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Approximation by sub-matrix means of multiple Fourier series in the Hölder metric

Xhevat Z. Krasniqi

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Abstract In this paper some results on approximation by sub-matrix means of multiple Fourier series in the Hölder metric are obtained. Our results are applicable for a wider class of sequences and give a better degree of approximation than those presented previously by others.

1 Introduction

Let $f(x,y)$ be an integrable in the sense of Lebesgue over the square $[-\pi,\pi]\times[-\pi,\pi]:=[-\pi,\pi]^2$ and 2π periodic with respekt x and y. We recall that the double Fourier series of the function $f(x,y)$ is defined by

$$f(x,y)\sim\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\Big[a_{mn}\cos mx\cos ny+b_{mn}\sin mx\cos ny$$
$$+c_{mn}\cos mx\sin ny+d_{mn}\sin mx\sin ny\Big],$$

where

$$\lambda_{mn}=\begin{cases}1/4, & \text{if } m=n=0,\\ 1/2, & \text{if } m>0, n=0\vee m=0, n>0,\\ 1, & \text{if } m>0, n>0,\end{cases}$$

and

$$a_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\cos mu\cos nv\,du\,dv,$$
$$b_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\sin mu\cos nv\,du\,dv,$$
$$c_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\cos mu\sin nv\,du\,dv,$$
$$d_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\sin mu\sin nv\,du\,dv,$$

are the Fourier coefficients of the function $f(x,y)$, for $m=0,1,2,\dots$ and $n=0,1,2,\dots$.

The sequence $\{s_{m,n}(f;x,y)\}_{m=1,n=1}^{+\infty,+\infty}$ represents the sequence of partial sums of the double Fourier series which can be rewritten in integral form by

$$s_{m,n}(x,y):=s_{m,n}(f;x,y):=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x+u,y+v)D_m(u)D_n(v)\,du\,dv,$$

where the Dirichlet's kernel is defined by

$$D_s(t):=\frac{1}{2}+\sum_{r=1}^{s}\sin rt=\frac{\sin\left(s+\frac{1}{2}\right)t}{2\sin\frac{t}{2}},\ (s=1,2,\dots).$$

Beyond this, let

$$t_{m,n}(x,y) := t_{m,n}(f;A,B;x,y) := \sum_{i=0}^{m}\sum_{j=0}^{n} a_{m,i}b_{n,j}s_{m,n}(x,y), \quad m,n \geq 0,$$

where $A := (a_{m,i})$ and $B := (b_{n,j})$ are two lower triangular infinite matrices such that

$$a_{m,i} = \begin{cases} \geq 0, & \text{if } i \leq m, \\ 0, & \text{if } i > m; \end{cases} \quad (i,m = 0,1,2,\dots) \quad \text{and} \quad \sum_{i=0}^{m} a_{m,i} = 1, \tag{1.1}$$

and

$$b_{n,j} = \begin{cases} \geq 0, & \text{if } j \leq n, \\ 0, & \text{if } j > n; \end{cases} \quad (j,n = 0,1,2,\dots) \quad \text{and} \quad \sum_{j=0}^{n} b_{n,j} = 1. \tag{1.2}$$

Next definition gives the notion of (A,B) summability of a double Fourier series (see [3]).

The double Fourier series of the function $f(x,y)$ is said to be (A,B) summable to a finite number s , if $\tau_{m,n}(x,y) \to s$ as $m,n \to \infty$. The conditions of regularity for double matrix summability are given by

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{m,i}b_{n,j} \to 1 \quad \text{as} \quad m,n \to \infty,$$

$$\lim_{m,n} \sum_{j=0}^{n} |a_{m,i}b_{n,j}| = 0 \quad \text{for each} \quad i = 1,2,\dots,$$

$$\lim_{m,n} \sum_{i=0}^{m} |a_{m,i}b_{n,j}| = 0 \quad \text{for each} \quad j = 1,2,\dots.$$

Now, we need to recall some notations and definitions.

The Hölder (or Lipschitz) class $H_{(\alpha,\beta)}$ (or Lip(α,β)) contains continuous functions $f(x,y)$ 2π-periodic with respect to both variables x and y. It is defined by (see [2])

$$H_{(\alpha,\beta)} := \left\{ f : |f(x,y;z,w)| := |f(x,y) - f(z,w)| \leq K\left(|x-z|^{\alpha} + |y-w|^{\beta}\right)\right\}$$

for some $\alpha, \beta > 0$ and for all x,y,z,w, where K is a positive constant which may depend on f, but not on x,y,z,w. The $H_{(\alpha,\beta)}$ class is a Banach space with the norm $\|(\cdot)\|_{(\alpha,\beta)}$ defined by

$$\|f\|_{(\alpha,\beta)} = \|f\|_C + \sup_{x\neq z, y\neq w} \Delta^{(\alpha,\beta)} f(x,y;z,w),$$

where

$$\Delta^{(\alpha,\beta)} f(x,y;z,w) = \frac{|f(x,y;z,w)|}{|x-z|^{\alpha} + |y-w|^{\beta}}, \quad x \neq u, y \neq v,$$

by convention $\Delta^{(0,0)} f(x,y;z,w) = 0$ and

$$\|f\|_C = \sup_{(x,y)\in[-\pi,\pi]^2} |f(x,y)|.$$

Throughout this paper, for two positive quantities u and v, we write $u = \mathcal{O}(v)$ instead of $u \leq Kv$, where K is an absolute positive constant.

In [2] the degree of approximation of the function $f(x,y)$ belonging to Hölder (Lipschitz) class by matrix means of double Fourier series has been determined in consistency with the norm $\|(\cdot)\|_{(\alpha,\beta)}$.

Let us reword that result:

Theorem 1.1 ([2]). *Assume $A := (a_{m,i})$ and $B := (b_{n,j})$ are two lower triangular matrices, where $\{a_{m,i}\}$ and $\{b_{n,j}\}$ are non-decreasing sequences with respect to $i \leq m$ and $j \leq n$, satisfying the conditions (1.1) and (1.2) respectively, and the double matrix method (A,B) is* *regular. If $f(x,y)$ is a 2π-periodic function in x and y, Lebesgue integrable in $[-\pi,\pi]^2$, and belonging to the class $H_{(\alpha,\beta)}$ for $0<\alpha,\beta\le 1$, then*

$$\|t_{m,n}-f\|_{(\alpha,\beta)}=\mathcal{O}(1)\begin{cases}\frac{1}{(m+1)^\alpha}+\frac{1}{(n+1)^\beta}, & \text{if} \quad 0<\alpha<1,\ 0<\beta<1,\\ \frac{\log(\pi(m+1))}{m+1}+\frac{\log(\pi(n+1))}{n+1}, & \text{if} \quad \alpha=\beta=1,\end{cases} \tag{1.3}$$

for all $m,n=0,1,2,\ldots$.

In the same paper the little Lipschitz class $\text{lip}(\alpha,\beta)$ has been defined too. Indeed, it said that a function $f(x,y)\in\text{lip}(\alpha,\beta)$ if

$$\lim_{z\to x, w\to y}\frac{|f(x,y;z,w)|}{|x-z|^\alpha+|y-w|^\beta}=0,$$

uniformly in (x,y).

It already has been pointed out (see [2]) that for little Lipschitz class $\text{lip}(\alpha,\beta)$, $0<\alpha,\beta\le 1$, analogous statement with Theorem 1.1 holds true when $\mathcal{O}(1)$ is replaced with $o(1)$. Moreover, the results mentioned so far are extended to multiple Fourier series as well.

For our purposes we need still some notations and notions.

A sequence $\mathbf{c}:=\{c_n\}$ of nonnegative numbers tending to zero is called of Rest Bounded Variation, or briefly $\mathbf{c}\in RBVS$, if it has the property

$$\sum_{n=m}^{\infty}|c_n-c_{n+1}|\le K(\mathbf{c})c_m \tag{1.4}$$

for all natural numbers m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$ (see [5]).

Let $\mathbb{F}$ be an infinite subset of $\mathbb{N}$ and $\mathbb{F}$ as range of strictly increasing sequence of positive integers, say $\mathbb{F}=\{\lambda(n)\}_{n=1}^\infty$. The Cesàro sub-method C_λ is defined as

$$(C_\lambda x)_n=\frac{1}{\lambda(n)}\sum_{k=1}^{\lambda(n)}x_k, \quad (n=1,2,\ldots),$$

where $\{x_k\}$ is a sequence of real or complex numbers. Therefore, C_λ-method yields a subsequence of the Cesàro method C_1 and hence it is regular for any λ. C_λ-matrix is obtained by deleting a set of rows from Cesàro matrix. The basic properties of C_λ-method can be found in [1] and [8].

More general means than C_λ-means has been considered, see [6], and the following transformation has also been defined:

$$t_n^\lambda(f_1;x):=\sum_{k=0}^{\lambda(n)}a_{\lambda(n),k}s_k(f_1;x),$$

where $(a_{n,k})$ is an infinite lower triangular regular matrix with non-negative entries with row sums 1, and $s_k(f_1;x)$ denotes the partial sums of the single Fourier series of the function f_1.

Now we define the following trigonometric polynomials

$$t_{m,n}^\lambda(f;x,y):=\sum_{i=0}^{\lambda(m)}\sum_{j=0}^{\lambda(n)}a_{\lambda(m),i}b_{\lambda(n),j}s_{i,j}(f;x,y),$$

where $(a_{n,i})$ and $(b_{m,j})$ two lower triangular regular matrix satisfying the conditions (1.1) and (1.2) respectively.

It is the main aim of this paper to prove an analogous statement as Theorem 1.1 metric using the trigonometric polynomials $t_{m,n}^\lambda(f;x,y)$ instead of $t_{m,n}(f;x,y)$ and instead of conditions $\{a_{m,i}\}$ and $\{b_{n,j}\}$ are non-decreasing sequences with respect to $i\le m$ and $j\le n$ we use conditions $\{a_{m,i}\}\in RBVS$ and $\{b_{n,j}\}\in RBVS$ with respect to $i\le m$ and $j\le n$ respectively. Our technique used for the proof of our results has some in common with proofs in [2], but it has also some differences. As we will see our results give sharper degree of approximation since those are not expressed simply in terms of m and n, but in terms of strict increasing sequences $\lambda(m)$ and $\lambda(n)$.

To do this we need next notations

$$\begin{aligned}\Psi(u,v):=&\Psi(x,y;u,v):=\frac{1}{4}\big[f(x+u,y+v)\\&+f(x+u,y-v)+f(x-u,y+v)+f(x-u,y-v)-4f(x,y)\big]\\F(u,v):=&\Phi(u,v)-\Psi(u,v),\ \text{where}\ \Phi(u,v):=\Psi(z,w;u,v),\\A_{\lambda(n),k}:=&\sum_{r=0}^{k}\widetilde{c}_{\lambda(n),r},\quad K_n^{\lambda}(t):=\frac{1}{\pi}\sum_{k=0}^{\lambda(n)}\widetilde{c}_{\lambda(n),k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}},\end{aligned}$$

where $(\widetilde{c}_{p,q})$ is a lower triangular regular matrix.

Closing this section, it should be noted here that if $f\in H_{\alpha,\beta}$, then

$$|F(u,v)|=\mathcal{O}\left(|x-z|^{\alpha}+|y-w|^{\beta}\right). \tag{1.5}$$

Next section has been devoted to some helpful lemmas.

2 Auxiliary Lemmas

In this section we are going to prove some estimates for $|K_n^{\lambda}(t)|$, which we need afterwards for the proofs of the main results.

Lemma 2.1. *Let $(\widetilde{c}_{n,k})$ be a lower triangular regular matrix with non-negative entries. Then for $0<t\leq\frac{1}{\lambda(n)+1}$*

$$|K_n^{\lambda}(t)|=\mathcal{O}(\lambda(n)+1).$$

Proof. Applying the elementary inequality $\sin\alpha\leq\alpha$, Jordan's inequality $\sin\beta\geq\frac{2}{\pi}\beta$ for $\beta\in[0,\frac{\pi}{2}]$, and our assumptions, we have

$$\begin{aligned}|K_n^{\lambda}(t)|&\leq\frac{1}{2\pi}\sum_{k=0}^{\lambda(n)}\left|\widetilde{c}_{\lambda(n),k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|\\&\leq\frac{1}{2}\sum_{k=0}^{\lambda(n)}\widetilde{c}_{\lambda(n),k}\frac{\left(k+\frac{1}{2}\right)t}{t}\leq\frac{1}{4}(2\lambda(n)+1)\sum_{k=0}^{\lambda(n)}\widetilde{c}_{\lambda(n),k}=\mathcal{O}(\lambda(n)+1).\end{aligned}$$

□

Lemma 2.2. *Let $(\widetilde{c}_{n,k})$ be a lower triangular regular matrix with non-negative entries and $\{\widetilde{c}_{n,k}\}\in RBVS$ with respect to k. Then for $\frac{1}{\lambda(n)+1}<t\leq\pi$*

$$|K_n^{\lambda}(t)|=\mathcal{O}\left(\frac{A_{\lambda(n),\tau}}{t}\right),$$

where τ denotes the integer part of $1/t$.

Proof. Using Jordan's inequality $\sin\beta\geq\frac{2}{\pi}\beta$ for $\beta\in[0,\frac{\pi}{2}]$, and our assumptions, we have

$$\begin{aligned}|K_n^{\lambda}(t)|&=\left|\frac{1}{2\pi}\sum_{k=0}^{\lambda(n)}\widetilde{c}_{\lambda(n),k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|\leq\frac{1}{2t}\left|\sum_{k=0}^{\lambda(n)}\widetilde{c}_{\lambda(n),k}\,\mathrm{Im}\,e^{i\left(k+\frac{1}{2}\right)t}\right|\\&\leq\frac{1}{2t}\left|e^{\frac{it}{2}}\sum_{k=0}^{\lambda(n)}\widetilde{c}_{\lambda(n),k}e^{ikt}\right|=\frac{1}{2t}\left|\sum_{k=0}^{\lambda(n)}\widetilde{c}_{\lambda(n),k}e^{ikt}\right|.\end{aligned}$$

Similar to [9], see Lemma 2, we obtain

$$\begin{aligned}
\left|\sum_{k=0}^{\lambda(n)} \widetilde{c}_{\lambda(n),k} e^{ikt}\right| &\leq \left|\sum_{k=0}^{\tau-1} \widetilde{c}_{\lambda(n),k} e^{ikt}\right| + \left|\sum_{k=\tau}^{\lambda(n)} \widetilde{c}_{\lambda(n),k} e^{ikt}\right| \\
&\leq \sum_{k=0}^{\tau-1} \widetilde{c}_{\lambda(n),k} + 2\widetilde{c}_{\lambda(n),\tau} \times \max_{\tau\leq k\leq\lambda(n)} \left|\sum_{s=0}^{k} e^{ist}\right| \\
&= A_{\lambda(n),\tau-1} + 2\widetilde{c}_{\lambda(n),\tau} \times \max_{\tau\leq k\leq\lambda(n)} \left|\frac{1-e^{i(s+1)t}}{1-e^{it}}\right| \\
&\leq A_{\lambda(n),\tau-1} + \frac{4\widetilde{c}_{\lambda(n),\tau}}{\sqrt{(1-\cos t)^2 + (\sin t)^2}} \\
&= A_{\lambda(n),\tau-1} + \frac{2\widetilde{c}_{\lambda(n),\tau}}{\sin\frac{t}{2}} \\
&\leq A_{\lambda(n),\tau-1} + \frac{2\pi\widetilde{c}_{\lambda(n),\tau}}{t} \\
&\leq A_{\lambda(n),\tau-1} + 2\pi(\tau+1)\widetilde{c}_{\lambda(n),\tau}.
\end{aligned}$$

Since $\{\widetilde{c}_{n,j}\} \in RBVS$ with respect to j, then for $0 \leq s \leq \tau$ we get

$$\begin{aligned}
\widetilde{c}_{\lambda(n),\tau} &\leq \sum_{k=\tau}^{\infty} \left|\widetilde{c}_{\lambda(n),k} - \widetilde{c}_{\lambda(n),k+1}\right| \\
&\leq \sum_{k=s}^{\infty} \left|\widetilde{c}_{\lambda(n),k} - \widetilde{c}_{\lambda(n),k+1}\right| \leq K(\mathbf{c})\widetilde{c}_{\lambda(n),s},
\end{aligned}$$

so $\widetilde{c}_{\lambda(n),\tau} \leq K(\mathbf{c})\widetilde{c}_{\lambda(n),s}$, and therefore

$$\begin{aligned}
|K_n^\lambda(t)| &\leq \frac{1}{2t}\left|\sum_{k=0}^{\lambda(n)} \widetilde{c}_{\lambda(n),k} e^{ikt}\right| \leq \frac{1}{2t}\left(A_{\lambda(n),\tau-1} + 2\pi(\tau+1)\widetilde{c}_{\lambda(n),\tau}\right) \\
&\leq \frac{1}{2t}\left(A_{\lambda(n),\tau-1} + 2\pi\widetilde{c}_{\lambda(n),\tau}\sum_{s=0}^{\tau} 1\right) \\
&\leq \frac{1}{2t}\left(A_{\lambda(n),\tau-1} + 2\pi K(\mathbf{c})\sum_{s=0}^{\tau}\widetilde{c}_{\lambda(n),s}\right) = \mathcal{O}\left(\frac{A_{\lambda(n),\tau}}{t}\right).
\end{aligned}$$

□

3 Main Results

Theorem 3.1. *Assume $A := (a_{m,i})$ and $B := (b_{n,j})$ are two lower triangular matrices, where $\{a_{m,i}\} \in RBVS$ and $\{b_{n,j}\} \in RBVS$ with respect to $i \leq m$ and $j \leq n$, satisfying the conditions (1.1) and (1.2) respectively, and the double matrix method (A, B) is regular. If $f(x,y)$ is a 2π-periodic function in x and y, Lebesgue integrable in $[-\pi,\pi]^2$, and belonging to the class $H_{(\alpha,\beta)}$ for $0 < \alpha, \beta \leq 1$, then*

$$\|t_{m,n}^\lambda - f\|_{(\alpha,\beta)} = \mathcal{O}(1)\begin{cases} \frac{1}{(\lambda(m)+1)^\alpha} + \frac{1}{(\lambda(n)+1)^\beta}, & \text{if} \quad 0<\alpha<1,\ 0<\beta<1, \\ \frac{\log(\pi(\lambda(m)+1))}{\lambda(m)+1} + \frac{\log(\pi(\lambda(n)+1))}{\lambda(n)+1}, & \text{if} \quad \alpha=\beta=1, \end{cases} \tag{3.1}$$

for all $m, n = 0, 1, 2, \ldots$.

Proof. Using the well-known equality

$$\int_{-\pi}^{\pi} \frac{\sin\left(s+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}dt = 1, \ \ (s = 1, 2, \dots),$$

and some elementary calculations, we get

$$s_{i,j}(f; x, y) - f(x, y) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \Psi(u, v)\frac{\sin\left(i+\frac{1}{2}\right)u \cdot \sin\left(j+\frac{1}{2}\right)v}{\sin\frac{t}{2}\cdot\sin\frac{t}{2}}dudv. \tag{3.2}$$

Based on (3.2) and the double matrix means $t^{\lambda}_{m,n}(f; x, y)$ of $s_{i,j}(f; x, y)$, we can write

$$\begin{aligned} t^{\lambda}_{m,n}(f; x, y) - f(x, y) &= \sum_{i=0}^{\lambda(m)}\sum_{j=0}^{\lambda(n)} a_{\lambda(m),i}b_{\lambda(n),j}\left[s_{i,j}(f; x, y) - f(x, y)\right] \\ &= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\Psi(u, v)\sum_{i=0}^{\lambda(m)}\sum_{j=0}^{\lambda(n)} a_{\lambda(m),i}b_{\lambda(n),j}\frac{\sin\left(i+\frac{1}{2}\right)u \cdot \sin\left(j+\frac{1}{2}\right)v}{\sin\frac{t}{2}\cdot\sin\frac{t}{2}}dudv \\ &= \int_0^{\pi}\int_0^{\pi}\Psi(u, v)K^{\lambda}_m(u)K^{\lambda}_n(v)dudv. \end{aligned}$$

Firstly, we estimate the quantity $|t^{\lambda}_{m,n}(f; x, y) - f(x, y) - [t^{\lambda}_{m,n}(f; z, w) - f(z, w)]|$. Indeed,

$$\begin{aligned} &|t^{\lambda}_{m,n}(f; x,y) - f(x, y) - [t^{\lambda}_{m,n}(f; z, w) - f(z, w)]| \\ &\quad = \left|\int_0^{\pi}\int_0^{\pi} F(u, v)K^{\lambda}_m(u)K^{\lambda}_n(v)dudv\right| \\ &\quad \leq \Bigg(\int_0^{\frac{1}{\lambda(m)+1}}\int_0^{\frac{1}{\lambda(n)+1}} + \int_0^{\frac{1}{\lambda(m)+1}}\int_{\frac{1}{\lambda(n)+1}}^{\pi} \\ &\qquad + \int_{\frac{1}{\lambda(m)+1}}^{\pi}\int_0^{\frac{1}{\lambda(n)+1}} + \int_{\frac{1}{\lambda(m)+1}}^{\pi}\int_{\frac{1}{\lambda(n)+1}}^{\pi}\Bigg)|F(u, v)||K^{\lambda}_m(u)||K^{\lambda}_n(v)|dudv \\ &\quad := I^{(1)}_{m,n} + I^{(2)}_{m,n} + I^{(3)}_{m,n} + I^{(4)}_{m,n}. \end{aligned} \tag{3.3}$$

Using (1.5) and Lemma 2.1 twice, we have

$$\begin{aligned} I^{(1)}_{m,n} &= \int_0^{\frac{1}{\lambda(m)+1}}\int_0^{\frac{1}{\lambda(n)+1}} |F(u, v)||K^{\lambda}_m(u)||K^{\lambda}_n(v)|dudv \\ &= \mathcal{O}\left[(\lambda(m)+1)(\lambda(n)+1)\right]\int_0^{\frac{1}{\lambda(m)+1}}\int_0^{\frac{1}{\lambda(n)+1}} |F(u, v)|dudv \\ &= \mathcal{O}\left(|x - z|^{\alpha} + |y - w|^{\beta}\right). \end{aligned} \tag{3.4}$$

In order to estimate $I^{(2)}_{m,n}$ we use Lemmas 2.1–2.2 and (1.5), which imply

$$\begin{aligned} I^{(2)}_{m,n} &= \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^{\pi} |F(u,v)||K^{\lambda}_m(u)||K^{\lambda}_n(v)|dudv \\ &= \mathcal{O}\left[\lambda(m)+1\right] \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^{\pi} |F(u,v)| \frac{B_{\lambda(n),[1/v]}}{v} dudv \\ &= \mathcal{O}\left(|x-z|^{\alpha}+|y-w|^{\beta}\right) \int_{\frac{1}{\lambda(n)+1}}^{\pi} \frac{B_{\lambda(n),[1/v]}}{v} dv \\ &= \mathcal{O}\left(|x-z|^{\alpha}+|y-w|^{\beta}\right) \int_{\frac{1}{\pi}}^{\lambda(n)+1} \frac{B_{\lambda(n),t}}{t} dt \\ &= \mathcal{O}\left(|x-z|^{\alpha}+|y-w|^{\beta}\right) \frac{B_{\lambda(n),\lambda(n)+1}}{\lambda(n)+1} \int_{\frac{1}{\pi}}^{\lambda(n)+1} dt \\ &= \mathcal{O}\left(|x-z|^{\alpha}+|y-w|^{\beta}\right), \end{aligned} \tag{3.5}$$

since $\frac{A_{\lambda(n),t}}{t}$ is a monotone increasing function with respect to $t \in \left[\frac{1}{\pi}, \lambda(n)+1\right]$.

Using Lemmas 2.1–2.2 again and (1.5), we have proved that

$$I^{(3)}_{m,n} = \mathcal{O}\left(|x-z|^{\alpha}+|y-w|^{\beta}\right), \tag{3.6}$$

as well as

$$I^{(4)}_{m,n} = \mathcal{O}\left(|x-z|^{\alpha}+|y-w|^{\beta}\right). \tag{3.7}$$

Inserting (3.4)-(3.7) into (3.3) we obtain

$$\sup_{x\neq z, y\neq w} \frac{|t^{\lambda}_{m,n}(f;x,y)-f(x,y)-[t^{\lambda}_{m,n}(f;z,w)-f(z,w)]|}{|x-z|^{\alpha}+|y-w|^{\beta}} = \mathcal{O}(1). \tag{3.8}$$

Now, it is clear that we can write

$$\begin{aligned} &|t^{\lambda}_{m,n}(f;x,y)-f(x,y)| \\ &= \left| \int_0^{\pi}\int_0^{\pi} \Psi(u,v)K^{\lambda}_m(u)K^{\lambda}_n(v)dudv \right| \\ &\leq \left(\int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} + \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^{\pi} \right. \\ &\quad \left. + \int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_0^{\frac{1}{\lambda(n)+1}} + \int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_{\frac{1}{\lambda(n)+1}}^{\pi} \right) |\Psi(u,v)||K^{\lambda}_m(u)||K^{\lambda}_n(v)|dudv \\ &:= J^{(1)}_{m,n} + J^{(2)}_{m,n} + J^{(3)}_{m,n} + J^{(4)}_{m,n}. \end{aligned} \tag{3.9}$$

It is verified in [4] that if $f \in H(\alpha,\beta)$ then $\Psi \in H(\alpha,\beta)$ as well i.e. $|\Psi(u,v)| = \mathcal{O}\left(|u|^{\alpha}+|v|^{\beta}\right)$. Whence, using Lemma 2.1 we have

$$\begin{aligned} J^{(1)}_{m,n} &= \int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} |\Psi(u,v)||K^{\lambda}_m(u)||K^{\lambda}_n(v)|dudv \\ &= \mathcal{O}\left[(\lambda(m)+1)(\lambda(n)+1)\right] \int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} (u^{\alpha}+v^{\beta})dudv \\ &= \mathcal{O}\left((\lambda(m)+1)^{-\alpha}+(\lambda(n)+1)^{-\beta}\right). \end{aligned} \tag{3.10}$$

Next, Lemmas 2.1–2.2 and $|\Psi(u,v)|=\mathcal{O}\left(|u|^{\alpha}+|v|^{\beta}\right)$ imply

$$\begin{aligned}
J^{(2)}_{m,n}=&\int_0^{\frac{1}{\lambda(m)+1}}\int_{\frac{1}{\lambda(n)+1}}^{\pi}|\Psi(u,v)||K_m^{\lambda}(u)||K_n^{\lambda}(v)|dudv\\
=&\mathcal{O}\left[\lambda(m)+1\right]\int_0^{\frac{1}{\lambda(m)+1}}\int_{\frac{1}{\lambda(n)+1}}^{\pi}(u^{\alpha}+v^{\beta})\frac{B_{\lambda(n),[1/v]}}{v}dudv\\
=&\mathcal{O}\left[\lambda(m)+1\right]\left[\int_0^{\frac{1}{\lambda(n)+1}}u^{\alpha}du\int_{\frac{1}{\lambda(n)+1}}^{\pi}\frac{B_{\lambda(n),[1/v]}}{v}dv\right.\\
&\left.+\int_0^{\frac{1}{\lambda(n)+1}}du\int_{\frac{1}{\lambda(n)+1}}^{\pi}v^{\beta-1}B_{\lambda(n),[1/v]}dv\right]\\
=&\mathcal{O}\left[\lambda(m)+1\right]\left[[\lambda(m)+1]^{-\alpha-1}\int_{\frac{1}{\pi}}^{\lambda(n)+1}\frac{B_{\lambda(n),t}}{t}dt+\frac{1}{\lambda(m)+1}\int_{\frac{1}{\pi}}^{\lambda(n)+1}\frac{B_{\lambda(n),t}}{t^{\beta+1}}dt\right]\\
=&\mathcal{O}\left[\lambda(m)+1\right]\left[[\lambda(m)+1]^{-\alpha-1}+\frac{1}{\lambda(m)+1}\cdot\frac{B_{\lambda(n),\lambda(n)+1}}{\lambda(n)+1}\int_{\frac{1}{\pi}}^{\lambda(n)+1}\frac{dt}{t^{\beta}}\right]\\
=&\mathcal{O}\begin{cases}\frac{1}{(\lambda(m)+1)^{\alpha}}+\frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if}\quad 0<\alpha\le 1,\ 0<\beta<1,\\ \frac{1}{(\lambda(m)+1)^{\alpha}}+\frac{\log(\pi(\lambda(n)+1))}{\lambda(n)+1}, & \text{if}\quad 0<\alpha\le 1,\ \beta=1,\end{cases}
\end{aligned}$$

since $\frac{B_{\lambda(n),v}}{v}$ is a monotone increasing function with respect to $t\in\left[\frac{1}{\pi},\lambda(n)+1\right]$.

Again, using Lemmas 2.1–2.2, $|\Psi(u,v)|=\mathcal{O}\left(|u|^{\alpha}+|v|^{\beta}\right)$ and monotonicity of $\frac{A_{\lambda(m),u}}{u}$ we get

$$\begin{aligned}
J^{(3)}_{m,n}=&\int_{\frac{1}{\lambda(m)+1}}^{\pi}\int_0^{\frac{1}{\lambda(n)+1}}|\Psi(u,v)||K_m^{\lambda}(u)||K_n^{\lambda}(v)|dudv\\
=&\mathcal{O}\left[\lambda(n)+1\right]\int_{\frac{1}{\lambda(m)+1}}^{\pi}\int_0^{\frac{1}{\lambda(n)+1}}(u^{\alpha}+v^{\beta})\frac{A_{\lambda(m),[1/u]}}{u}dudv\\
=&\mathcal{O}\left[\lambda(n)+1\right]\left[\frac{1}{\lambda(n)+1}\int_{\frac{1}{\lambda(m)+1}}^{\pi}u^{\alpha-1}A_{\lambda(m),[1/u]}du\right.\\
&\left.+\int_{\frac{1}{\lambda(m)+1}}^{\pi}\frac{A_{\lambda(m),[1/u]}}{u}du\int_0^{\frac{1}{\lambda(n)+1}}v^{\beta}dv\right]\\
=&\mathcal{O}\left[\lambda(n)+1\right]\left[\frac{1}{\lambda(n)+1}\int_{\frac{1}{\pi}}^{\lambda(m)+1}\frac{A_{\lambda(m),t}}{t^{\alpha+1}}dt\right.\\
&\left.+[\lambda(n)+1]^{-\beta-1}\int_{\frac{1}{\pi}}^{\lambda(m)+1}\frac{A_{\lambda(m),t}}{t}dt\right]\\
=&\mathcal{O}\left[\lambda(n)+1\right]\left[\frac{1}{\lambda(n)+1}\cdot\frac{A_{\lambda(m),\lambda(m)+1}}{\lambda(m)+1}\int_{\frac{1}{\pi}}^{\lambda(m)+1}\frac{dt}{t^{\alpha}}+[\lambda(n)+1]^{-\beta-1}\right]\\
=&\mathcal{O}\begin{cases}\frac{1}{(\lambda(m)+1)^{\alpha}}+\frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if}\quad 0<\alpha<1,\ 0<\beta\le 1,\\ \frac{\log(\pi(\lambda(m)+1))}{\lambda(m)+1}+\frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if}\quad \alpha=1,\ 0<\beta\le 1.\end{cases}
\end{aligned}\tag{3.12}$$

Now, using only Lemma 2.2, $|\Psi(u,v)|=\mathcal{O}\left(|u|^{\alpha}+|v|^{\beta}\right)$ and monotonicity of $\frac{A_{\lambda(m),u}}{u}$ and

$\frac{B_{\lambda(n),v}}{v}$ we obtain

$$
\begin{aligned}
J^{(4)}_{m,n} =& \int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_{\frac{1}{\lambda(n)+1}}^{\pi} |\Psi(u,v)||K^{\lambda}_m(u)||K^{\lambda}_n(v)|dudv \\
=&\mathcal{O}\left[\int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_{\frac{1}{\lambda(n)+1}}^{\pi} (u^{\alpha}+v^{\beta})\frac{A_{\lambda(m),[1/u]}B_{\lambda(n),[1/v]}}{uv}dudv\right] \\
=&\mathcal{O}\Bigg[\int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_{\frac{1}{\lambda(n)+1}}^{\pi} u^{\alpha-1}A_{\lambda(m),[1/u]}\frac{B_{\lambda(n),[1/v]}}{v}dudv \\
&+\int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_{\frac{1}{\lambda(n)+1}}^{\pi} v^{\beta-1}B_{\lambda(n),[1/v]}\frac{A_{\lambda(m),[1/u]}}{u}dudv\Bigg] \qquad (3.13)\\
=&\mathcal{O}\Bigg[\int_{\frac{1}{\pi}}^{\lambda(m)+1}\frac{A_{\lambda(m),t}}{t^{\alpha+1}}dt\int_{\frac{1}{\pi}}^{\lambda(n)+1}\frac{B_{\lambda(n),s}}{s}ds \\
&+\int_{\frac{1}{\pi}}^{\lambda(m)+1}\frac{A_{\lambda(m),t}}{t}dt\int_{\frac{1}{\pi}}^{\lambda(n)+1}\frac{B_{\lambda(n),s}}{s^{\beta+1}}ds\Bigg] \\
=&\mathcal{O}\begin{cases}\frac{1}{(\lambda(m)+1)^{\alpha}}+\frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if} \quad 0<\alpha<1,\ 0<\beta<1,\\ \frac{\log(\pi(\lambda(m)+1))}{\lambda(m)+1}+\frac{\log(\pi(\lambda(n)+1))}{\lambda(n)+1}, & \text{if} \quad \alpha=\beta=1.\end{cases}
\end{aligned}
$$

Inserting (3.10)-(3.13) into (3.9) we obtain

$$
|t^{\lambda}_{m,n}(f;x,y)-f(x,y)| = \mathcal{O}\begin{cases}\frac{1}{(\lambda(m)+1)^{\alpha}}+\frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if} \quad 0<\alpha<1,\ 0<\beta<1,\\ \frac{\log(\pi(\lambda(m)+1))}{\lambda(m)+1}+\frac{\log(\pi(\lambda(n)+1))}{\lambda(n)+1}, & \text{if} \quad \alpha=\beta=1.\end{cases}
$$

Subsequently,

$$
\|t^{\lambda}_{m,n}(f)-f\|_C = \mathcal{O}\begin{cases}\frac{1}{(\lambda(m)+1)^{\alpha}}+\frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if} \quad 0<\alpha<1,\ 0<\beta<1,\\ \frac{\log(\pi(\lambda(m)+1))}{\lambda(m)+1}+\frac{\log(\pi(\lambda(n)+1))}{\lambda(n)+1}, & \text{if} \quad \alpha=\beta=1.\end{cases}
$$

With this the proof is completed. □

Corollary 3.2. *If* $a_{m,i} = \frac{1}{\lambda(m)+1}$, $\forall i$ *and* $b_{n,j} = \frac{1}{\lambda(n)+1}$, $\forall j$, *then the degree of approximation of* $f \in H_{(\alpha,\beta)}$ *for* $0<\alpha,\beta\leq 1$, *is given by*

$$
\|t^{\lambda}_{m,n}-f\|_{(\alpha,\beta)} = \mathcal{O}(1)\begin{cases}\frac{1}{(\lambda(m)+1)^{\alpha}}+\frac{1}{(\lambda(n)+1)^{\beta}}, & \textit{if} \quad 0<\alpha<1,\ 0<\beta<1,\\ \frac{\log(\pi(\lambda(m)+1))}{\lambda(m)+1}+\frac{\log(\pi(\lambda(n)+1))}{\lambda(n)+1}, & \textit{if} \quad \alpha=\beta=1,\end{cases} \tag{3.14}
$$

for all $m,n=0,1,2,\ldots$.

Remark 3.3. Note that our results are sharper than those obtained in [2], since $\lambda(m)\geq m$, $\lambda(n)\geq n$ and for $0<\alpha<1,\ 0<\beta<1$ hold $\frac{1}{(\lambda(m)+1)^{\alpha}}\leq\frac{1}{(m+1)^{\alpha}}$ and $\frac{1}{(\lambda(n)+1)^{\beta}}\leq\frac{1}{(n+1)^{\beta}}$.

Remark 3.4. In particilar case, if we take $\lambda(m)=m$ and $\lambda(n)=n$, we exactly obtain the results proved in [2].

Corollary 3.5. *If* $a_{m,i} = \frac{p_{\lambda(m)-i}}{P_{\lambda(m)}}$, $\forall i$ *and* $b_{n,j} = \frac{q_{\lambda(n)-j}}{Q_{\lambda(n)}}$, $\forall j$, *then the degree of approximation of* $f \in H_{(\alpha,\beta)}$ *for* $0<\alpha,\beta\leq 1$, *is given by*

$$
\|N^{\lambda}_{m,n}-f\|_{(\alpha,\beta)} = \mathcal{O}(1)\begin{cases}\frac{1}{(\lambda(m)+1)^{\alpha}}+\frac{1}{(\lambda(n)+1)^{\beta}}, & \textit{if} \quad 0<\alpha<1,\ 0<\beta<1,\\ \frac{\log(\pi(\lambda(m)+1))}{\lambda(m)+1}+\frac{\log(\pi(\lambda(n)+1))}{\lambda(n)+1}, & \textit{if} \quad \alpha=\beta=1,\end{cases} \tag{3.15}
$$

for all $m,n=0,1,2,\ldots$, *where*

$$
N^{\lambda}_{m,n}(f;x,y) := \frac{1}{P_{\lambda(m)}Q_{\lambda(n)}}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{\lambda(m)-i}q_{\lambda(n)-j}s_{m,n}(x,y).
$$

Remark 3.6. If we take $\lambda(m) = m$ and $\lambda(n) = n$ in Corollary 3.5 we obtain Corollary 2.3 of [2]. Even in this case our results are sharper as we discussed above.

Remark 3.7. The results obtained here can be extended to multiple Fourier series of three or more dimensions.

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Author information

Xhevat Z. Krasniqi, University of Prishtina "Hasan Prishtina", Faculty of Education, Department of Mathematics and Informatics, Avenue "Mother Theresa " no. 5, 10000 Prishtina, Kosovo.
E-mail: xhevat.krasniqi@uni-pr.edu

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