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Common Fixed Point Theorems for Weakly Compatible Self-Mappings under Contraction Conditions in Complex Valued b-Metric Spaces

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Abstract We establish some common fixed point theorems for four weakly compatible self-mappings satisfying generalized contraction under rational expressions in complex valued b-metric spaces. Our results unify, extend and improve some results in the literature (see [3]). Our results supported by an example.

1 Introduction

Fixed point theory is one of the useful theories for determining the existence and uniqueness of solutions to many mathematical equations in mathematical science, engineering and applications in other fields. Banach's contraction principle is one of the most important relations in the study of non-linear equations and many applications in mathematical science and engineering fields. There are many extensions and generalizations were made by using different contractive conditions on an ambient space. These contractive conditions play very important role to show the existence and uniqueness of a fixed point. The concept of metric space was developed and generalized as rectangular metric spaces, pseudo metric spaces, probabilistic metric spaces, D-metric spaces, Fuzzy metric spaces, cone metric spaces, 2-metric spaces and G-metric spaces etc.. see [1,6,9,10,11,12,13,17].

In 1998, Czerwik [8] introduced the concept of b-metric spaces that were considered more general than ordinary metric spaces. Recently, Rao. et al. [14] introduced the concept of complex valued b-metric spaces which is more general than complex valued metric spaces that was presented by Azam et al. [5]

2 Preliminaries

In this section, we begin with some basic notations and definitions that will be very important and useful for our results.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \operatorname{Re}(z_1) \le \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \le \operatorname{Im}(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions hold:

(p_1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
(p_2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(j_2)$,
(p_3) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
(p_4) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (p_1),(p_2) and (p_3) is satisfied and we write $z_1 \prec z_2$ if only (p_3) is satisfied.

Definition 2.1 [16] The "max" function of the partial order $\precsim$ is defined as follows:

(i) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$.
(ii) $z_1 \precsim \max\{z_2, z_3\} \Rightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$.
(iii) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ or $|z_1| \leq |z_2|$.

Definition 2.2 [4] Let X be a nonempty set and $s \geq 1$ be a real number. A mapping $d : X \times X \to \mathbb{C}$ is called a b-metric on X if for all $x, y, z \in X$, the following axioms are satisfied:

(BM_1) $d(x, y) = 0$ if and only if $x = y$,
(BM_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(BM_3) $d(x, y) \leq s\,[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then the pair (X, d) is called a b-metric space.

Example 2.1 [15] Let (X, d) be a metric space and $\rho(x, y) = \big(d(x, y)\big)^q$, where $q > 1$ is a given real number. Then (X, ρ) is a b-metric space with $s = 2^{q-1}$.

Definition 2.3 [14] Let X be a nonempty set with a real number $s \geq 1$ and $\mathbb{C}$ be the set of all complex numbers. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$, the following conditions are satisfied:

(CBM_1) $0 \precsim d(x, y)$ and $d(x, y) = 0$ iff $x = y$,
(CBM_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(CBM_3) $d(x, y) \precsim s\,[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then (X, d) is called a complex valued b-metric space.

Example 2.2 [14] Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = |x - y|^2 + i\,|x - y|^2 \quad \forall\ x, y \in X,$$

Then (X, d) is called a complex valued b-metric space with $s = 2$.

Definition 2.4 [7] Let X be a complex valued metric space and (S, T) be a pair of self-mappings. The pair (S, T) is said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$. i.e., they commute at their coincidence points.

Definition 2.5 [5] Let $\{x_n\}$ be a sequence in a complex valued metric space (X, d) and $x \in X$. Then

(i) x is called the limit of $\{x_n\}$ if for every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec \varepsilon$ for all $n > n_0$ and we can write $\lim_{n\to\infty} x_n = x$.

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec \varepsilon$ for all $n > n_0$, where $m \in \mathbb{N}$.

(iii) (X, d) is said to be a complete complex valued metric space if every Cauchy sequence is convergent in (X, d).

Lemma 2.1 [5] Let (X, d) be a complex valued metric space. Then a sequence $\{x_n\}$ in X converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.2 [5] Let (X, d) be a complex valued metric space. Then a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

3 Main Results

Now, we will prove our main theorem:

Theorem 3.1 Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $F, G, S, T : X \to X$ be four self-mappings satisfying:

$$d(Fx, Gy) \precsim \beta \max\left\{d(Sx, Ty)\,,\ \frac{d(Sx, Fx)\, d(Ty, Gy)}{1 + d(Fx, Gy)}\right\} \tag{1}$$

for all $x, y \in X$, where β is a non-negative real with $0 \leq \beta < 1$. If $F(X) \subseteq T(X)$ and $G(X) \subseteq S(X)$, then F, G, S and T have a coincidence point. In addition, if the pairs (F, S) and (G, T) are weakly compatible, then there exists a unique common fixed point of the four self-mappings.

Proof. Let x_0 be arbitrary point in X. Since $F(X) \subseteq T(X)$ and $G(X) \subseteq S(X)$, we can define the sequence $\{x_n\}$ such that,

$$\begin{cases} x_{2n+1} = Fx_{2n} = Tx_{2n+1} \\ x_{2n+2} = Gx_{2n+1} = Sx_{2n+2} \end{cases}, \tag{2}$$

for all $n \in \mathbb{N}$.

Then, from (1) and (2), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Fx_{2n}, Gx_{2n+1}) \\ &\precsim \beta \max\left\{d(Sx_{2n}, Tx_{2n+1})\,,\ \frac{d(Sx_{2n}, Fx_{2n})\, d(Tx_{2n+1}, Gx_{2n+1})}{1 + d(Fx_{2n}, Gx_{2n+1})}\right\} \\ &\precsim \beta \max\left\{d(x_{2n}, x_{2n+1})\,,\ \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})}\right\} \\ &= \beta\, d(x_{2n}, x_{2n+1}), \end{aligned}$$

where $d(x_{2n+1}, x_{2n+2}) \leq 1 + d(x_{2n+1}, x_{2n+2})$.

That is,

$$d(x_{2n+1}, x_{2n+2}) \precsim \beta\, d(x_{2n}, x_{2n+1}). \tag{3}$$

By a similar way,

$$\begin{aligned} d(x_{2n+2}, x_{2n+3}) &= d(Gx_{2n+1}, Fx_{2n+2}) \\ &\precsim \beta \max\left\{d(Sx_{2n+2}, Tx_{2n+1})\,,\ \frac{d(Sx_{2n+2}, Fx_{2n+2})\, d(Tx_{2n+1}, Gx_{2n+1})}{1 + d(Fx_{2n+2}, Gx_{2n+1})}\right\} \\ &\precsim \beta \max\left\{d(x_{2n+2}, x_{2n+1})\,,\ \frac{d(x_{2n+2}, x_{2n+3})\, d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+3}, x_{2n+2})}\right\} \\ &\precsim \beta \max\left\{d(x_{2n+1}, x_{2n+2})\,,\ \frac{d(x_{2n+2}, x_{2n+3})\, d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+2}, x_{2n+3})}\right\} \\ &\precsim \beta \max\left\{d(x_{2n+1}, x_{2n+2})\,,\ d(x_{2n+1}, x_{2n+2})\right\} \\ &= \beta\, d(x_{2n+1}, x_{2n+2}). \end{aligned}$$

This implies that

$$d(x_{2n+2}, x_{2n+3}) \precsim \beta d(x_{2n+1}, x_{2n+2}), \tag{4}$$

for all $n \in \mathbb{N}$.

On repeating this process, we find that

$$d(x_{2n+1},x_{2n+2}) \precsim \beta d(x_{2n},x_{2n+1}) \precsim \beta^2 d(x_{2n-1},x_{2n}) \precsim \ldots \precsim \beta^{2n+1} d(x_0,x_1). \quad (5)$$

Hence, we have

$$d(x_{n+1},x_{n+2}) \precsim \beta d(x_n,x_{n+1}) \precsim \beta^2 d(x_{n-1},x_n) \precsim \ldots \precsim \beta^{n+1} d(x_0,x_1). \quad (6)$$

Therefore, for $n, m \in \mathbb{N}$, one can write

$$\begin{aligned} d(x_m,x_{m+n}) &\precsim s\,[\,d(x_m,x_{m+1}) + d(x_{m+1},x_{m+n})\,] \\ &\precsim s\,d(x_m,x_{m+1}) + s^2\,[\,d(x_{m+1},x_{m+2}) + d(x_{m+2},x_{m+n})\,] \\ &\precsim s\,d(x_m,x_{m+1}) + s^2\,d(x_{m+1},x_{m+2}) + \ldots \\ &\qquad + s^{n-1}\,d(x_{m+n-2},x_{m+n-1}) + s^n\,d(x_{m+n-1},x_{m+n}) \\ &\precsim s\beta^m\,d(x_0,x_1) + s^2\beta^{m+1}\,d(x_0,x_1) + \ldots + s^n\beta^{m+n-1}\,d(x_0,x_1) \\ &\precsim s\beta^m\big(1 + s\beta + (s\beta)^2 + \ldots + (s\beta)^{n-1}\big)\,d(x_0,x_1) \\ &\precsim \frac{s\beta^m}{1-s\beta}\,d(x_0,x_1) \longrightarrow 0, \text{ as } m \longrightarrow \infty. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is complete, then there exists $v \in X$ such that $x_n \longrightarrow v$ as $n \longrightarrow \infty$. Then from (2), one can write

$$\lim_{n\longrightarrow\infty} Fx_{2n} = \lim_{n\longrightarrow\infty} Gx_{2n+1} = v$$

$$\text{and } \lim_{n\longrightarrow\infty} Sx_{2n} = \lim_{n\longrightarrow\infty} Tx_{2n+1} = v.$$

Thus,

$$\lim_{n\longrightarrow\infty} Fx_{2n} = \lim_{n\longrightarrow\infty} Gx_{2n+1} = \lim_{n\longrightarrow\infty} Sx_{2n} = \lim_{n\longrightarrow\infty} Tx_{2n+1} = v. \quad (7)$$

Since $F(X) \subseteq T(X)$, there exists $u \in X$ such that

$$Tu = v. \quad (8)$$

We show that $Gu = Tu$, then from (1) we get

$$\begin{aligned} d(v,Gu) &\precsim s\,[d(v,Fx_{2n}) + d(Fx_{2n},Gu)] \\ &\precsim s\,d(v,Fx_{2n}) + s\beta \max\left\{d(Sx_{2n},Tu)\,,\ \frac{d(Sx_{2n},Fx_{2n})\,d(Tu,Gu)}{1+d(Fx_{2n},Gu)}\right\} \\ &\precsim s\,d(v,x_{2n+1}) + s\beta \max\left\{d(x_{2n},v)\,,\ \frac{d(x_{2n},x_{2n+1})\,d(v,Gu)}{1+d(x_{2n+1},Gu)}\right\}. \end{aligned} \quad (9)$$

Taking the limit as $n \longrightarrow \infty$ in (9) and using (7), we get $d(v,Gu) \precsim 0$, then $|d(v,Gu)| \leq 0$. This implies that $Gu = v$ and

$$v = Gu = Tu. \quad (10)$$

Hence u is a coincidence point of G and T.

Similarly, since $G(X) \subseteq S(X)$, we can show that

$$v = Fw = Sw. \quad (11)$$

Thus, w is a coincidence point of F and S.

Since the pairs (F, S) and (G, T) are weakly compatible, one can say

$$GTu = TGu \quad \text{and} \quad FSw = SFw. \qquad (12)$$

Substituting from (10) and (11) in (12), we have

$$Gv = Tv \quad \text{and} \quad Fv = Sv, \qquad (13)$$

i.e., $v \in X$ is a coincidence point for the four self-mappings.

Next, we show that v is a common fixed point of G, T, F and S. From the equation (1), we have

$$d(Fv, Gu) \precsim \beta \max\left\{d(Sv, Tu)\,,\ \frac{d(Sv, Fv)\, d(Tu, Gu)}{1 + d(Fv, Gu)}\right\}.$$

Consequently, from (10), (11) and (13), we have

$$d(Fv, v) \precsim \beta \max\left\{d(Fv, v)\,,\ \frac{d(Fv, Fv)\, d(v, v)}{1 + d(Fv, v)}\right\}.$$

Thus, $(1 - \beta)\, d(Fv, v) \precsim 0$, then $|d(Fv, v)| \leq 0$. i.e., $Fv = v$, then according to (13), we get

$$v = Fv = Sv. \qquad (14)$$

By a similar method and using (14), one can show

$$v = Gv = Tv. \qquad (15)$$

Hence v is a common fixed point for our self-mappings.

To show the uniqueness: Let $v^* \neq v$ be another common fixed point of the four self-mappings, then from (1), we get

$$\begin{aligned} d(v, v^*) &= d(Fv, Gv^*) \\ &\precsim \beta \max\left\{d(Sv, Tv^*)\,,\ \frac{d(Sv, Fv)\, d(Tv^*, Gv^*)}{1 + d(Fv, Gv^*)}\right\}. \end{aligned}$$

This implies that

$$(1 - \beta)\, |d(v, v^*)| \leq 0.$$

since $0 \leq \beta < 1$, then, $|d(v, v^*)| = 0$. i.e., $v = v^*$, then v is a unique common fixed point of F, G, S and T. This completes the proof.

Now, if we take $S = T = I$, (where I is the identity mapping) in Theorem 3.1, we obtain the following corollary.

Corollary 3.1 [[3] ,Theorem 1] Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $F, G : X \to X$ be self-mappings satisfy:

$$d(Fx, Gy) \precsim \beta \max\left\{d(x, y)\,,\ \frac{d(x, Fx)\, d(y, Gy)}{1 + d(Fx, Gy)}\right\},$$

for all $x, y \in X$, where β is a non-negative real with $0 \leq \beta < 1$. Then F and G have a unique common fixed point.

Taking $F = G$ in Corollary 3.1, we get the following result.

Corollary 3.2 [[3] ,Corollary 1] Let (X,d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $F: X \to X$ be a self-mapping satisfy:

$$d(Fx,Fy) \precsim \beta \max\left\{d(x,y)\,,\ \frac{d(x,Fx)\,d(y,Fy)}{1+d(Fx,Fy)}\right\},$$

for all $x,y \in X$, where β is a non-negative real with $0 \leq \beta < 1$. Then F has a unique fixed point.

If we put $s = 1$, we obtain other corollary.

Corollary 3.3 [[3] ,Theorem 1] Let (X,d) be a complete complex valued metric space and $F,G: X \to X$ be self-mappings satisfy:

$$d(Fx,Gy) \precsim \beta \max\left\{d(x,y)\,,\ \frac{d(x,Fx)\,d(y,Gy)}{1+d(Fx,Gy)}\right\},$$

for all $x,y \in X$, where β is a non-negative real with $0 \leq \beta < 1$. Then F and G have a unique common fixed point.

Now, we prove the following theorem under other contractive condition.

Theorem 3.2 Let (X,d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and $F,G,S,T: X \to X$ be four self-mappings satisfying:

$$d(Fx,Gy) \precsim \beta \max\left\{d(Sx,Ty)\,,\ \frac{d(Sx,Gy)\,d(Ty,Fx)}{1+d(Fx,Gy)}\right\} \tag{16}$$

for all $x,y \in X$, where β is a non-negative real with $0 \leq \beta < 1$. If $F(X) \subseteq T(X)$ and $G(X) \subseteq S(X)$, then F,G,S and T have a coincidence point. In addition, if the pairs (F,S) and (G,T) are weakly compatible, then there exists a unique common fixed point of the four self-mappings.

Proof. Let x_0 be arbitrary point in X. Since $F(X) \subseteq T(X)$ and $G(X) \subseteq S(X)$, we can write the sequence $\{x_n\}$ as (2). Consequently, from (16) and (2), for all $n \in \mathbb{N}$, we get

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) &= d(Fx_{2n},Gx_{2n+1}) \\
&\precsim \beta \max\left\{d(Sx_{2n},Tx_{2n+1})\,,\ \frac{d(Sx_{2n},Gx_{2n+1})\,d(Tx_{2n+1},Fx_{2n})}{1+d(Fx_{2n},Gx_{2n+1})}\right\} \\
&\precsim \beta \max\left\{d(x_{2n},x_{2n+1})\,,\ \frac{d(x_{2n},x_{2n+2})\,d(x_{2n+1},x_{2n+1})}{1+d(x_{2n+1},x_{2n+2})}\right\} \\
&\precsim \beta \max\{d(x_{2n},x_{2n+1})\,,\ 0\} \\
&= \beta\, d(x_{2n},x_{2n+1}),
\end{aligned}$$

Then, we have (3). Repeating this process, we get (4) and (5).

Also, for $n,m \in \mathbb{N}$, we obtain

$$\begin{aligned}
d(x_m,x_{m+n}) &\precsim s\,[\,d(x_m,x_{m+1}) + d(x_{m+1},x_{m+n})\,] \\
&\precsim s\,d(x_m,x_{m+1}) + s^2\,[\,d(x_{m+1},x_{m+2}) + d(x_{m+2},x_{m+n})\,] \\
&\precsim s\,d(x_m,x_{m+1}) + s^2\,d(x_{m+1},x_{m+2}) + \ldots \\
&\qquad + s^{n-1}\,d(x_{m+n-2},x_{m+n-1}) + s^n\,d(x_{m+n-1},x_{m+n})
\end{aligned}$$

$$\precsim s\beta^m\, d(x_0,x_1) \,+\, s^2\beta^{m+1}\, d(x_0,x_1) \,+\, ... \,+\, s^n\beta^{m+n-1}\, d(x_0,x_1)$$
$$\precsim s\beta^m\big(1+s\beta+(s\beta)^2+...+(s\beta)^{n-1}\big)\, d(x_0,x_1)$$
$$\precsim \frac{s\beta^m}{1-s\beta}\, d(x_0,x_1) \longrightarrow 0, \text{ as } m \longrightarrow \infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is complete, then there exists $v \in X$ such that $x_n \longrightarrow v$ as $n \longrightarrow \infty$. Consequently, from (2), we write

$$\lim_{n\longrightarrow\infty} Fx_{2n} \;=\; \lim_{n\longrightarrow\infty} Gx_{2n+1} \;=\; v$$
$$\text{and } \lim_{n\longrightarrow\infty} Sx_{2n} \;=\; \lim_{n\longrightarrow\infty} Tx_{2n+1} \;=\; v.$$

Then, we get (7). Since $F(X) \subseteq T(X)$, there exists $u \in X$ such that (8) is satisfied.

Now, we show that $Gu = Tu$, then from (16), we get

$$d(v,Gu) \precsim s\,[d(v,Fx_{2n})+d(Fx_{2n},Gu)]$$
$$\precsim s\,d(v,Fx_{2n}) \,+\, s\beta\, \max\left\{d(Sx_{2n},Tu)\,,\, \frac{d(Sx_{2n},Gu)\, d(Tu,Fx_{2n})}{1+d(Fx_{2n},Gu)}\right\}$$
$$\precsim s\,d(v,x_{2n+1})+s\beta\max\left\{d(x_{2n},v)\,,\, \frac{d(x_{2n},Gu)\, d(v,x_{2n+1})}{1+d(x_{2n+1},Gu)}\right\}. \tag{17}$$

Taking the limit as $n \longrightarrow \infty$ in (17) and using (7), we have $d(v,Gu) \precsim 0$, then $|d(v,Gu)| \le 0$. i.e., $Gu = v$ and we obtain (10). Consequently, in this case u became a coincidence point of G and T.

Similarly, since $G(X) \subseteq S(X)$, we can deduce (11). Then, w also became a coincidence point of F and S.

Substituting from (10) and (11) in the weakly compatible condition of the pairs (F,S) and (G,T)[in Eq.(12)], we get (13).

Next, we show that v is a common fixed point of G, T, F and S. From the equation (16), we have

$$d(Fv,Gu) \precsim \beta\, \max\left\{d(Sv,Tu)\,,\, \frac{d(Sv,Gu)\, d(Tu,Fv)}{1+d(Fv,Gu)}\right\}$$

Using (10), (11) and (13), the previous inequality became

$$d(Fv,v) \precsim \beta\, \max\left\{d(Fv,v)\,,\, \frac{d(Fv,v)\, d(v,Fv)}{1+d(Fv,v)}\right\}.$$

Thus, $(1-\beta)\, d(Fv,v) \precsim 0$, then $|d(Fv,v)| \le 0$. i.e., $Fv = v$, then according to (13), we get (14). By a similar way and using (14), we obtain the equation (15) that show v is a common fixed point for our self-mappings.

For the uniqueness, let $v^* \neq v$ be another common fixed point of the four self-mappings, then from (16), we have

$$d(v,v^*) \;=\; d(Fv,Gv^*)$$

$$\precsim \beta \max\left\{ d(Sv,Tv^*)\,,\ \frac{d(Sv,Gv^*)\,d(Tv^*,Fv)}{1+d(Fv,Gv^*)}\right\}.$$

This implies that $(1-\beta)\,|d(v,v^*)| \le 0$. Since $0 \le \beta < 1$, then, $|d(v,v^*)| = 0$. i.e., $v = v^*$, then v is a unique common fixed point of F, G, S and T. Consequently, the proof is complete.

Using another rational expression, we state and show the following theorem.

Theorem 3.3 Let (X,d) be a complete complex valued b-metric space with coefficient $s \ge 1$ and $F, G, S, T : X \to X$ be four self-mappings satisfying:

$$d(Fx,Gy) \precsim \beta \max\left\{ d(Sx,Ty)\,,\ \frac{d^2(Sx,Gy)+d^2(Ty,Fx)}{d(Sx,Gy)+d(Ty,Fx)}\right\} \qquad (18)$$

for all $x, y \in X$, where β is a non-negative real with $0 \le \beta < 1$. If $F(X) \subseteq T(X)$ and $G(X) \subseteq S(X)$, then F, G, S and T have a coincidence point. In addition, if the pairs (F,S) and (G,T) are weakly compatible, then there exists a unique common fixed point of the four self-mappings.

Proof. Let x_0 be arbitrary point in X. Since $F(X) \subseteq T(X)$ and $G(X) \subseteq S(X)$, we can write the sequence $\{x_n\}$ as (2). Consequently, from (18) and (2), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) &= d(Fx_{2n},Gx_{2n+1})\\
&\precsim \beta \max\left\{ d(Sx_{2n},Tx_{2n+1})\,,\ \frac{d^2(Sx_{2n},Gx_{2n+1})+d^2(Tx_{2n+1},Fx_{2n})}{d(Sx_{2n},Gx_{2n+1})+d(Tx_{2n+1},Fx_{2n})}\right\}\\
&\precsim \beta \max\left\{ d(x_{2n},x_{2n+1})\,,\ \frac{d^2(x_{2n},x_{2n+2})+d^2(x_{2n+1},x_{2n+1})}{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}\right\}\\
&\precsim \beta \max\{d(x_{2n},x_{2n+1})\,,\ d(x_{2n},x_{2n+2})\}\\
&\precsim \beta \max\{\, d(x_{2n},x_{2n+1})\,,\ s[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})]\,\}\\
&= s\beta\ [d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})].
\end{aligned}$$

This implies that,

$$d(x_{2n+1},x_{2n+2}) \precsim \mu\, d(x_{2n},x_{2n+1}),$$

where $\mu = \frac{s\beta}{1-s\beta}$.

Similarly,

$$\begin{aligned}
d(x_{2n+2},x_{2n+3}) &= d(Gx_{2n+1},Fx_{2n+2})\\
&\precsim \beta \max\left\{ d(Sx_{2n+2},Tx_{2n+1})\,,\ \frac{d^2(Sx_{2n+2},Gx_{2n+1})+d^2(Tx_{2n+1},Fx_{2n+2})}{d(Sx_{2n+2},Gx_{2n+1})+d(Tx_{2n+1},Fx_{2n+2})}\right\}\\
&\precsim \beta \max\left\{ d(x_{2n+2},x_{2n+1})\,,\ \frac{d^2(x_{2n+2},x_{2n+2})+d^2(x_{2n+1},x_{2n+3})}{d(x_{2n+2},x_{2n+2})+d(x_{2n+1},x_{2n+3})}\right\}\\
&\precsim \beta \max\{d(x_{2n+1},x_{2n+2})\,,\ d(x_{2n+1},x_{2n+3})\}\\
&\precsim \beta \max\{\, d(x_{2n+1},x_{2n+2})\,,\ s[d(x_{2n+1},x_{2n+2})+d(x_{2n+2},x_{2n+3})]\,\}\\
&= s\beta\ [d(x_{2n+1},x_{2n+2})+d(x_{2n+2},x_{2n+3})].
\end{aligned}$$

Then,

$$d(x_{2n+2},x_{2n+3}) \precsim \mu\, d(x_{2n+1},x_{2n+2}),$$

On continuing this process, we get (5).

Also, for $n, m \in \mathbb{N}$, we obtain

$$\begin{aligned} d(x_m, x_{m+n}) &\precsim s\,[\,d(x_m, x_{m+1}) \;+\; d(x_{m+1}, x_{m+n})\,] \\ &\precsim s\,d(x_m, x_{m+1}) \;+\; s^2\,[\,d(x_{m+1}, x_{m+2}) \;+\; d(x_{m+2}, x_{m+n})\,] \\ &\precsim s\,d(x_m, x_{m+1}) \;+\; s^2\,d(x_{m+1}, x_{m+2}) \;+\; \ldots \\ &\qquad\qquad + s^{n-1}\,d(x_{m+n-2}, x_{m+n-1}) \;+\; s^n\,d(x_{m+n-1}, x_{m+n}) \\ &\precsim s\beta^m\,d(x_0, x_1) \;+\; s^2\beta^{m+1}\,d(x_0, x_1) \;+\; \ldots \;+\; s^n\beta^{m+n-1}\,d(x_0, x_1) \\ &\precsim s\beta^m\big(1 + s\beta + (s\beta)^2 + \ldots + (s\beta)^{n-1}\big)\,d(x_0, x_1) \\ &\precsim \frac{s\beta^m}{1 - s\beta}\,d(x_0, x_1) \longrightarrow 0, \text{ as } m \longrightarrow \infty. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, then there exists $v \in X$ such that $x_n \longrightarrow v$ as $n \longrightarrow \infty$. Consequently, from (2), we write

$$\lim_{n\longrightarrow\infty} Fx_{2n} \;=\; \lim_{n\longrightarrow\infty} Gx_{2n+1} \;=\; v$$
$$\text{and} \lim_{n\longrightarrow\infty} Sx_{2n} \;=\; \lim_{n\longrightarrow\infty} Tx_{2n+1} \;=\; v.$$

Then, we get (7). Since $F(X) \subseteq T(X)$, there exists $u \in X$ such that (8) is satisfied.

Now, we show that $Gu = Tu$, then from (18), we have

$$\begin{aligned} d(v, Gu) &\precsim s\,[d(v, Fx_{2n}) + d(Fx_{2n}, Gu)] \\ &\precsim s\,d(v, Fx_{2n}) \;+\; s\beta\, \max\left\{d(Sx_{2n}, Tu)\,,\; \frac{d^2(Sx_{2n}, Gu) + d^2(Tu, Fx_{2n})}{d(Sx_{2n}, Gu) + d(Tu, Fx_{2n})}\right\} \\ &\precsim s\,d(v, x_{2n+1}) \;+\; s\beta\, \max\left\{d(x_{2n}, v)\,,\; \frac{d^2(x_{2n}, Gu) + d^2(v, x_{2n+1})}{d(x_{2n}, Gu) + d(v, x_{2n+1})}\right\}. \end{aligned} \tag{19}$$

Taking the limit as $n \longrightarrow \infty$ in (17) and using (7), we have $d(v, Gu) \precsim 0$, where $1 - \beta \neq 0$, then $|d(v, Gu)| \leq 0$. i.e., $Gu = v$ and we obtain (10). Similarly, since $G(X) \subseteq S(X)$, we can deduce (11).

Substituting from (10) and (11) in the weakly compatible condition of the pairs (F, S) and (G, T)[in Eq.(12)], we get (13).

Next, we show that v is a common fixed point of G, T, F and S. From the equation (18), we have

$$d(Fv, Gu) \precsim \beta\, \max\left\{d(Sv, Tu)\,,\; \frac{d^2(Sv, Gu) + d^2(Tu, Fv)}{d(Sv, Gu) + d(Tu, Fv)}\right\}.$$

Using (10), (11) and (13), we have

$$d(Fv, v) \precsim \beta\, \max\left\{d(Fv, v)\,,\; \frac{d^2(Fv, v) + d^2(v, Fv)}{d(Fv, v) + d(v, Fv)}\right\}.$$

This implies that, $(1 - \beta)\,d(Fv, v) \precsim 0$, then $|d(Fv, v)| \leq 0$. i.e., $Fv = v$, then according to (13), we get (14). By a similar way and using (14), we obtain the equation (15) that show v is a common fixed point for our self-mappings.

For the uniqueness, let $v^* \neq v$ be another common fixed point of the four self-mappings, then from (18), we have

$$d(v, v^*) = d(Fv, Gv^*)$$
$$\precsim \beta \max \left\{ d(Sv, Tv^*) \, , \, \frac{d^2(Sv, Gv^*) + d^2(Tv^*, Fv)}{d(Sv, Gv^*) + d(Tv^*, Fv)} \right\}.$$

This implies that $(1 - \beta) |d(v, v^*)| \leq 0$. Since $0 \leq \beta < 1$, then, $|d(v, v^*)| = 0$. i.e., $v = v^*$, then v is a unique common fixed point of F, G, S and T. Consequently, the proof is complete.

Example 3.1 Let $X = (0, 4]$ be a complex valued b-metric space with a complex valued b-metric $d(x, y) = |x - y|^2 + i\, |x - y|^2$ for all $x, y \in X$ and F, G, S and T be self-mappings of X, defined as follow:

$$Fx = \begin{cases} \frac{1}{4} & \text{if } x \in (0,2) \\ 1 & \text{if } x \in [2,4] \end{cases} \quad \text{and} \quad Gx = \begin{cases} \frac{1}{4} & \text{if } x \in (0,2) \\ 1 & \text{if } x \in [2,4] \end{cases}$$

$$Sx = \begin{cases} \frac{1}{4} & \text{if } x \in (0,2) \\ 1 & \text{if } x = 2 \\ \frac{x}{2} - 1 & \text{if } x \in (2,4] \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{1}{4} & \text{if } x \in (0,2) \\ 1 & \text{if } x = 2 \\ \frac{x}{2} + 1 & \text{if } x \in (2,4] \end{cases}$$

This implies that

$$F(X) = \left\{ \frac{1}{4}, 1 \right\}, \quad G(X) = \left\{ \frac{1}{4}, 1 \right\}, \quad S(X) = (0, 1], \quad T(X) = (0, 3].$$

Therefore, one can say

$$F(X) \subseteq T(X) \quad \text{and} \quad G(X) \subseteq S(X).$$

Also, the previous complex valued b-metric can be written as:

$$d(x, y) = \sqrt{2}\, e^{i \frac{\pi}{4}} \; |x - y|^2$$

Next, to prove the inequality (1) in Theorem 3.1 is satisfied, there are three cases:

Case 1: Let $x, y \in (0, 2)$, then

$$Fx = Fy = Gx = Gy = \frac{1}{4} \quad \text{and} \quad Sx = Sy = Tx = Ty = \frac{1}{4}.$$

Now, we can show that

L.H.S. $= d(Fx, Gy) = 0.$

R.H.S. $= \beta \max \left\{ d(Sx, Ty) \, , \, \frac{d(Sx, Fx)\, d(Ty, Gy)}{1 + d(Fx, Gy)} \right\} = 0.$

Clearly, both of sides are equivalent and both of them are equal to zero.

Case 2: Let $x = y = 2$, then

$$Fx = Fy = Gx = Gy = 1 \quad \text{and} \quad Sx = Sy = Tx = Ty = 1.$$

Now, one can show that

L.H.S. $= d(Fx, Gy) = 0.$

R.H.S. $= \beta \max\left\{ d(Sx, Ty) \, , \, \frac{d(Sx, Fx)\, d(Ty, Gy)}{1 + d(Fx, Gy)} \right\} = 0.$

Clearly, both of sides are equivalent and both of them are equal to zero.

Case 3: Let $x, y \in (2, 4]$, then

$$Fx = Fy = Gx = Gy = 1, \quad Sx = Sy = \frac{x}{2} - 1 \quad \text{and} \quad Tx = Ty = \frac{x}{2} + 1.$$

Now, one can show that

L.H.S. $= d(Fx, Gy) = 0.$

R.H.S. $= \beta \max\left\{ d(Sx, Ty) \, , \, \frac{d(Sx, Fx)\, d(Ty, Gy)}{1 + d(Fx, Gy)} \right\} = \sqrt{2}\, \beta\, e^{i\frac{\pi}{4}} \left| \frac{x}{2} - \frac{y}{2} - 2 \right|^2.$

Clearly, right hand side depends on the value of x, y and β whereas $0 \leq \beta < 1$, so one can say that

$$d(Fx, Gy) \precsim \beta \max\left\{ d(Sx, Ty) \, , \, \frac{d(Sx, Fx)\, d(Ty, Gy)}{1 + d(Fx, Gy)} \right\}.$$

Consequently, from the previous cases, we deduce

$$d(Fx, Gy) \precsim \beta \max\left\{ d(Sx, Ty) \, , \, \frac{d(Sx, Fx)\, d(Ty, Gy)}{1 + d(Fx, Gy)} \right\}.$$

Also, the pairs (F, S) and (G, T) are weakly compatible.

From the other hand, let $\{x_n\} = \{1 + \frac{1}{2n-1}\}_{n \geq 1}$ and $\{x_n^*\} = \{1 + \frac{1}{2n+1}\}_{n \geq 1}$ be two sequences in X. Then, we find that

$$\lim_{n \to \infty} Fx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Gx_n^* = \lim_{n \to \infty} Tx_n^* = \frac{1}{4} \in X.$$

Therefore, $\frac{1}{4}$ is a unique common fixed point of F, G, S and T.

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