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PRESERVATION CONDITIONS FOR INFINITE INTEGER SEQUENCE ITERATED GENERATING FUNCTION SCHEMES

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Abstract The concept of iterated generating functions has existed for some time now—these are polynomials that, subject to a recursive process, continuously retain (preserve) and develop more and more lead terms whose coefficients correspond to elements of an infinite sequence. In this paper we focus on the formulation of conditions governing the preserving nature of schemes producing such polynomials, with examples provided that reveal a variety of iterated generating function behaviours.

1 Introduction

1.1 Background

This paper has its origins in results published in 1999 [5], when the notion of an iterated generating function—though not couched in such a term explicitly—was introduced through an iterative procedure creating Catalan number subsequences of ever increasing length. Chinese academic J. Luo had previously reported the observation upon studying the writings of scholar Antu Ming (c.1692-1763), interpreting it first (1988) with reference to vector multiplication, and then (1993) in the context of polynomials; the latter was adopted for convenience in [5], where an inductive proof verified the method put forward and provided motivation for further work.

The celebrated Catalan sequence $\{c_n\}_0^\infty = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{1, 1, 2, 5, 14, \ldots\}$ (see the O.E.I.S. Sequence No. A000108 [7]) has an ordinary generating function $C(x)$ satisfying the equation $0 = xC^2(x) - C(x) + 1$. Almost a decade later, it was noted by Clapperton *et al.* [1] that the recursive algorithm of [5] masked what was merely the simple discretisation $C_{r+1}(x) = 1 + xC_r^2(x)$ of this governing equation which for $r \geq 0$ (given $C_0(x) = 0$) returns successive polynomials (that is, generating functions) $C_1(x) = c_0$, $C_2(x) = c_0 + c_1x$, $C_3(x) = c_0 + c_1x + c_2x^2 + x^3$, $C_4(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + 6x^4 + 6x^5 + 4x^6 + x^7, \ldots$, mapping to a set of finite sequences $\{c_0\}$, $\{c_0, c_1\}$, $\{c_0, c_1, c_2, 1\}$, $\{c_0, c_1, c_2, c_3, 6, 6, 4, 1\}$, and so on, with a single Catalan number added each time within an exponentially increasing string of terms. The authors went on to describe (Theorem, p. 119), and prove formally in two contrasting ways, a means to generate iteratively an *arbitrary finite* ‘target’ sequence (as opposed to an infinite one such as the Catalan sequence), with examples of different type included to demonstrate robustness of the simple procedure devised.

A recurrence rule of general form $G_{r+1}(x) = H(x, G_r(x))$ (H polynomial in $G_r(x)$, with functional (polynomial) coefficients in x) is—based on the known equation governing a generating function $G(x)$ for a particular infinite sequence—easily set up, it seems, through mere re-arrangement and discretisation, outputting a series of iterated generating functions (polynomials) whose agreeing coefficients as target sequence terms progress linearly—in the above example for the Catalan sequence, the rate is 1. The analysis of [1] was an interesting aside to other, more fundamental, questions concerning infinite sequences delivered by means of an input-output mechanism controlling iterated generating functions, and these were taken up in parallel work [2] where further examination of discretised systems was made. Beginning with some simple illustrations of naturally occurring schemes that produce known sequences in various ways iteratively, the 'fitting' of a general six-parameter algorithm (with $H(x, G_r(x)) = (\alpha + \beta x)G_r^2(x) + (\gamma + \delta x)G_r(x) + \varepsilon + \eta x$ quadratic in $G_r(x)$ and with linear polynomial coefficients) to three different sequences was conducted by hand in an attempt to gain insight into the manner in which anticipated schemes are yielded (or not) as a consequence of the imposed recursion $G_{r+1}(x) = H(x, G_r(x))$. The polynomial $H(x, G_r(x))$ needed to be over-specified, of course, in order to accommodate expected final forms, and the technique of 'term matching' driving the emergent formulations was revealing in itself as an exercise in iterated generating function scheme construction from first principles. While useful, however, algebraic intractability demanded that the evaluation of the free variables $\alpha, \ldots, \eta$ be automated so as to offer an opportunity for exploratory calculations on a larger scale. Higher order polynomials $H(x, G_r(x))$ in $G_r(x)$ (and with more degrees of freedom within functional coefficients) were able to be deployed computationally, the results from which led to the discovery of so called Catalan polynomials seen to be integral to an interesting and novel scheme formulated to attain a pre-set (finite) number of Catalan sequence terms before 'failing' (a result formulated empirically and formally proved [2, Theorem, p. 17]).

Since the 2008 works [1, 2] the phenomenon of iterated generating functions has not been discussed in the literature to any great extent, the most pertinent article comprising a short account offered by Jarvis [4] within the framework of metric space theory which throws new light on it. Other papers, where they appear, have focused elsewhere. It was, for instance, remarked in [1] that not all sequences can be produced by means of iterated generating functions since there are only a countable number of schemes and an uncountable number of sequences in existence. This fact was proved in the Appendix of [3], for completeness, as part of a paper designed to identify a class of unrealisable, or 'impossible', sequences (possessing the characteristic of triple exponentiality); the theme was developed in [6], where a certain class of lacunary sequences (linked to automatic sequences, and transcendental numbers) was also shown to be unachievable.

1.2 This Paper

Any successful iterated generating function scheme must exhibit the feature of 'preservation' in the sense that polynomial terms whose coefficients match target sequence elements are *preserved* (that is, retained and not altered) in the next iterated polynomial as determined by the particular controlling recurrence law—in general, at least one new 'correct' term will be added at each step (and then preserved), obeying a linear, as opposed to exponential, convergence (that is, rate of addition); new sequences can themselves be generated, of course. This paper looks more closely at the notion of preservation, and in particular the manner in which its salient features may be described mathematically as conditions on a scheme. Accordingly, numerous examples are offered so as to furnish the theory and assist the reader in understanding the nature of iterated generating functions beyond the necessarily general discussion above.

2 Theory and Analysis

2.1 Notation and Terminology

Consider polynomials $F_1(x), F_2(x), F_3(x), \ldots, \in \mathbb{Z}[x]$. We say that the polynomial sequence $\{F_1(x), F_2(x), F_3(x), \ldots\}$ exhibits preservation if

$$F_{i+1}(x) - F_i(x) = x^i \Delta_i(x), \qquad i \geq 1, \tag{2.1}$$

where $\Delta_i(x) \in \mathbb{Z}[x]$. This means that for any value of i, neighboring polynomials $F_i(x), F_{i+1}(x)$ locally agree in terms at least up to and including those in x^{i-1}. The preservation property of the sequence is said to be a globally *strict* one if (2.1) holds with the constant term of $\Delta_i(x)$ non-zero for all $i \geq 1$. A sequence may, on the other hand, be globally *non-strict*, in which case, for any i, however many terms in $x^0, x^1, x^2, \ldots$, are absent in $\Delta_i(x)$ increases the level of agreement between $F_i(x), F_{i+1}(x)$ in that instance to include terms in $x^i, x^{i+1}, x^{i+2}, \ldots$ Note that both strict and non-strict types of local preservation may also be present within a polynomial sequence.

We denote by $\mathcal{S}(x,y)$ a bi-variate polynomial in $\mathbb{Z}[x,y]$, observing (where it contains a non-zero constant term) it can always be decomposed into constituent parts

$$\mathcal{S}(x,y) = xyg(x,y) + f(x) + r(y), \tag{2.2}$$

with $g(x,y) \in \mathbb{Z}[x,y]$, $0 \neq f(x) \in \mathbb{Z}[x]$ and, for $a_1, a_2, \ldots, a_n \in \mathbb{Z}$, $r(y) = \sum_{j=1}^{n} a_j y^j = a_1 y + a_2 y^2 + \cdots + a_n y^n = \mathcal{S}(0,y) - f(0) \in \mathbb{Z}[y]$ (the stand alone constant within $\mathcal{S}(x,y)$ is assigned to $f(x)$). For example, $\mathcal{S}(x,y) = 3+2x+8y+xy+xy^5+x^2y+x^3y^5+6x^8y^{10}+3x^4+2y^3$ is represented as shown, for which $g(x,y) = 1 + x + x^2y^4 + 6x^7y^9 + y^4$, $f(x) = 3 + 2x + 3x^4$ and $r(y) = 8y + 2y^3$. The function $\mathcal{S}(x,y) = (\alpha + \beta x)y^2 + (\gamma + \delta x)y + \varepsilon + \eta x$ mentioned in the Introduction has $g(x,y) = \delta + \beta y$, $f(x) = \varepsilon + \eta x$ and $r(y) = \gamma y + \alpha y^2$.

A sequence of iterated generating functions is produced through a recursive scheme captured by the notation $\{\mathcal{S}(x,y), F_1(x)\}$. That is to say, successive polynomials in the sequence are, given $F_1(x)$, delivered using an input-output law

$$F_{i+1}(x) = \mathcal{S}(x, F_i(x)) = xF_i(x)g(x, F_i(x)) + f(x) + r(F_i(x)), \qquad i \geq 1. \tag{2.3}$$

It is the preserving characteristics of these polynomials that we wish to examine here, based on the form of $\mathcal{S}(x,y)$ (2.2) through which the described recursive algorithm (2.3) is executed.

2.2 Main Results and Examples

Our findings take the form of two theorems which encapsulate the mathematics driving iterated generating functions in the context of preservation, the second of which we illustrate with some examples. We begin with the first of our main results.

Theorem 2.1. *If the sequence of polynomials $\{F_1(x) = f(x), F_2(x), F_3(x), \ldots\}$ yielded by a scheme $\{\mathcal{S}(x,y), F_1(x)\}$ exhibits preservation and, for some $\gamma \geq 1$, $\Delta_\gamma(x) \in \mathbb{Z}[x]$ has a non-zero constant (as lead term), then the following Conditions C(i),C(ii) hold:*

- C(i): $0 = \sum_{j=1}^{n} a_j f^j(0) = r(f(0))$;
- C(ii): $0 = \sum_{j=1}^{n} j a_j f^{j-1}(0) = r'(f(0))$.

The proof of C(i) is straightforward.

Proof. Noting that $F_1(x) = f(x) \neq 0$ (or else, if $f(x)$ were zero, each of $F_2(x), F_3(x), F_4(x), \ldots,$ would also be zero by repeated use of (2.3)), and with constant term $[x^0]\{F_1(x)\} = F_1(0) = f(0)$, consider $F_2(x) = \mathcal{S}(x, F_1(x)) = \mathcal{S}(x, f(x)) = xf(x)g(x, f(x)) + f(x) + r(f(x))$. Writing $f(x) = f(0) + x\hat{f}(x)$ (for some $\hat{f}(x) \in \mathbb{Z}[x]$), then $F_2(x) = xf(x)g(x, f(x)) + f(x) + \sum_{j=1}^{n} a_j [f(0) + x\hat{f}(x)]^j$, having a constant term identified readily as $f(0) + \sum_{j=1}^{n} a_j f^j(0) = f(0) + r(f(0))$ (alternatively, $[x^0]\{F_2(x)\} = F_2(0) = 0 \cdot f(0)g(0, f(0)) + f(0) + r(f(0)) = f(0) + r(f(0))$). However, $[x^0]\{F_2(x)\} = f(0)$ (arguing that the polynomials $F_1(x), F_2(x)$ may display strict or non-strict preservation locally between them, but the most restrictive case allows us to match only the constant terms), whence $f(0) + r(f(0)) = f(0)$ (equating the two forms) and Condition (i) is immediate. □

The proof of C(ii) is a great deal more involved.

Proof. We write, from (2.3) with $k \geq 2$ assumed,

$$F_{k+1}(x) = \mathcal{S}(x, F_k(x)) = xF_k(x)g(x, F_k(x)) + f(x) + r(F_k(x)). \tag{T.1}$$

Let the function $g(x,y) \in \mathbb{Z}[x,y]$ have, for $m \geq 0$, the form $g(x,y) = \sum_{l=0}^{m} g_l(x)y^l$ (with polynomial coefficients $g_0(x), g_1(x), \ldots, g_m(x) \in \mathbb{Z}[x]$), and consider the first r.h.s. element of

(T.1), being

$$\begin{aligned} xF_k(x)g(x,F_k(x)) &= x\sum_{l=0}^{m} g_l(x)F_k^{l+1}(x) \\ &= x\sum_{l=0}^{m} g_l(x)[F_{k-1}(x)+\{F_k(x)-F_{k-1}(x)\}]^{l+1} \\ &= x\sum_{l=0}^{m} g_l(x)[F_{k-1}(x)+x^{k-1}\Delta_{k-1}(x)]^{l+1} \end{aligned} \quad \text{(T.2)}$$

by preservation, for some $\Delta_{k-1}(x) \in \mathbb{Z}[x]$. Continuing,

$$\begin{aligned} &= x\sum_{l=0}^{m} g_l(x)\sum_{p=0}^{l+1}\binom{l+1}{p}F_{k-1}^{p}(x)[x^{k-1}\Delta_{k-1}(x)]^{l+1-p} \\ &= x\sum_{l=0}^{m} g_l(x)\left[\sum_{p=0}^{l}\binom{l+1}{p}F_{k-1}^{p}(x)[x^{k-1}\Delta_{k-1}(x)]^{l+1-p}+F_{k-1}^{l+1}(x)\right] \\ &= xF_{k-1}(x)\sum_{l=0}^{m} g_l(x)F_{k-1}^{l}(x)+x^kW_1(x), \end{aligned} \quad \text{(T.3)}$$

where $W_1(x) = W_1(x;k) = \sum_{l=0}^{m} g_l(x)\sum_{p=0}^{l}\binom{l+1}{p}F_{k-1}^{p}(x)x^{I_k(l,p)}\Delta_{k-1}^{l+1-p}(x) \in \mathbb{Z}[x]$ and integer $I_k(l,p) = (l-p)(k-1) \geq 0$. In other words,

$$xF_k(x)g(x,F_k(x)) = xF_{k-1}(x)g(x,F_{k-1}(x)) + x^kW_1(x). \quad \text{(T.4)}$$

We now turn to the final term of the r.h.s. of (T.1), which is

$$\begin{aligned} &r(F_k(x)) \\ &\quad = \sum_{j=1}^{n} a_jF_k^j(x) \\ &\quad = \sum_{j=1}^{n} a_j[F_{k-1}(x)+\{F_k(x)-F_{k-1}(x)\}]^j \\ &\quad = \sum_{j=1}^{n} a_j[F_{k-1}(x)+x^{k-1}\Delta_{k-1}(x)]^j \qquad \text{(by preservation)} \\ &\quad = \sum_{j=1}^{n} a_j\sum_{p=0}^{j}\binom{j}{p}F_{k-1}^{p}(x)[x^{k-1}\Delta_{k-1}(x)]^{j-p} \\ &\quad = \sum_{j=1}^{n} a_j\left[\sum_{p=0}^{j-2}\binom{j}{p}F_{k-1}^{p}(x)x^{(k-1)(j-p)}\Delta_{k-1}^{j-p}(x)+jF_{k-1}^{j-1}(x)x^{k-1}\Delta_{k-1}(x)+F_{k-1}^{j}(x)\right] \\ &\quad = \sum_{j=1}^{n} a_jF_{k-1}^{j}(x)+x^{k-1}\Delta_{k-1}(x)\sum_{j=1}^{n} ja_jF_{k-1}^{j-1}(x)+x^kW_2(x), \end{aligned} \quad \text{(T.5)}$$

where $W_2(x) = W_2(x;k) = \sum_{j=1}^{n} a_j\sum_{p=0}^{j-2}\binom{j}{p}F_{k-1}^{p}(x)x^{J_k(j,p)}\Delta_{k-1}^{j-p}(x) \in \mathbb{Z}[x]$ and integer $J_k(j,p) = (j-p)(k-1)-k \geq 0$. Thus,

$$r(F_k(x)) = r(F_{k-1}(x)) + x^{k-1}\Delta_{k-1}(x)r'(F_{k-1}(x)) + x^kW_2(x). \quad \text{(T.6)}$$

Combining (T.4) and (T.6), equation (T.1) reads, therefore,

$$\begin{aligned} F_{k+1}(x) &= xF_{k-1}(x)g(x,F_{k-1}(x)) + x^kW_1(x) + f(x) \\ &\qquad + r(F_{k-1}(x)) + x^{k-1}\Delta_{k-1}(x)r'(F_{k-1}(x)) + x^kW_2(x) \\ &= F_k(x) + x^{k-1}\Delta_{k-1}(x)r'(F_{k-1}(x)) + x^kW_3(x) \end{aligned} \quad \text{(T.7)}$$

on setting $W_3(x) = W_3(x;k) = W_1(x) + W_2(x) \in \mathbb{Z}[x]$ and re-applying (T.1). Since the polynomial sequence $\{F_1(x), F_2(x), F_3(x), \ldots\}$ is a preserving one, $[x^0]\{F_{k-1}(x)\} = [x^0]\{F_1(x)\} = [x^0]\{f(x)\} = f(0)$, so that, writing $F_{k-1}(x) = f(0) + x\hat{F}_{k-1}(x)$ (for some $\hat{F}_{k-1}(x) \in \mathbb{Z}[x]$), we next consider, from within (T.7), the term

$$\begin{aligned}
r'(F_{k-1}(x)) &= \sum_{j=1}^{n} ja_j F_{k-1}^{j-1}(x) \\
&= \sum_{j=1}^{n} ja_j [f(0) + x\hat{F}_{k-1}(x)]^{j-1} \\
&= a_1 + \sum_{j=2}^{n} ja_j [f(0) + x\hat{F}_{k-1}(x)]^{j-1} \\
&= a_1 + \sum_{j=1}^{n-1} (j+1)a_{j+1} [f(0) + x\hat{F}_{k-1}(x)]^{j} \\
&= a_1 + \sum_{j=1}^{n-1} (j+1)a_{j+1} \sum_{p=0}^{j} \binom{j}{p} [x\hat{F}_{k-1}(x)]^p f^{j-p}(0) \\
&= a_1 + \sum_{j=1}^{n-1} (j+1)a_{j+1} \left[f^j(0) + \sum_{p=1}^{j} \binom{j}{p} [x\hat{F}_{k-1}(x)]^p f^{j-p}(0) \right] \\
&= a_1 + \sum_{j=1}^{n-1} (j+1)a_{j+1} [f^j(0) + xW_4(x)] \\
&= a_1 + \sum_{j=1}^{n-1} (j+1)a_{j+1} f^j(0) + x \sum_{j=1}^{n-1} (j+1)a_{j+1} W_4(x) \\
&= a_1 + \sum_{j=2}^{n} ja_j f^{j-1}(0) + xW_5(x) \\
&= \sum_{j=1}^{n} ja_j f^{j-1}(0) + xW_5(x) \\
&= r'(f(0)) + xW_5(x), \qquad \text{(T.8)}
\end{aligned}$$

where $W_4(x) = W_4(x;k,j) = \sum_{p=1}^{j} \binom{j}{p} x^{p-1} \hat{F}_{k-1}^{p}(x) f^{j-p}(0) \in \mathbb{Z}[x]$ and $W_5(x) = W_5(x;k) = \sum_{j=1}^{n-1}(j+1)a_{j+1}W_4(x) \in \mathbb{Z}[x]$. Equation (T.7) can now be written as

$$\begin{aligned}
F_{k+1}(x) - F_k(x) &= x^{k-1}\Delta_{k-1}(x)[r'(f(0)) + xW_5(x)] + x^k W_3(x) \\
&= x^{k-1}\Delta_{k-1}(x) r'(f(0)) + x^k W_6(x), \qquad \text{(T.9)}
\end{aligned}$$

with $W_6(x) = W_6(x;k) = \Delta_{k-1}(x)W_5(x) + W_3(x) \in \mathbb{Z}[x]$, and our proof nears completion. Suppose there exists a polynomial $\Delta_\gamma(x) \in \mathbb{Z}[x]$ with non-zero constant term $\Delta_\gamma(0) = a_\gamma$, say. Then $\Delta_\gamma(x)$ is expressible as $\Delta_\gamma(x) = a_\gamma + x\hat{\Delta}_\gamma(x)$ with $\hat{\Delta}_\gamma(x) \in \mathbb{Z}[x]$. For $k = \gamma + 1$ (which, since $k \geq 2$, means that $\gamma \geq 1$), (T.9) now reads, in this instance,

$$\begin{aligned}
F_{\gamma+2}(x) - F_{\gamma+1}(x) &= x^\gamma \Delta_\gamma(x) r'(f(0)) + x^{\gamma+1} W_6(x) \\
&= x^\gamma [a_\gamma + x\hat{\Delta}_\gamma(x)] r'(f(0)) + x^{\gamma+1} W_6(x) \\
&= a_\gamma x^\gamma r'(f(0)) + x^{\gamma+1} W_7(x), \qquad \text{(T.10)}
\end{aligned}$$

where $W_7(x) = W_7(x;k,\gamma) = \hat{\Delta}_\gamma(x) r'(f(0)) + W_6(x) \in \mathbb{Z}[x]$. However, we know that by preservation $F_{\gamma+2}(x) - F_{\gamma+1}(x) = x^{\gamma+1}\Delta_{\gamma+1}(x)$ ($\Delta_{\gamma+1}(x) \in \mathbb{Z}[x]$), so that, by comparison with (T.10), $a_\gamma x^\gamma r'(f(0)) = 0$ (and $\Delta_{\gamma+1}(x) = W_7(x)$); since $a_\gamma \neq 0$ by assumption, it follows that $r'(f(0)) = 0$. □

Our second main result is now stated and established. Note that Theorems 2.1 and 2.2 almost combine to constitute a single result with necessary and sufficient parts.

Theorem 2.2. *Choose $F_1(x) = f(x) \neq 0$ as the initial polynomial of a sequence of iterated generating functions $\{F_1(x) = f(x), F_2(x), F_3(x), \ldots\}$ produced by a scheme $\{\mathcal{S}(x,y), F_1(x)\}$. Then, if Conditions C(i),C(ii) hold, the sequence exhibits preservation.*

Proof. From the proof of C(i) of Theorem 2.1 we recall that $[x^0]\{F_2(x)\} = f(0) + r(f(0)) = f(0)$ assuming C(i) holds. Thus the polynomials $F_1(x)$ and $F_2(x)$ have the same constant term $f(0)$, and $F_2(x) - F_1(x) = x\Delta_1(x)$ for some $\Delta_1(x) \in \mathbb{Z}[x]$. We need to show that, for all $i \geq 1$, $F_{i+1}(x) - F_i(x) = x^i\Delta_i(x)$ (with $\Delta_i(x) \in \mathbb{Z}[x]$) to establish preservation, and we argue inductively.

Having seen it to be true for $i = 1$, we assume the result holds for some $i = k \geq 1$ and consider

$$F_{k+2}(x) - F_{k+1}(x) = x^k\Delta_k(x)r'(f(0)) + x^{k+1}W_8(x) \tag{T.11}$$

directly from by (T.9) (*having invoked the inductive hypothesis on the way to it*[1]), where $W_8(x) = W_8(x;k) = W_6(x;k+1) \in \mathbb{Z}[x]$. If C(ii) is also satisfied then $F_{k+2}(x) - F_{k+1}(x)$ contracts to $x^{k+1}W_8(x)$ and, this being of the correct form, the inductive step is upheld. □

This completes the technical proofs we need, and we move on to give examples (supplemented by some remarks and observations)—these have been carefully chosen, from a large number of test runs made, to highlight those facets of iterated generating functions as established by Theorems 2.1 and 2.2.

Remark 2.1. If C(i) holds, but C(ii) does not, then we find that it is the constant term $f(0)$ of $f(x)$ that is *alone* held within the polynomial sequence $\{F_1(x), F_2(x), F_3(x), \ldots\}$ and, in the sense that no more terms are added and preserved, in such a case there is no element of genuine preservation to the sequence as it progresses. Noting that $[x^0]\{F_1(x)\} = f(0)$, then $[x^0]\{F_2(x)\} = f(0)$ by C(i) (as seen in the proof of Theorem 2.2 above). Now $F_3(x) = xF_2(x)g(x, F_2(x)) + f(x) + r(F_2(x))$, with $[x^0]\{F_3(x)\} = F_3(0) = f(0) + r(F_2(0)) = f(0) + r(f(0)) = f(0)$ again by C(i). It follows, therefore, that $[x^0]\{F_4(x)\} = [x^0]\{F_5(x)\} = \cdots = f(0)$; it is left as a simple reader exercise to see that terms in x, x^2, x^3, and so on, are (unless by exception) at continual variance within the subsequent procession of polynomials, and so not preserved.

Example 1. With regard to Remark 2.1 then, by way of an example, on choosing $\mathcal{S}(x,y) \in \mathbb{Z}[x,y]$ (2.2) for which $g(x,y) = x + xy - xy^2$, $f(x) = 2 - x + 3x^2$ and $r(y) = 4y + 2y^2 - 2y^3$ ($\Rightarrow a_1 = 4,\ a_2 = 2,\ a_3 = -2$), the scheme $\{\mathcal{S}(x,y), F_1(x)\}$ returns, by (2.3) with $F_1(x) = f(x) = 2-x+3x^2$, polynomials $F_2(x) = 2+11x-45x^2+69x^3-134x^4+\cdots$, $F_3(x) = 2-133x-669x^2+6333x^3-1442x^4+\cdots$, $F_4(x) = 2+1595x-168861x^2+2850669x^3+83307358x^4+\cdots$, $F_5(x) = 2-19141x-23413917x^2-2762993043x^3+2200431692446x^4+\cdots$, with but $f(0) = 2$ invariant throughout the polynomials generated. This is as expected, and is seen in further polynomials beyond $F_5(x)$, since $r(f(0)) = r(2) = a_1 \cdot 2^1 + a_2 \cdot 2^2 + a_3 \cdot 2^3 = 2a_1 + 4a_2 + 8a_3 = 2(4) + 4(2) + 8(-2) = 8 + 8 - 16 = 0$, while $r'(f(0)) = r'(2) = a_1 \cdot 2^0 + 2a_2 \cdot 2^1 + 3a_3 \cdot 2^2 = a_1 + 4a_2 + 12a_3 = 4 + 4(2) + 12(-2) = 4 + 8 - 24 = -12 \neq 0$.

Example 2. A further example emphasises the observation. With $g(x,y) = x + x^2 + 2xy - 2y^3$, $f(x) = -1 + 4x^2$ and $r(y) = y + 5y^2 + 6y^3 + 2y^4$ ($a_1 = 1,\ a_2 = 5,\ a_3 = 6,\ a_4 = 2$), then $r(f(0)) = r(-1) = -a_1+a_2-a_3+a_4 = -1+5-6+2 = 0$, but $r'(-1) = a_1-2a_2+3a_3-4a_4 = 1-10+18-8 = 1 \neq 0$. Given $F_1(x) = f(x)$, the next few polynomial forms are as predicted (preserving only the term $f(0) = -1$)—we find that $F_2(x) = -1-2x+9x^2+31x^3-28x^4+\cdots$,

[1]This, perhaps, needs some clarification. We consider $F_{k+2}(x) = xF_{k+1}(x)g(x, F_{k+1}(x)) + f(x) + r(F_{k+1}(x))$ by (2.3). The terms $xF_{k+1}(x)g(x, F_{k+1}(x))$ and $r(F_{k+1}(x))$ are dealt with in the same way as for the Theorem 2.1 proof of C(ii), with the (twice taken) step $F_k(x) - F_{k-1}(x) \to x^{k-1}\Delta_{k-1}(x)$ made previously under the assumption of preservation (that is, *en route* to (T.2),(T.5)) now, with $k \to k+1$, *admissible by the inductive hypothesis*. The only other prior application of assumed preservation is seen to have led (just after (T.7)) to $F_{k-1}(x) = f(0) + x\hat{F}_{k-1}(x)$, which now necessitates that $F_k(x) = f(0) + x\hat{F}_k(x)$ holds—this is readily argued independently (by extending the argument that $[x^0]\{F_1(x)\} = [x^0]\{F_2(x)\} = f(0)$ to show $[x^0]\{F_3(x)\} = [x^0]\{F_4(x)\} = \cdots = f(0)$; see also Remark 2.1), giving that the general polynomial $F_k(x)$ has, for any general $k \geq 1$, initial term $f(0)$. Thus, we arrive at our version of (T.9), *being (T.11)*, ready to apply C(ii) and so complete the proof of Theorem 2.2.

$F_3(x) = -1-4x-6x^2+112x^3+426x^4+\cdots$, $F_4(x) = -1-6x-49x^2-37x^3+2228x^4+\cdots$, $F_5(x) = -1-8x-128x^2-1000x^3+3692x^4+\cdots$; again, this particular occurrence (of minimal constancy) persists throughout additional polynomials computed.

Remark 2.2. If neither C(i) nor C(ii) hold (as seen, for instance, by slightly modifying $r(y)$ in Example 2 to $r(y) = y + 5y^2 + 6y^3 + 3y^4$, so that $r(-1) = 1 \neq 0$ and $r'(-1) = -3 \neq 0$), or if C(ii) holds but not C(i) (as seen with $r(y) = y + 5y^2 + 11y^3 + 6y^4$ in Example 2, for which $r(-1) = -1 \neq 0$ and $r'(-1) = 0$), then any associated scheme $\{\mathcal{S}(x, y), f(x)\}$ exhibits no preservation.

Example 3 (Non-Strict Preservation). The scheme of Example 1, but now with $r(y) = 4y - 4y^2 + y^3$ so that C(i) and C(ii) are satisfied ($r(2) = r'(2) = 0$), is used to illustrate non-strict preservation. Given $F_1(x) = f(x) = 2 - x + 3x^2$, computations yield

$$\begin{aligned} F_2(x) &= 2 - x + 3x^2 - 6x^3 + x^4 + 4x^5 - 27x^6 + 27x^7 - 27x^8, \\ F_3(x) &= 2 - x + 3x^2 - 6x^3 + 25x^4 - 48x^5 + 85x^6 + 63x^7 + \cdots, \\ F_4(x) &= 2 - x + 3x^2 - 6x^3 + 25x^4 - 144x^5 + 485x^6 - 1569x^7 + \cdots, \\ F_5(x) &= 2 - x + 3x^2 - 6x^3 + 25x^4 - 144x^5 + 869x^6 - 3937x^7 + \cdots, \end{aligned} \tag{2.4}$$

and so on, with global preservation proceeding according to the (non-strict) difference law $F_{i+1}(x) - F_i(x) = x^{i+2}\Delta_i(x)$ for $i \geq 1$. Iterated generating functions arising from the process produce ever more terms of the infinite sequence $\{2, -1, 3, -6, 25, -144, 869, -5473, 35461, -234727, \ldots\}$.

Example 4 (Mixed Strict & Non-Strict Preservation). Staying now with Example 2 but choosing $r(y) = y + 5y^2 + 7y^3 + 3y^4$, then C(i) and C(ii) are satisfied ($r(-1) = r'(-1) = 0$) and, given $F_1(x) = f(x) = -1 + 4x^2$, subsequent polynomials follow thus:

$$\begin{aligned} F_2(x) &= -1 - 2x + 5x^2 + 31x^3 + 20x^4 - 188x^5 - 288x^6 + 512x^7 + \cdots, \\ F_3(x) &= -1 - 2x - 3x^2 - 3x^3 - 35x^4 + 538x^5 + 7730x^6 + 17517x^7 + \cdots, \\ F_4(x) &= -1 - 2x - 3x^2 - 3x^3 + 53x^4 + 232x^5 + 1485x^6 - 9238x^7 + \cdots, \\ F_5(x) &= -1 - 2x - 3x^2 - 3x^3 + 53x^4 + 232x^5 - 891x^6 - 12152x^7 + \cdots, \end{aligned} \tag{2.5}$$

and so on. The notable feature here is the fact that, after creating and retaining an additional single term (that is, $-2x$) in moving from $F_1(x)$ to $F_2(x)$, therefafter *two* terms are added and preserved per iterate. After two initial iterations exhibiting locally strict preservation between the pairs $F_1(x), F_2(x)$ and $F_2(x), F_3(x)$, non-strict preservation occurs thereafter which is governed by the difference law $F_{i+1}(x) - F_i(x) = x^{2(i-1)}\Delta_i(x)$ for $i = 3, 4, 5, \ldots$ Iterated generating functions arising from the scheme produce ever more terms of a sequence $\{-1, -2, -3, -3, 53, 232, -891, -12152, -16662, 433590, \ldots\}$; the interested reader is referred to Appendix A for a detailed explanation of this two term addition.

As a point of interest, we note that changing $f(x)$ even slightly within Example 4 is enough to produce new results which are worth a mention. With $f(x) = -1$ (so that $f(0) = -1$ still, and preservation conditions remain satisfied) then initial strict pairwise local preservation between $F_1(x), F_2(x)$ and $F_3(x)$ is thereafter superseded by non-strict global preservation (governed by the same difference law) as terms are added, two at a time for $F_3(x)$ onwards, towards the sequence $\{-1, -2, -7, -3, 193, 1028, -4963, -90104, -203346, \ldots\}$. For $f(x) = -1 + x + 2x^3$, on the other hand, a sequence $\{-1, -1, -5, -23, -76, 46, 3274, 30348, 159824, \ldots\}$ is achieved, one new term at a time, under globally strict preservation. The reason for these outcomes, being a little subtle, is explained in Appendix B.

Remark 2.3. Remark 2.1 is consistent even in the case when $f(0) = 0$. Suppose $g(x, y) = 2x + y$, $f(x) = x + 2x^2$ and $r(y) = y + 2y^2 + 3y^3$, for example ($r(0) = 0$, $r'(0) = 1 \neq 0$, so that C(i) holds but not C(ii)). Then the zero constant term in $F_1(x) = f(x)$ remains absent also in $F_2(x), F_3(x), F_4(x), \ldots$ (and so is retained by default). Any term(s) not present at the start of $f(x)$ are included as one or more zero(s) in $F_1(x) = f(x)$ and are subsequently preserved.

2.3 Further Observations and Examples

If $r(y)$ is a non-trivial degree $n \geq 1$ polynomial $r(y) = a_1 y + a_2 y^2 + \cdots + a_n y^n$ we can infer, from consideration of the first few values of n, some particular results for a scheme $\{\mathcal{S}(x,y), f(x)\}$.

Case $n = 1$**.** With $r(f(0)) = a_1 f(0)$ and $r'(f(0)) = a_1$, we conclude trivially that the scheme is preserving if $a_1 = 0$ (so that $r(y)$ is identically zero), or else it is non-preserving (if $a_1 \neq 0$, then only C(ii) fails when $f(0) = 0$, and both C(i),C(ii) fail when $f(0) \neq 0$).

Case $n = 2$**.** With $r(f(0)) = a_1 f(0) + a_2 f^2(0)$ and $r'(f(0)) = a_1 + 2a_2 f(0)$, we see that (a) if $f(0) = 0$ the scheme is a preserving one if $a_1 = 0$ also (so that $r(y)$ may take the form $r(y) = a_2 y^2$), or else (b) if $f(0) \neq 0$ preservation occurs only when $a_1 = a_2 = 0$ (discounted) and $r(y) = 0$ (C(i),C(ii) both fail when either $a_1 = 0$ and $a_2 \neq 0$, or $a_1 \neq 0$ and $a_2 = 0$).

An example of a Case (a) preserving scheme is that for which $g(x,y) = y$, $f(x) = x$ and $r(y) = y^2$. The starting polynomial $F_1(x) = f(x)$ produces others which collectively map to the series of subsequences $\{\mathbf{0},\mathbf{1}\}, \{\mathbf{0},\mathbf{1},\mathbf{1},1\}, \{\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{3},5,5,3,1\}, \{\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{9},23,45,75,109,\ldots\}$, $\{\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{9},\mathbf{31},97,263,649,\ldots\},\ldots,$ containing entries (shown in bold) of the O.E.I.S. Sequence No. A052709 $\{0,1,1,3,9,31,113,431,1697,6847,\ldots\}$ and continuously adding a single term (per iterate). An example of a Case (b) preserving scheme has $g(x,y) = 2x$, $f(x) = 1$ and $r(y) = 0$, producing iterated generating function coefficients of Sequence No. A077957 as follows (adding two terms per iteration, and devoid of any non-contributing terms at each stage): $\{\mathbf{1}\}, \{\mathbf{1},\mathbf{0},\mathbf{2}\}, \{\mathbf{1},\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{4}\}, \{\mathbf{1},\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{8}\}, \{\mathbf{1},\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{8},\mathbf{0},\mathbf{16}\},\ldots$ Both sets of iterations adhere to globally non-strict difference rules which are easily determined as $F_{i+1}(x) - F_i(x) = x^{i+1}\Delta_i(x)$ (Case (a)) and $F_{i+1}(x) - F_i(x) = x^{2i}\Delta_i(x)$ (Case (b)).

These examples are set up using the *ordinary generating functions of the respective infinite sequences* (see [2, pp. 7–8]), as now explained for clarity. In Case (a) Sequence No. A052709 has a governing equation $0 = (1+x)G^2(x) - G(x) + x$ which can be re-arranged and discretised to give a recurrence $G_{r+1}(x) = (1+x)G_r^2(x) + x = H(x, G_r(x))$; this corresponds to a scheme $\{\mathcal{S}(x,y), x\}$ for which $\mathcal{S}(x,y) = (1+x)y^2 + x = xy(y) + x + y^2$ in line with (2.2). In Case (b) the governing equation is simply $0 = (1-2x^2)G(x) - 1$ for Sequence No. A077957, whose discretised version $G_{r+1}(x) = 2x^2 G_r(x) + 1 = H(x, G_r(x))$ aligns itself with a scheme $\{\mathcal{S}(x,y), 1\}$ where $\mathcal{S}(x,y) = 2x^2 y + 1 = xy(2x) + 1$.

Case $n = 3$**.** With $r(f(0)) = a_1 f(0) + a_2 f^2(0) + a_3 f^3(0)$ and $r'(f(0)) = a_1 + 2a_2 f(0) + 3a_3 f^2(0)$, we see that if $f(0) = 0$ the scheme is preserving if $a_1 = 0$ also (so that $r(y)$ can assume the form $r(y) = a_2 y^2 + a_3 y^3$); an example in which $g(x,y) = 2x + y$, $f(x) = x + 2x^2$ and $r(y) = y^2 + 2y^3$ is found to deliver $\{\mathbf{0},\mathbf{1},2\}, \{\mathbf{0},\mathbf{1},\mathbf{3},9,24,28,16\}, \{\mathbf{0},\mathbf{1},\mathbf{3},\mathbf{11},57,255,953,2973,8290,\ldots\}, \{\mathbf{0},\mathbf{1},\mathbf{3},\mathbf{11},\mathbf{61},353,2059,11075,54797,\ldots\}, \{\mathbf{0},\mathbf{1},\mathbf{3},\mathbf{11},\mathbf{61},\mathbf{361},2319,15111,96273,\ldots\}$, and so on; these preserving subsequences follow globally non-strict preservation (for $i \geq 1$, $F_{i+1}(x) - F_i(x) = x^{i+1}\Delta_i(x)$). If $f(0) \neq 0$ preservation requires $a_1 = f^2(0)a_3$ and $a_2 = -2f(0)a_3$, as in Example 3 where $a_3 = 1$ (and $f(0) = 2$) and the conditions for preservation would need $a_1 = 4a_3 = 4$ and $a_2 = -4a_3 = -4$, which are satisfied (in Example 1, however (with $f(0) = 2$ still, and $a_3 = -2$), neither criteria hold and we see no preservation as a result).

In general, if $f(0) = 0$ it is clear that, for $n \geq 2$, $a_1 = 0$ within the degree n polynomial $r(y)$ is sufficient for a scheme $\{\mathcal{S}(x,y), f(x)\}$ to be a preserving one whose iterated generating functions add at least one term to each subsequence string. The absence of $r(y)$ is, regardless of the value of $f(0)$, sufficient alone to yield a preserving scheme. To finish this section, and so the paper, we give some examples, using well-known sequences, in which $r(y) = 0$ and C(i),C(ii) are satisfied trivially.

2.4 Other Examples ($r(y) = 0$)

We have seen, in the Introduction, how the generating function equation for the Catalan sequence gives rise to a scheme $\{\mathcal{S}(x,y), 1\}$ where $\mathcal{S}(x,y) = xy^2 + 1 = xy(y) + 1$ and iterated generating functions obey globally strict preservation (adding one sequence term within each new polynomial produced). Two similar schemes, of form $\{\mathcal{S}(x,y), 1\}$, that also exhibit the same type of preservation, are those associated with the Motzkin and (Large) Schröder sequences (respectively, A001006 $\{1,1,2,4,9,\ldots\}$ and A006318 $\{1,2,6,22,90,\ldots\}$) for which $\mathcal{S}(x,y) = xy(1+xy)+1$ and $\mathcal{S}(x,y) = xy(1+y)+1$. A slight variation is the Fibonacci sequence (A000045 $\{0,1,1,2,3,\ldots\}$), whose scheme $\{\mathcal{S}(x,y), x\}$ has $\mathcal{S}(x,y) = xy(1+x)+x$ and is globally non-strict (caused by the two-term start 0,1 from $F_1(x) = f(x) = x$, as seen previously in the example for A052709)—adding one sequence term per iterate. Many other examples could be included here, such as that where $g(x,y) = -(3x^3+2y+x^2y^4-y^5)$ and $f(x) = -(2x^2+6x^5)$ (with $r(y) = 0$) which, obeying the globally non-strict law $F_{i+1}(x) - F_i(x) = x^{3i+2}\Delta_i(x)$ $(i \geq 1)$, yields preserving subsequences (showing three-term addition per iteration) $\{\mathbf{0},\mathbf{0},-\mathbf{2},0,0,-6\}$, $\{\mathbf{0},\mathbf{0},-\mathbf{2},\mathbf{0},\mathbf{0},-\mathbf{14},6,0,\ldots\}$, $\{\mathbf{0},\mathbf{0},-\mathbf{2},\mathbf{0},\mathbf{0},-\mathbf{14},\mathbf{6},\mathbf{0},-\mathbf{112},90,-18,\ldots\}$, $\{\mathbf{0},\mathbf{0},-\mathbf{2},\mathbf{0},\mathbf{0},-\mathbf{14},\mathbf{6},\mathbf{0},-\mathbf{112},\mathbf{90},-\mathbf{18},-1288,1392,-390,\ldots\}$.

3 Summary

In this paper an analytical framework has been provided for the formal study of iterated generating functions, illustrated by a variety of examples to show behaviour types and aid understanding of the essential processes underpinning them. The two conditions of Theorem 2.1 are those critical in determining sequence term preservation, or otherwise, and should the constant term within $\mathcal{S}(x,y)$ (2.2) be assigned to the function $r(y)$ (instead of $f(x)$) they can be re-cast neatly in relation to a so called *super stable fixed point* of $r(y)$ (this observation led to the approach taken in [6] where a class of 0-1 binary sequences were identified that have a basic (lacunary) property and cannot be reached by the iterated generating function methodology). The model presented here has allowed us to develop some previous exploratory work and reveal something of the fundamental nature of iterated generating functions, giving theoretical insight into some of the subtleties they can possess; future analysis promises to offer up yet more of their characteristics.

As a final comment, note that the notion of preservation is not just restricted to schemes for which $\mathcal{S}(x,y)$ lies in $\mathbb{Z}[x,y]$ (the presentation is merely made here in the context of *integer* sequences since these are prevalent in discrete mathematics), as illustrated for completeness in Appendix C where preserving schemes are set up to deliver subsequences containing elements from $\mathbb{Q}$ and $\mathbb{C}$.

Appendix A

Here we explain the non-strict preservation evident in Example 4 of Section 2.2. To do this we consider, using (2.3) as a starting point (recall that $g(x,y) = x+x^2+2xy-2y^3$, $f(x) = -1+4x^2$ and $r(y) = y+5y^2+7y^3+3y^4$),

$$\begin{aligned}
&F_{n+2}(x) - F_{n+1}(x) \\
&\quad= \mathcal{S}(x,F_{n+1}(x)) - \mathcal{S}(x,F_n(x)) \\
&\quad= xF_{n+1}(x)g(x,F_{n+1}(x)) + f(x) + r(F_{n+1}(x)) \\
&\qquad\qquad - [xF_n(x)g(x,F_n(x)) + f(x) + r(F_n(x))] \\
&\quad= xF_{n+1}(x)[x+x^2+2xF_{n+1}(x) - 2F_{n+1}^3(x)] \\
&\qquad + F_{n+1}(x) + 5F_{n+1}^2(x) + 7F_{n+1}^3(x) + 3F_{n+1}^4(x) \\
&\qquad - \{xF_n(x)[x+x^2+2xF_n(x) - 2F_n^3(x)] \\
&\qquad\qquad + F_n(x) + 5F_n^2(x) + 7F_n^3(x) + 3F_n^4(x)\}, \qquad \text{(A.1)}
\end{aligned}$$

where the terms appear in their natural order. Writing $F_{n+1}^3(x) - F_n^3(x)$ in the form $[F_{n+1}(x) - F_n(x)][F_{n+1}^2(x)+F_{n+1}(x)F_n(x)+F_n^2(x)]$ and $F_{n+1}^4(x)-F_n^4(x) = [F_{n+1}(x)-F_n(x)][F_{n+1}(x)+F_n(x)][F_{n+1}^2(x)+F_n^2(x)]$, the r.h.s. of (A.1) can, with a little work, be factored as

$$F_{n+2}(x) - F_{n+1}(x) = [F_{n+1}(x) - F_n(x)]L(x), \qquad \text{(A.2)}$$

where

$$\begin{aligned} L(x) &= 1 + x^2(1+x) + (2x^2+5)[F_{n+1}(x) + F_n(x)] \\ &\quad - (2x-3)[F_{n+1}(x) + F_n(x)][F_{n+1}^2(x) + F_n^2(x)] \\ &\quad + 7[F_{n+1}^2(x) + F_{n+1}(x)F_n(x) + F_n^2(x)]. \end{aligned} \tag{A.3}$$

Now since $F_2(x) = -1 - 2x + \cdots$,

$$F_n(x) = -1 - 2x + O(x^2) \quad \text{and} \quad F_{n+1}(x) = -1 - 2x + O(x^2), \qquad n \geq 2, \tag{A.4}$$

by (pre-established) preservation. Thus, with each of the functional products

$$F_{n+1}^2(x), F_{n+1}(x)F_n(x), F_n^2(x) = 1 + 4x + O(x^2) \qquad n \geq 2, \tag{A.5}$$

and the sum $F_{n+1}(x) + F_n(x) = -2 - 4x + O(x^2)$ $(n \geq 2)$, it is found, after some algebraic effort (omitted), that terms in x^0, x^1 *self-cancel* within $L(x)$, so that $L(x) = O(x^2)$ and (A.2) reads, for some $\Delta_n(x) \in \mathbb{Z}[x]$,

$$F_{n+2}(x) - F_{n+1}(x) = x^2[F_{n+1}(x) - F_n(x)]\Delta_n(x), \qquad n \geq 2, \tag{A.6}$$

which means that $F_{n+2}(x)$ and $F_{n+1}(x)$ have at least two more terms in common than do $F_{n+1}(x)$ and $F_n(x)$. Further calculations reveal that carrying extra terms in $F_{n+1}(x), F_n(x)$ indicates that $\Delta_n(x)$ has a non-zero constant term (enhancing (A.4) as $F_{n+1}(x), F_n(x) = -1 - 2x - 3x^2 + \cdots$ $(n \geq 3)$ gives $\Delta_n(x) = -27 + \cdots$, while modifying additionally as $F_{n+1}(x), F_n(x) = -1 - 2x - 3x^2 - 3x^3 + \cdots$ $(n \geq 3)$ gives $\Delta_n(x) = -27 - 127x + \cdots$), from which we conclude that the Example 4 scheme (after a 'start up' iteration) adds and preserves *precisely* two sequence terms per iterated generating function.

Appendix B

Here we consider the two (Section 2.2) modified versions of Example 4 (and confirm the conclusion of Example 4 itself), regarding the rate of term addition in the respective iterated generating functions developed.

Suppose (with $g(x,y) = x + x^2 + 2xy - 2y^3$, $r(y) = y + 5y^2 + 7y^3 + 3y^4$ still) Example 4 has a function $F_1(x) = f(x) = -1 + ax + \cdots$, for arbitrary a, as part of a scheme $\{\mathcal{S}(x,y), F_1(x)\}$. Following the same (rather lengthy) type of algebraic procedure as seen in Appendix A, it is found that for $n \geq 2$,

$$F_{n+1}(x), F_n(x) = -1 - (2-a)x + O(x^2) \tag{B.1}$$

(with $F_{n+1}^2(x), F_{n+1}(x)F_n(x), F_n^2(x) = 1 + 2(2-a)x + O(x^2)$ and $F_{n+1}(x) + F_n(x) = -2 - 2(2-a)x + O(x^2)$), and in turn

$$F_{n+2}(x) - F_{n+1}(x) = x[F_{n+1}(x) - F_n(x)][4a + O(x)], \qquad n \geq 2, \tag{B.2}$$

so that, for $a \neq 0$, $F_{n+2}(x)$ and $F_{n+1}(x)$ have exactly one more term in common than do $F_{n+1}(x)$ and $F_n(x)$. Thus, when $a = 0$ (when, for instance, $f(x) = -1$ (or $f(x) = -1 + 4x^2$ of Example 4)), $F_{n+2}(x) - F_{n+1}(x) = x[F_{n+1}(x) - F_n(x)]O(x) = x^2[F_{n+1}(x) - F_n(x)]\Delta_n(x)$ $(\Delta_n(x) \in \mathbb{Z}[x]; \Delta_n(0) \neq 0)$, and two sequence terms are added per iterated generating function. The case when $f(x) = -1 + x + 2x^3$ corresponds to $a = 1$, and here $F_{n+2}(x) - F_{n+1}(x) = x[F_{n+1}(x) - F_n(x)]\Delta_n(x)$ (where $\Delta_n(x) = 4 + O(x)$), resulting in just a single added term per iteration of the scheme $\{\mathcal{S}(x,y), F_1(x)\}$.

Appendix C

The subsequences of rationals $\{\mathbf{\frac{1}{2}}, -\frac{1}{3}, \frac{1}{4}\}, \{\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{6}}, -\frac{59}{36}, \frac{907}{432}, -\frac{1039}{576}, \frac{35}{48}, \ldots\}, \{\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{6}}, -\mathbf{\frac{113}{144}}, -\frac{1615}{432}, \frac{57905}{5184}, -\frac{7658863168386153}{1125899906842624}, \ldots\}, \{\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{6}}, -\mathbf{\frac{113}{144}}, -\mathbf{\frac{1877}{864}}, -\frac{98813}{27648}, \frac{5557720768179465}{140737488355328}, \ldots\}, \{\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{6}}, -\mathbf{\frac{113}{144}}, -\mathbf{\frac{1877}{864}}, -\mathbf{\frac{6479}{9216}}, \frac{3031718930469555}{562949953421312}, \ldots\}, \ldots$, are given by the preserving scheme $\{\mathcal{S}(x,y), F_1(x)\}$

for which $F_1(x) = f(x) = \frac{1}{2} - \frac{1}{3}x + \frac{1}{4}x^2$, $g(x,y) = \frac{1}{4} - \frac{2}{3}x + \frac{3}{2}y - 4xy$ and $r(y) = \frac{1}{8}y - \frac{1}{2}y^2 + \frac{1}{2}y^3$; the scheme (with $r(\frac{1}{2}) = r'(\frac{1}{2}) = 0$) satisfies C(i),C(ii),[2] and is a globally strict one adding one term per polynomial iterate.

A further preserving scheme is seen with $F_1(x) = f(x) = 1 - 3i + (3-5i)x \in \mathbb{C}[x]$, $g(x,y) = 5 + ix + (2-3i)y + 2xy \in \mathbb{C}[x,y]$, and $r(y) = -2(41-13i)y + 4(8+11i)y^2 + (5-7i)y^3 \in \mathbb{C}[y]$ (it is left as a reader exercise to check that $0 = r(f(0)) = r'(f(0))$, with $f(0) = 1-3i$); computer output shows, as expected, terms added and preserved accordingly, forming the progressive sequence $\{1-3i, -26-8i, -563-19591i, 26638965-9996649i, 32846150608+39846030240i, \ldots\}$ in line with strict global preservation.

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[2] Equivalently, note that the Section 2.3 Case $n = 3$ criteria relating a_1, a_2, a_3 are satisfied: given $a_3 = [y^3]\{r(s)\} = 1/2$, it is seen that $1/8 = a_1 = f^2(0)a_3 = (1/2)^2 a_3 = (1/4)a_3$ and $-1/2 = a_2 = -2f(0)a_3 = -2(1/2)a_3 = -a_3$.