$, $i = 1, 2$.

We have that

$$\begin{aligned} 0 &= L([P_1^*, P_2]_\xi) \\ &= [L(P_1)^*, P_2]_\xi + [P_1^*, L(P_2)]_\xi \\ &= L(P_1)^* P_2 - \xi P_2 L(P_1)^* + P_1 L(P_2) - \xi L(P_2) P_1. \end{aligned} \tag{2.8}$$

Multiplying both sides of the equation (2.8) by P_1 and P_2 from the left and right, respectively, we obtain that

$$P_1 L(P_1)^* P_2 + P_1 L(P_2) P_2 = 0. \tag{2.9}$$

Similarly since $\xi \neq 0$, we can get

$$P_2 L(P_1)^* P_1 + P_2 L(P_2) P_1 = 0. \tag{2.10}$$

By using the equations (2.6) and (2.7) in the proof of Claim 11, we have that

$$P_1 L(P_1) P_2 + P_1 L(P_2) P_2 = 0 \tag{2.11}$$

and

$$P_2 L(P_1) P_1 + P_2 L(P_2) P_1 = 0. \tag{2.12}$$

If we add the equations (2.11) and (2.12), then we get $L(P_1) + L(P_2) = 0$. Thus $\Delta(P_1) = L(P_1) - [P_1, T_0] = 0$ and $\Delta(P_2) = L(P_2) + L(P_1) = 0$.

Claim 13. $\Delta(\mathcal{M}_{ij}) \subseteq \mathcal{M}_{ij},\ 1 \leq i \neq j \leq 2$.

For any $B_{ij} \in \mathcal{M}_{ij}$, $1 \leq i \neq j \leq 2$, we have

$$\begin{aligned} \Delta(B_{ij}) &= \Delta([P_i^*, B_{ij}]_\xi) \\ &= [P_i^*, \Delta(B_{ij})]_\xi = P_i \Delta(B_{ij}) - \xi \Delta(B_{ij}) P_i. \end{aligned}$$

Then,

$$P_i \Delta(B_{ij}) P_i = P_j \Delta(B_{ij}) P_j = 0. \tag{2.13}$$

Moreover, if $\xi \neq -1$, then we have $P_j \Delta(B_{ij}) P_i = 0$.

Assume that $\xi = -1$. For every $A_{ii} \in \mathcal{M}_{ii}$, $B_{ij} \in \mathcal{M}_{ij}$,

$$\begin{aligned} \Delta(A_{ii}^* B_{ij}) &= \Delta([A_{ii}^*, B_{ij}]_{-1}) \\ &= [\Delta(A_{ii})^*, B_{ij}]_{-1} + [A_{ii}^*, \Delta(B_{ij})]_{-1} \\ &= \Delta(A_{ii})^* B_{ij} + B_{ij} \Delta(A_{ii})^* + A_{ii}^* \Delta(B_{ij}) + \Delta(B_{ij}) A_{ii}^*. \end{aligned}$$

It follows from (2.13) that,

$$P_j \Delta(A_{ii}^* B_{ij}) P_i = P_j \Delta(B_{ij}) A_{ii}^* P_i = \Delta(B_{ij}) A_{ii}^*. \tag{2.14}$$

Then for every N_{ii},

$$P_j \Delta(N_{ii}^* A_{ii}^* B_{ij}) P_i = \Delta(B_{ij}) N_{ii}^* A_{ii}^*. \tag{2.15}$$

On the other hand,

$$P_j \Delta(N_{ii}^* A_{ii}^* B_{ij}) P_i = \Delta(A_{ii}^* B_{ij}) N_{ii}^*.$$

By (2.14), we also have $\Delta(A_{ii}^* B_{ij}) N_{ii}^* = \Delta(B_{ij}) A_{ii}^* N_{ii}^*$ since

$$\Delta(A_{ii}^* B_{ij}) P_i = (I - P_i) \Delta(A_{ii}^* B_{ij}) P_i = P_j \Delta(A_{ii}^* B_{ij}) P_i = \Delta(B_{ij}) A_{ii}^*.$$

It means that

$$P_j \Delta(N_{ii}^* A_{ii}^* B_{ij}) P_i = \Delta(A_{ii}^* B_{ij}) N_{ii}^* = \Delta(B_{ij}) A_{ii}^* N_{ii}^*. \tag{2.16}$$

From (2.15) and (2.16), we have

$$\Delta(B_{ij}) [N_{ii}^*, A_{ii}^*] = 0.$$

Now replacing N_{ii} by $N_{ii}R_{ii}$ where $R_{ii} \in \mathcal{M}_{ii}$, we obtain

$$\Delta(B_{ij})R_{ii}^*[N_{ii}^*, A_{ii}^*] = 0.$$

By Lemma 1.1(ii), $\Delta(B_{ij})P_i^* = 0$. Hence $P_j\Delta(B_{ij})P_i = 0$ for all $\xi \in \mathbb{C}$. Thus we get $\Delta(B_{ij}) \in \mathcal{M}_{ij}$ for all $B_{ij} \in \mathcal{M}_{ij}$.

Claim 14. $\Delta(\mathcal{M}_{ii}) \subseteq \mathcal{M}_{ii},\ i = 1, 2.$

We have that

$$\begin{aligned}
0 &= \Delta(P_i) = \Delta\Big(\Big[I^*, \frac{1}{1-\xi}P_i\Big]_\xi\Big) \\
&= \Big[\Delta(I)^*, \frac{1}{1-\xi}P_i\Big]_\xi + \Big[I^*, \Delta\Big(\frac{1}{1-\xi}P_i\Big)\Big]_\xi \\
&= \frac{1}{1-\xi}\Delta([I^*, P_i]_\xi) + \Big[I^*, \Delta\Big(\frac{1}{1-\xi}P_i\Big)\Big]_\xi \\
&= \frac{1}{1-\xi}\Delta((1-\xi)P_i) + (1-\xi)\Delta\Big(\frac{1}{1-\xi}P_i\Big).
\end{aligned}$$

On the other hand, we get

$$\Delta((1-\xi)P_i) = \Delta([P_i^*, P_i]_\xi) = 0.$$

Hence $\Delta\Big(\frac{1}{1-\xi}P_i\Big) = 0$. For any $A_{ii} \in \mathcal{M}_{ii}$,

$$\begin{aligned}
\Delta(A_{ii}) &= \Delta\Big(\Big[\Big(\frac{1}{1-\bar{\xi}}P_i\Big)^*, A_{ii}\Big]_\xi\Big) \\
&= \Big[\frac{1}{1-\xi}P_i, \Delta(A_{ii})\Big]_\xi \\
&= \frac{1}{1-\xi}P_i\Delta(A_{ii}) - \xi\Delta(A_{ii})\frac{1}{1-\xi}P_i \\
&= \frac{1}{1-\xi}(P_i\Delta(A_{ii}) - \xi\Delta(A_{ii})P_i).
\end{aligned}$$

Thus we have $\Delta(A_{ii}) \in \mathcal{M}_{ii}$, $i = 1, 2$.

Now, we will show that $\Delta(AB) = \Delta(A)B + A\Delta(B)$ for every $A, B \in \mathcal{M}$, that is Δ is an additive derivation.

Claim 15. For any $A_{ii} \in \mathcal{M}_{ii}$, $A_{jj} \in \mathcal{M}_{jj}$, $B_{ij} \in \mathcal{M}_{ij}$, $1 \le i \ne j \le 2$, we have

$$\begin{aligned}
\Delta(A_{ii}B_{ij}) &= \Delta(A_{ii})B_{ij} + A_{ii}\Delta(B_{ij}), \\
\Delta(B_{ij}A_{jj}) &= \Delta(B_{ij})A_{jj} + B_{ij}\Delta(A_{jj}), \\
\Delta(B_{ij}^*) &= \Delta(B_{ij})^*.
\end{aligned}$$

We have that

$$\begin{aligned}
-\xi\Delta(A_{ii}B_{ji}^*) &= \Delta(-\xi A_{ii}B_{ji}^*) = \Delta([B_{ji}^*, A_{ii}]_\xi) \\
&= [\Delta(B_{ji})^*, A_{ii}]_\xi + [B_{ji}^*, \Delta(A_{ii})]_\xi \\
&= -\xi A_{ii}\Delta(B_{ji})^* - \xi\Delta(A_{ii})B_{ji}^*.
\end{aligned}$$

Since $\xi \ne 0$, we have $\Delta(A_{ii}B_{ji}^*) = A_{ii}\Delta(B_{ji})^* + \Delta(A_{ii})B_{ji}^*$. On the other hand, by using the equation $\Delta(P_i) = 0$, we get

$$\Delta(B_{ji}^*) = \Delta(P_iB_{ji}^*) = P_i\Delta(B_{ji})^* = \Delta(B_{ji})^*.$$

It follows from that

$$\begin{aligned}\Delta(A_{ii}B_{ij}) &= \Delta(A_{ii}(B_{ij}^*)^*)\\ &= A_{ii}\Delta(B_{ij}^*)^* + \Delta(A_{ii})B_{ij}\\ &= A_{ii}\Delta(B_{ij}) + \Delta(A_{ii})B_{ij}.\end{aligned}$$

Similarly, we have $\Delta(B_{ij}A_{jj}) = \Delta(B_{ij})A_{jj} + B_{ij}\Delta(A_{jj})$.

Claim 16. For any $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$, $i = 1, 2$, we have

$$\begin{aligned}\Delta(A_{ii}B_{ii}) &= \Delta(A_{ii})B_{ii} + A_{ii}\Delta(B_{ii}),\\ \Delta(A_{ii}^*) &= \Delta(A_{ii})^*.\end{aligned}$$

For any $C_{ij} \in \mathcal{M}_{ij}$, $i \neq j$, it follows from Claim 15 that

$$\begin{aligned}&\Delta(A_{ii}B_{ii}^*)C_{ij} + A_{ii}B_{ii}^*\Delta(C_{ij})\\ =\;& \Delta(A_{ii}B_{ii}^*C_{ij})\\ =\;& \Delta(A_{ii})B_{ii}^*C_{ij} + A_{ii}\Delta(B_{ii}^*C_{ij})\\ =\;& \Delta(A_{ii})B_{ii}^*C_{ij} + A_{ii}\Delta([B_{ii}^*, C_{ij}]_\xi)\\ =\;& \Delta(A_{ii})B_{ii}^*C_{ij} + A_{ii}([\Delta(B_{ii})^*, C_{ij}]_\xi) + A_{ii}([B_{ii}^*, \Delta(C_{ij})]_\xi)\\ =\;& \Delta(A_{ii})B_{ii}^*C_{ij} + A_{ii}\Delta(B_{ii})^*C_{ij} + A_{ii}B_{ii}^*\Delta(C_{ij}).\end{aligned}$$

Thus $(\Delta(A_{ii}B_{ii}^*) - \Delta(A_{ii})B_{ii}^* - A_{ii}\Delta(B_{ii})^*)C_{ij} = 0$, for all $C_{ij} \in \mathcal{M}_{ij}$. Then we have

$$(\Delta(A_{ii}B_{ii}^*) - \Delta(A_{ii})B_{ii}^* - A_{ii}\Delta(B_{ii})^*)P_iCP_j = 0$$

for all $C \in \mathcal{M}$. It follows from Lemma 1.1(iii) that

$$\Delta(A_{ii}B_{ii}^*) = \Delta(A_{ii})B_{ii}^* + A_{ii}\Delta(B_{ii})^*.$$

By using the above equation, we also have

$$\Delta(A_{ii}^*) = \Delta(P_iA_{ii}^*) = P_i\Delta(A_{ii})^* = \Delta(A_{ii})^*$$

since $\Delta(P_i) = 0$. Hence

$$\begin{aligned}\Delta(A_{ii}B_{ii}) &= \Delta(A_{ii}(B_{ii}^*)^*)\\ &= \Delta(A_{ii})B_{ii} + A_{ii}\Delta(B_{ii}^*)^*\\ &= \Delta(A_{ii})B_{ii} + A_{ii}\Delta(B_{ii}).\end{aligned}$$

Claim 17. For any $A_{ij} \in \mathcal{M}_{ij}$, $B_{ji} \in \mathcal{M}_{ji}$, $1 \leq i \neq j \leq 2$, we have

$$\Delta(A_{ij}B_{ji}) = \Delta(A_{ij})B_{ji} + A_{ij}\Delta(B_{ji}).$$

For any $C_{ij} \in \mathcal{M}_{ij}$, $i \neq j$, it follows from Claim 2 and Claim 15 that

$$\begin{aligned}&\Delta(A_{ij}B_{ij}^*)C_{ij} + A_{ij}B_{ij}^*\Delta(C_{ij})\\ =\;& \Delta(A_{ij}B_{ij}^*C_{ij})\\ =\;& \Delta(A_{ij})B_{ij}^*C_{ij} + A_{ij}\Delta(B_{ij}^*C_{ij})\\ =\;& \Delta(A_{ij})B_{ij}^*C_{ij} + A_{ij}\Delta([B_{ij}^*, C_{ij}]_\xi) + \xi A_{ij}\Delta(C_{ij}B_{ij}^*)\\ =\;& \Delta(A_{ij})B_{ij}^*C_{ij} + A_{ij}([\Delta(B_{ij})^*, C_{ij}]_\xi) + A_{ij}([B_{ij}^*, \Delta(C_{ij})]_\xi)\\ =\;& \Delta(A_{ij})B_{ij}^*C_{ij} + A_{ij}\Delta(B_{ij})^*C_{ij} + A_{ij}B_{ij}^*\Delta(C_{ij}).\end{aligned}$$

Then $(\Delta(A_{ij}B_{ij}^*) - \Delta(A_{ij})B_{ij}^* - A_{ij}\Delta(B_{ij})^*)C_{ij} = 0$ for all $C_{ij} \in \mathcal{M}_{ij}$. Hence we get

$$(\Delta(A_{ij}B_{ij}^*) - \Delta(A_{ij})B_{ij}^* - A_{ij}\Delta(B_{ij})^*)P_iCP_j = 0$$

for all $C \in \mathcal{M}$. It follows from Lemma 1.1(iii) that

$$\Delta(A_{ij}B_{ij}^*) = \Delta(A_{ij})B_{ij}^* + A_{ij}\Delta(B_{ij})^*.$$

Since $\Delta(B_{ij}^*) = \Delta(B_{ij})^*$, we have

$$\begin{aligned} \Delta(A_{ij}B_{ji}) &= \Delta(A_{ij}(B_{ji}^*)^*) \\ &= \Delta(A_{ij})B_{ji} + A_{ij}\Delta(B_{ji}^*)^* \\ &= \Delta(A_{ij})B_{ji} + A_{ij}\Delta(B_{ji}). \end{aligned}$$

Claim 18. Δ is an additive derivation.

For any $A = \sum_{i,j=1}^{2} A_{ij},\ B = \sum_{i,j=1}^{2} B_{ij} \in \mathcal{M}$, we have

$$\begin{aligned} \Delta(AB) &= \Delta(A_{11}B_{11}) + \Delta(A_{11}B_{12}) + \Delta(A_{12}B_{21}) + \Delta(A_{12}B_{22}) + \Delta(A_{21}B_{11}) \\ &\quad + \Delta(A_{21}B_{12}) + \Delta(A_{22}B_{21}) + \Delta(A_{22}B_{22}) \\ &= \Delta(A_{11})B_{11} + A_{11}\Delta(B_{11}) + \Delta(A_{11})B_{12} + A_{11}\Delta(B_{12}) + \Delta(A_{12})B_{21} \\ &\quad + A_{12}\Delta(B_{21}) + \Delta(A_{12})B_{22} + A_{12}\Delta(B_{22}) + \Delta(A_{21})B_{11} + A_{21}\Delta(B_{11}) \\ &\quad + \Delta(A_{21})B_{12} + A_{21}\Delta(B_{12}) + \Delta(A_{22})B_{21} + A_{22}\Delta(B_{21}) + \Delta(A_{22})B_{22} \\ &\quad + A_{22}\Delta(B_{22}) \\ &= \Delta(A_{11})(B_{11} + B_{12}) + \Delta(A_{12})(B_{21} + B_{22}) + \Delta(A_{21})(B_{11} + B_{12}) \\ &\quad + \Delta(A_{22})(B_{21} + B_{22}) + A_{11}(\Delta(B_{11}) + \Delta(B_{12})) + A_{12}(\Delta(B_{21}) + \Delta(B_{22})) \\ &\quad + A_{21}(\Delta(B_{11}) + \Delta(B_{12})) + A_{22}(\Delta(B_{21}) + \Delta(B_{22})) \\ &= (\Delta(A_{11}) + \Delta(A_{21}))(B_{11} + B_{12}) + (\Delta(A_{12}) + \Delta(A_{22}))(B_{21} + B_{22}) \\ &\quad + (A_{11} + A_{21})(\Delta(B_{11}) + \Delta(B_{12})) + (A_{12} + A_{22})(\Delta(B_{21}) + \Delta(B_{22})) \\ &= \Delta(A)B + A\Delta(B). \end{aligned}$$

Hence Δ is an additive derivation.

By the definition of the mapping Δ, we obtain that L is an additive derivation. Finally, we need to prove that $L(A^*) = L(A)^*$ for all $A \in \mathcal{M}$.

From Claim 15 and Claim 16, we get

$$\begin{aligned} \Delta(A^*) &= \Delta(A_{11}^*) + \Delta(A_{12}^*) + \Delta(A_{21}^*) + \Delta(A_{22}^*) \\ &= \Delta(A_{11})^* + \Delta(A_{12})^* + \Delta(A_{21})^* + \Delta(A_{22})^* \\ &= (\Delta(A_{11}) + \Delta(A_{12}) + \Delta(A_{21}) + \Delta(A_{22}))^* \\ &= \Delta(A)^* \end{aligned}$$

for all $A \in \mathcal{M}$. Then by using $T_0^* = -T_0$, we have that

$$\begin{aligned} L(A^*) - [A^*, T_0] &= \Delta(A^*) = \Delta(A)^* \\ &= (L(A) - [A, T_0])^* \\ &= L(A)^* - (AT_0 - T_0A)^* \\ &= L(A)^* - (-T_0A^* + A^*T_0) \\ &= L(A)^* - [A^*, T_0] \end{aligned}$$

for all $A \in \mathcal{M}$. That is $L(A^*) = L(A)^*$ for all $A \in \mathcal{M}$.

Hence we obtain that L is an additive $*$-derivation, as desired. □

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