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Almost Conformal Ricci solitons in (k,μ)-Paracontact metric manifolds

Mohd Danish Siddiqi

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Abstract: (k,μ)-Paracontact mertric manifolds admitting the almost conformal Ricci Solitons and gradient shrinking Ricci soliton have been studied. We prove the non-existence of almost conformal Ricci soliton in a (k,μ)-paracontact metric manifold M has been established under certain condition.

1 Introduction

In recent years the pioneering works of R. S. Hamilton [8] towards the solution of the Poincare conjecture in dimension 3 have produced a flourishing activity in the research of self similar solutions, or solitons, of the Ricci flow. The study of the geometry of solitons, in particular their classification in dimension 3, has been essential in providing a positive answer to the conjecture; however in higher dimension and in the complete, possibly noncompact case, the understanding of the geometry and the classification of solitons seems to remain a desired goal for a not too proximate future. In the generic case a soliton structure on the Riemannian manifold (M,g) is the choice of a smooth vector field X on M and a real constant λ satisfying the structural requirement

$$Ric+\frac{1}{2}\mathcal{L}_X g=\lambda g, \tag{1.1}$$

where Ric is the Ricci tensor of the metric g and $\mathcal{L}_X g$ is the Lie derivative of this latter in the direction of X. In what follows we shall refer to λ as to the soliton constant. The soliton is called expanding, steady or shrinking if, respectively, $\lambda>0$, $\lambda=0$ or $\lambda>0$. When X is the gradient of a potential $\psi\in C^\infty(M)$, the soliton is called a gradient Ricci soliton [5] and the previous equation (1.1) takes the form

$$\nabla\nabla\psi=S+\lambda g. \tag{1.2}$$

Both equations (1.1) and (1.2) can be considered as perturbations of the Einstein equation

$$Ric=\lambda g. \tag{1.3}$$

and reduce to this latter in case X or $\nabla\psi$ are Killing vector fields. When $X=0$ or ψ is constant we call the underlying Einstein manifold a trivial Ricci soliton.

Definition 1.1. A Ricci soliton (g,V,λ) on a Riemannian manifold is defined by

$$\mathcal{L}_V g+2S+2\lambda=0, \tag{1.4}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda<0,\lambda=0$ and $\lambda>0$, respectively.

Ricci solitons, in the context of general relativity, have been studied by M. Ali and Z. Ahsan (see [17], [18], [19]). A. E. Fischer [7] introduced a new concept called conformal Ricci flow which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. These new equations are given by

$$\frac{\partial t}{\partial t} = -2S - \left(p + \frac{2}{n}\right) g, \tag{1.5}$$

where $R(g) = -1$ and p is a non-dynamical scalar field(time dependent scalar field), $R(g)$ is the scalar curvature of the manifold and n is the dimension of the manifold M.

The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics and because of this analogy the time dependent scalar field p is called a conformal pressure and, as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint. The equilibrium points of the conformal Ricci flow equations are Einstein metrics with Einstein constant $\frac{-1}{n}$. Thus the conformal pressure p is zero at an equilibrium point and positive otherwise.

In 2015, N. Basu and A. Bhattacharyya [1] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\mathcal{L}_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g = 0, \tag{1.6}$$

where λ is a constant. Recently, in 2018 M. D. Siddiqi [20] study conformal η-Ricci solitons in δ-Lorentzian trans-Sasakian manifolds which also colely realted to this paper.

The concept of almost Ricci soliton was first introduced by S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti in 2010 [12]. R. Sharma has also done excellent work in almost Ricci soliton [14]. A Riemannian manifold (M^n, g) is an almost Ricci soliton [13] if there exists a complete vector field X and a smooth soliton function λ such that $\lambda : M^n \longrightarrow \mathbb{R}$ satisfying

$$R_{ij} + \frac{1}{2}(X_{i;j} + X_{j;i}) = \lambda g_{ij}, \tag{1.7}$$

where R_{ij} and $X_{i;j} + X_{j;i}$ stands for the Ricci tensor and the Lie derivative $\mathcal{L}_X g$ in local coordinates respectively. It will called shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

Now a gradient Ricci soliton on a Riemannian manifold (M^n, g_{ij}) is defined by [6]

$$S + Hess\psi = \rho g, \tag{1.8}$$

where $Hess\psi = \nabla\nabla\psi$ and for some constant ρ and for a smooth function ψ on M. ψ is called a potential function of the Ricci soliton and ∇ is the Levi-Civita connection on M. In particular a gradient shrinking Ricci soliton [2] satisfies the equation,

$$S + Hess\psi - \frac{1}{2\tau} g = 0, \tag{1.9}$$

where $\tau = T - t$ and T is the maximal time of the solution. Again for conformal Ricci soliton if the vector field is the gradient of a function f, then we call it as a conformal gradient shrinking Ricci soliton [5]. For conformal gradient shrinking Ricci soliton the equation is

$$S + Hess\psi - \left(\frac{1}{2\tau} - \frac{2}{n} - p\right) g \tag{1.10}$$

where $\tau = T - t$ and T is the maximal time of the solution and ψ is the Ricci potential function.

On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has manifold connections with the other fields of pure mathematics, and substantial applications in applied areas such as mechanics, optics, phase space of dynamical system, thermodynamics and control theory.

R. Sharma [13] initiated the study Ricci almost solitons in K-contact geometry. Recently, Calvaruso and Perrone [4] studied Ricci solitons in three-dimensional paracontact geometry. R. Sharam [14] also, studied some propetries of K-contact and (k, μ)-contact geometry. Therefore, in the present paper we studied the almost Conformal Ricci solitons in (k, μ)-paracontact metric manifolds.

2 Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be an almost paracontact manifold if it admits an almost paracontact structure (ϕ, ξ, η), where ϕ is a $(1,1)$- tensor field, ξ a vector field and its dual 1-form η and for any vector field X on M satisfying [9]

$$\phi^2 X = X - \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi) = 0, \tag{2.2}$$

the tensor field ϕ induces an almost paracomplex structure on each fibre of $D = ker(\eta)$, that is, the eigen distributions D^+_ϕ and D^-_ϕ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have same dimension n.

An almost paracontact structure is said to be normal [15] if and only if the $(1,2)$-type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ. If an almost paracontact manifold M equipped with a pseudo-Riemannian metric g of signature $(n+1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all smooth vector fields on the manifold M, then (M, g) is called an almost paracontact metric manifold. An almost paracontact structure is said to be a paracontact structure if

$$g(X, \phi Y) = d\eta(X, Y) \tag{2.4}$$

where g is the associated metric [15]. For any almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits (at least, locally) a ϕ-basis [15], that is, a pseudo orthonormal basis of vector fields of the form $\{\xi, E_1, E_2, ..., E_n, \phi E_1, \phi E_2, ..., \phi E_n\}$, where $\xi, E_1, E_2, ..., E_n$ are space-like vector fields and then, by (2.3) vector fields $\phi E_1, \phi E_2, ..., \phi E_n$ are time-like. In a paracontact metric manifold there exists a symmetric, trace-free $(1,1)$-tensor $h = \frac{1}{2}\mathcal{L}_\xi \phi$ satisfying [15]

$$\phi h + h\phi = 0, \qquad h\xi = 0, \tag{2.5}$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{2.6}$$

where ∇ is Levi-Civita connection of the pseudo-Riemannian manifold and for all $X \in \chi(M)$. It is clear that the tensor h satisfies $h = 0$ if and only if ξ is a Killing vector field and then (ϕ, ξ, η, g) is said to be a K-paracontact manifold. An almost paracontact manifold is said to be para-Sasakian if and only if the following condition holds [15]

$$(\nabla_X \phi)Y = -g(X, Y) + \eta(Y)X \tag{2.7}$$

for any $X, Y \in \chi(M)$. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(X, Y)\xi = -[\eta(Y)X - \eta(X)Y] \tag{2.8}$$

for any $X, Y \in \chi(M)$, but unlike contact metric geometry the relation (2.8) does not imply that the paracontact manifold is para-Sasakian manifold. Every para Sasakian manifold is a K-paracontact manifold, but the converse is not always true, as it is shown in three dimensional case. Paracontact metric manifolds have been studied by Cappelletti-Montano et al ([3], [4]), Martin-Molina ([10], [11]) and many others.

According to Cappelletti-Montano et al [3] we have the following definition.

Definition 2.1. A paracontact metric manifold is said to be (k, μ)-paracontact manifold if the curvature tensor R satisfies

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.9}$$

for all vector fields $X, Y \in \chi(M)$ and k, μ are real constants.

In a (k, μ)-paracontact manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, $n > 1$, the following relations hold [6]:

$$h^2 = (k+1)\phi^2 \tag{2.10}$$

$$(\nabla_X \phi)Y = -g[X - hX, Y]\xi + \eta(Y)[X - hX], \qquad for \quad k \neq -1, \tag{2.11}$$

$$(\nabla_X h)Y = -[(1+k)g(X, \phi Y) + g(X, \phi hY)]\xi \tag{2.12}$$

$$+\eta(Y)\phi h(hX - X) - \mu\eta(X)\phi hY, \qquad for \quad k \neq -1,$$

$$QX = [2(n-1) + \mu]X + [2(n-1) + \mu]hY \tag{2.13}$$

$$+[2(n-1) + n(2k - \mu)]\eta(X)\xi, \quad for \quad k \neq 1,,$$

$$S(X, \xi) = 2nk\eta(X) \tag{2.14}$$

$$Q\xi = 2nk\xi \tag{2.15}$$

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi \tag{2.16}$$

for any vector fields $X, Y \in \chi(M)$, where Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$. Making use of (2.6) we have

$$(\nabla_X \eta)Y = g(X, \phi Y) + g(\phi hX, Y) \tag{2.17}$$

for all vector fields $X, Y \in \chi(M)$.
For the main results first of all we need the following Lemmas:

Lemma 2.2. *([16], Theorem 3.3) Let M^{2n+1}, $n > 1$, be a paracontact metric manifold satisfies $R(X, Y)\xi = 0$, for all $X, Y \in \chi(M)$. Then M^{2n+1} is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.*

Lemma 2.3. *([3]) Let (M, ϕ, ξ, η, g) be a (k, μ)-paracontact metric manifold. Then for any vector fields $X, Y \in \chi(M)$ we have*

$$(\nabla_X \phi h)Y = g(h^2X - hX, Y)\xi + \eta(Y)[h^2X - hX] \tag{2.18}$$

$$-\mu\eta(X)hY, \quad for \quad k > -1$$

$$(\nabla_X \phi h)Y = (1+k)g(X, Y)\xi - g(hX, Y)\xi + \eta(Y)(h^2X - hX) \tag{2.19}$$

$$-\mu\eta(X)hY, \quad for \quad k < -1.$$

3 Almost conformal Ricci Solitons in (k, μ)-paracontact Metric Manifolds

In this section we discuss almost conformal Ricci solitons in (k, μ)-paracontact manifolds. We prove the following:

Theorem 3.1. *There does not exist almost conformal Ricci soliton in a (k, μ)-paracontact metric manifold M^{2n+1} $(n > 1)$ whose potential vector field is the Reeb vector field ξ with $k < 1$ or $k > 1$.*

Proof. Let a (k, μ)-paracontact metric manifold admits a Ricci almost soliton. Then we have from (1.6)

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) - \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X, Y) = 0. \tag{3.1}$$

Which is equivalent to

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + 2S(X, Y) \tag{3.2}$$

$$-\left[2\lambda-\left(p+\frac{2}{n}\right)\right]g(X,Y)=0$$

Using (2.6) in (3.2) we get

$$g(-\phi X+\phi hX,Y)+g(-\phi Y+\phi hY,X)+2S(X,Y) \tag{3.3}$$

$$-\left[2\lambda-\left(p+\frac{2}{n}\right)\right]g(X,Y)=0$$

This implies

$$g(\phi hX,Y)+S(X,Y)-\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]g(X,Y)=0 \tag{3.4}$$

that is

$$\phi hY+QY-\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]Y=0. \tag{3.5}$$

Now, taking covariantly derivative of (3.5) with respect to X.

$$(\nabla_X\phi hY)+\nabla_X QY-\frac{1}{2}X\left(2\lambda-\left(p+\frac{2}{n}\right)\right)Y=0. \tag{3.6}$$

Now we break our discussion in two cases:

Case(i): Let $k>1$. Applying (2.13) and (2.18) in (3.6) we have

$$(1+k)g(X,Y)\xi-g(hX,Y)\xi+(k+1)\eta(Y)X-(k+1)\eta(X)\eta(Y)\xi \tag{3.7}$$

$$-\eta(Y)hX-\mu\eta(X)hY+[2(n-1)+\mu](\nabla_X h)Y$$

$$+[2(n-1)+n(2k-\mu)][(\nabla_X\eta)(Y)\xi+\eta(Y)\nabla_X\xi-\frac{1}{2}X\left(2\lambda-\left(p+\frac{2}{n}\right)\right)Y=0.$$

Using (2.6), (2.12) and (2.17) we have

$$(1+k)g(X,Y)-g(hX,Y)\xi+(k+1)\eta(Y)X-(k+1)\eta(X)\eta(Y)\xi-\eta(Y)hX-\mu\eta(X)hY$$

$$-[2(n-1)+\mu][(k+1)g(X,\phi Y)\xi+g(X,\phi hY)\xi-(k+1)\eta(Y)\phi X+\eta(Y)\phi hX+\mu\eta(X)\phi hY]$$

$$+2(n-1)+n(2k-\mu)][g(X,\phi Y)\xi+g(\phi hX,Y)\xi-\eta(Y)\phi X+\eta(Y)\phi X+\eta(Y)]$$

$$-\frac{1}{2}X\left(2\lambda-\left(p+\frac{2}{n}\right)\right)Y=0. \tag{3.8}$$

Contracting X in (3.8) $X=\xi=e_i$ we get

$$(2n+1)(k+1)\eta(Y)=\frac{1}{2}\left(2\lambda-\left(p+\frac{2}{n}\right)\right)Y \tag{3.9}$$

Also putting $Y=Z=\xi$ in (3.4) yields $\lambda=2nk+\left(\frac{p}{2}+\frac{1}{n}\right)$, which is a constant. Applying this in (3.9) we have $k=-1$, which is a contradiction as we consider $k>-1$.
case(ii) Let $k<-1$. Making use of (2.13) and (2.19) in (3.5) we have

$$g((k+1)\phi^2X,Y)\xi-g(hX,Y)\xi+(k+1)\eta(Y)\phi^2X \tag{3.10}$$

$$-\eta(Y)hX-\mu\eta(X)hY+[2(n-1)+\mu](\nabla_X h)Y$$

$$+[2(n-1)+n(2k-\mu)][(\nabla_X\eta)(Y)\xi+\eta(Y)\nabla_X\xi]$$

$$-\frac{1}{2}X\left(2\lambda-\left(p+\frac{2}{n}\right)\right)Y=0.$$

Now, using equations (2.6), (2.12) and (2.17) in the above equation gives

$$g((k+1)\phi^2X,Y)\xi-g(hX,Y)\xi+(k+1)\eta(Y)\phi^2X \tag{3.11}$$

$$-\eta(Y)hX-\mu\eta(X)hY+[2(n-1)+\mu][(k+1)g(X,\phi Y)\xi+g(X,\phi hY)\xi$$

$$-(k+1)\eta(Y)\phi X+\eta(Y)\phi hX+\mu\eta(X)\phi hY]$$

$$+[2(n-1)+n(2k-\mu)][g(X,\phi Y)\xi+g(\phi hX,Y)\xi-\eta(Y)\phi X+\eta(Y)\phi hX]$$

$$-\frac{1}{2}X\left(2\lambda-\left(p+\frac{2}{n}\right)\right)Y=0.$$

Contracting X in (3.11) we get

$$2n(k+1)\eta(Y)-\frac{1}{2}\left(2\lambda-\left(p+\frac{2}{n}\right)\right)Y=0. \tag{3.12}$$

Again, putting $Y=Z=\xi$ in (3.4) yields $\lambda=2nk+\left(\frac{p}{2}+\frac{1}{n}\right)$, which is a constant. Applying this in (3.12) we have $k=-1$, which is a contradiction as we consider $k<-1$.
Combining the two cases our theorem follows. □

Theorem 3.2. *If a (k,μ)-paracontact metric manifold M^{2n+1} $(n>1)$ admits a almost conformal Ricci soliton for $k=-1$ whose potential vector field is the Reeb vector field ξ, then the almost conformal Ricci soliton is expanding with $Q\xi=-2n\xi$.*

Proof. Replacing Y by ξ in (3.6) we get $Q\xi=\frac{1}{2}\left(2\lambda-\left(p+\frac{2}{n}\right)\right)\xi$. On the other hand from (2.15) and $k=-1$ we have $Q\xi=-2n\xi$. Thus we obtain $\lambda=-2n+\left(\frac{p}{2}+\frac{1}{n}\right)$. This shows that the almost conformal Ricci soliton is expanding. □

Corollary 3.3. *The almost conformal Ricci soliton in a (k,μ)-paracontact metric manifold reduces to a Ricci soliton if $\lambda=constant$.*

4 Almost conformal gradient shrinking Ricci soliton on (k,μ)-paracontact metric manifold

A conformal gradient shrinking Ricci soliton equation is given by

$$S+\nabla\nabla f=\left(\frac{1}{2\tau}-\frac{2}{n}+p\right)g. \tag{4.1}$$

This reduces to

$$\nabla_Y Df+QY=\left(\frac{1}{2\tau}-\frac{2}{n}+p\right)Y \tag{4.2}$$

for ant $X\in\chi(M)$, where D is the gradient operator of g. From (4.2) it follows

$$\nabla_X\nabla_Y Df+\nabla_X QY=(\frac{1}{2\tau}-\frac{2}{3}-p)\nabla_X Y.$$

Now,

$$\begin{aligned} R(X,Y)Df &= \nabla_X\nabla_Y Df-\nabla_Y\nabla_X Df-\nabla_{[X,Y]}Df \\ &= (\frac{1}{2\tau}-\frac{2}{3}-p)[\nabla_X Y-\nabla_Y X-[X,Y]]-\nabla_X(QY)+\nabla_Y(QX)+Q[X,Y], \end{aligned}$$

where R is the curvature tensor.
Since ∇ is Levi-Civita connection, so from the above equation we get

$$R(X,Y)Df=-\nabla_X(QY)+\nabla_Y(QX)+Q[X,Y]=(\nabla_Y Q)X-(\nabla_X Q)Y. \tag{4.3}$$

Also

$$R(X,Y)Df=(\nabla_X Q)Y-(\nabla_Y Q)X-(YA)X+(XA)Y, \tag{4.4}$$

where $A=(\frac{1}{2\tau}-\frac{2}{3}-p)$.

Taking the covariant deriavtive of (2.13) along vector field X and using (2.17) we have

$$(\nabla_X Q)Y=\{2(n-1)+n(2k-\mu)\}\,[g(X,\phi Y)\xi+g(\phi hX,Y)\xi) \tag{4.5}$$

$$-\eta(Y)\phi X+\eta(Y)\phi hX]+\{2(n-1)+\mu\}\,(\nabla_X h)Y.$$

Applying (2.15) in (4.5) gives

$$(\nabla_X Q)Y-(\nabla_Y Q)X=[2(n-1)+\mu][-(k+1)[2g(X,\phi Y)\xi \tag{4.6}$$

$$+\eta(X)\phi Y-\eta(Y)\phi X]+(1-\mu)(\eta(X)\phi hY-\eta(Y)\phi hX)]+[2(n-1)+n(2k-\mu)]$$

$$\times[2g(X,\phi Y)\xi+\eta(X)\phi Y-\eta(Y)\phi X+\eta(Y)\phi hX-\eta(X)\phi hY].$$

Now, using (4.5) and (4.4), we obtain

$$R(X,Y)Df=[2(n-1)+\mu][(k+1)[2g(X,\phi Y)\xi+\eta(X)\phi Y-\eta(Y)\phi X] \tag{4.7}$$

$$+(1-\mu)(\eta(X)\phi hY-\eta(Y)\phi hX)]-[2(n-1)+n(2k-\mu)]$$

$$\times[2g(X,\phi Y)\xi+\eta(X)\phi Y-\eta(Y)\phi X+\eta(Y)\phi hX-\eta(X)\phi hY]-(YA)X+(XA)Y.$$

Taking inner product of (4.7) with ξ we obtain

$$g(R(X,Y)Df,\xi)=2(\mu-2k+\mu k+n\mu)g(X,)-YA\eta(X)+(XA)\eta(Y). \tag{4.8}$$

Substituting $X=\xi$ in (4.8) we get

$$g(R(\xi,Y)Df,\xi)=(\xi A)\eta(Y)-(YA). \tag{4.9}$$

Also from (2.9) it follows

$$R(\xi,Y)X=k[g(X,Y)\xi-\eta(X)Y]+\mu[g(hX,Y)\xi-\eta(X)hY]. \tag{4.10}$$

Taking the inner product of (4.10) with ξ gives

$$g(R(\xi,Y)Df,\xi)=kg(Y,Df-(\xi f)\xi)+\mu g(hY,Df) \tag{4.11}$$

In view of (4.9) and 4.11) we have

$$kg(Y,Df-\xi f)\xi)+\mu g(hY,Df)-(\xi A)\eta(Y)+(YA)=0, \tag{4.12}$$

from which we obtain

$$kDf-k(\xi f)\xi+\mu hDf+DA-(\xi A)\xi=0. \tag{4.13}$$

Contracting X in (4.4) and using the fact that the scalar curvature of the manifold is constant, we have

$$QDf=-2nDA.$$

Applying 4.13) gives

$$2nkDf+2n\mu hDf=QDf+2n(k(\xi f))+(\xi A))\xi \tag{4.14}$$

Taking inner product of (4.14) with ξ and since $Q\xi=2nk\xi$ it follows the

$$k(\xi f)+(\xi A)=0. \tag{4.15}$$

Using (4.17) in (4.14) we get

$$2nDf+2n\mu hDf=QDf. \tag{4.16}$$

Putting $X = \xi$ (4.2) we obtain

$$\nabla_\xi Df = \left[\left(\frac{1}{2\tau} - \frac{2}{n} + p\right) - 2nk\right]\xi \tag{4.17}$$

Replacing X by ϕX and Y by ϕY in (4.8) and (2.16), receptively, then comparing the right hand side we have

$$(\mu - 2k + \mu k + n\mu)g(\phi X, Y) = 0. \tag{4.18}$$

Since $d\eta \neq 0$, it follow that the above equation gives

$$k = \frac{\mu(n+1)}{2-\mu}. \tag{4.19}$$

Differentiating (4.14) along ξ implies

$$2nk_\xi Df + 2n\mu(\nabla_\xi h)Df + 2n\mu h(\nabla_\xi Df) = (\nabla_\xi Q)Df + Q(\nabla_\xi Df) \tag{4.20}$$

Now using (2.12), (4.5) and (4.20) we get

$$\mu[\mu(2n-1) - 2(n-1)]h\phi Df = 0. \tag{4.21}$$

Operating h on (4.21) and since $k \neq 1$ we obtains

$$\mu[\mu(2n-1) - 2(n-1)]\phi Df = 0. \tag{4.22}$$

Thus we consider the following cases:

Case1. If $\mu = 0$, then from (4.19) it follows that $k = 0$. Consequently (2.9) gives $R(X,Y)\xi = 0$. Therefore, using Lemma 2.2 we can state M^{2n+1} is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.

Case 2. If $\phi Df = 0$. Applying ϕ on both sides we obtain

$$Df = (\xi f)\xi. \tag{4.23}$$

Taking differentiation of (4.23) along any arbitrary vector field X, we have

$$\nabla_X Df = X(\xi f)\xi + (\xi f)(-\phi X + \phi h X).$$

Replacing X by ϕX and taking inner product with ϕY we have

$$g(\nabla_{\phi X} Df, \phi Y) = -(\xi f)[g(X, \phi Y) + g(hX, \phi Y]. \tag{4.24}$$

Interchanging X by Y in the above equation yields

$$g(\nabla_{\phi Y} Df, \phi X) = -(\xi f)[g(Y, \phi X) + g(hY, \phi X]. \tag{4.25}$$

Applying Poincare's lemma: On a contractible manifold, all closed forms are exact. Therefore $d^2 f(X,Y) = 0$, for all $X, Y \in \chi(M)$. From which we have

$$XY(f) - YX(f) - [X,Y]f = 0,$$

that is,

$$Xg(gradf, Y) - Yg(gradf, X) - g(gradf, [X,Y]) = 0.$$

This is equivalent to

$$\nabla Xg(gradf, Y) - g(gradf, \nabla XY) - g(gradf, X) + g(gradf, \nabla YX) = 0.$$

Since $\nabla g = 0$, the above equation yields

$$\nabla Xg(gradf, Y) - g(gradf, X) = 0,$$

that is, $g(\nabla XDf, Y) = g(\nabla YDf, X)$. Replacing X by ϕX and Y by ϕY in the foregoing equation we obtain $g(\nabla\phi XDf, \phi Y) = g(\nabla\phi YDf, \phi X)$. Applying this in (4.24) and (4.25) we have $(\xi f)g(X, \phi Y) = 0$, that is, $(\xi f)d\eta(X,Y) = 0$. Since $d\eta 6 = 0$, it follows that $\xi f = 0$.

Consequently from (4.23) we obtain $Df = 0$, this implies f is constant. Therefore from (4.1) we have

$$S(X, Y) = \left(\frac{1}{2\tau} - \frac{2}{n} + p\right) g(X, Y). \tag{4.26}$$

This shows the manifold is an Einstein manifold.
Case 3. If $\mu(2n-1)-2(n-1) = 0$, that is, $\mu = 2(n-1)2n-1$. Using (4.19) we get $k = n\frac{1}{n}$. From (2.13) and (4.19) we obtain

$$(2(1-n) + n\mu 2nk)(Df - ()\xi) + (2(n-1) + \mu - 2n\mu)hDf = 0. \tag{4.27}$$

Making use of $\mu = 2(n1)2n1$and $k = n\frac{1}{n}$ in the above equation and noticing $n > 1$ we have $Df = (\xi f)$. Proceeding in the same way as in Case 2 we obtain the manifold is an Einstein manifold.
Therefore now, we can state the following:

Theorem 4.1. *Let (M, g) be a $(2n+1)$-dimensional $(n > 1)$ (k, μ)-paracontact metric manifold with $k \neq -1$. If g is a gradient almost conformal Ricci soliton, then either the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4, or, M^{2n+1} is an Einstein manifold.*

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Author information

Mohd Danish Siddiqi, Department of Mathematics, Jazan University Faculty of Science, Jazan, Kingdom of Saudi Arabia.
E-mail: `anallintegral@gmaul.com; msiddiqi@jazanu.edu.sa`

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