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SOME CURVES ON THREE DIMENSIONAL LORENTZIAN TRANS-SASAKIAN MANIFOLDS

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Abstract The object of the present paper is to study some curves on three-dimensional trans-Sasakian manifolds with Lorentzian metric. Here we study biharmonic almost contact curves and slant curves on three-dimensional Lorentzian trans-Sasakian manifolds. We also consider C-loxodrome and C-parallel curves. An example is given.

1 Introduction

After the work of Baikoussis and Blair [1], the study of curves on contact manifolds has become a popular topic. They have studied Legendre curves on contact three-manifolds. In the study of contact manifolds, Legendre curves play an important role, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. In [2], the authors have studied Legendre curves on Lorentzian Sasakian manifolds. Trans-Sasakian manifolds form an important class of almost contact manifolds. It generalizes a large number of contact and almost contact manifolds. Recently, in the paper [4] Lorentzian trans-Sasakian manifolds was studied. For detailed references on Lorentzian trans-Sasakian manifolds we refer [4]. Recently, in [6] a large class of almost contact manifolds was studied admitting different types of curves. The present author has studied some curves on trans-Sasakian manifolds admitting semi-symmetric metric connections [14]. The present paper is organized as followes:

We give the required preliminaries and some basic results in Section 2. Section 3, contains the study of slant curves on Lorentzian trans-Sasakian manifolds. In Section 4, we study biharmonic almost contact curves on three-dimensional Lorentzian trans-Sasakian manifolds. Section 5, is devoted to study C-loxodrome and C-parallel slant curves. Finally we construct an example of three-dimensional Lorentzian trans-Sasakian manifold.

2 Priliminaries

Let M be a $(2n+1)$- dimensional connected differentiable manifold together with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Lorentzian metric such that

$$\phi^2(X) = X + \eta(X)\xi, \tag{2.1}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta\phi = 0, \quad \eta(\xi) = -1, \quad \eta(X) = g(X, \xi), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \quad \forall X, Y \in T(M). \tag{2.4}$$

Then M is called a Lorentzian trans-Sasakian manifold. Also a Lorentzian trans-Sasakian manifold M satifies

$$\nabla_X \xi = -\alpha(\phi X) - \beta(X + \eta(X)\xi), \tag{2.5}$$

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \tag{2.6}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. If $\alpha = 0$ and $\beta \in R$, the set of real numbers, then the manifold reduces to a Lorentzian $\beta-$Kenmotsu manifold studied by Yaliniz et al [15]. If $\beta = 0$ and $\alpha \in R$, then the manifold reduces to a Lorentzian α-Sasakian manifold studied by Yildiz, Turan and Murathan [16]. If $\alpha = 0$ and $\beta = 1$, then the manifold reduces to a Lorentzian Kenmotsu manifold introduced by Mihai, Oiaga and Rosca [8]. Furthermore, if $\beta = 0$ and $\alpha = 1$, then the manifold reduces to a Lorentzian Sasakian manifold studied by Ikawa and Erdogan [7]. Also Lorentzian para contact manifolds were introduced by Matsumoto [9] and futher studied by the authors [10], [11], [12]. Trans Lorentzian para Sasakian manifolds have been used by Gill and Dube [5].

Lemma 2.1. The Riemannian curvature tensor R in a three-dimensional Lorentzian trans-Sasakian manifold is given by [4]

$$\begin{aligned} R(X,Y)Z =& (\frac{r}{2} + 2\xi\beta - 2(\alpha^2+\beta^2) + 2\psi(\xi\alpha - 2\alpha\beta))[g(Y,Z)X - g(X,Z)Y] \\ &+ g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2) - 4\alpha\beta\psi)\eta(X)\xi \\ &+ \eta(X)(\phi(grad\,\alpha) - \psi(grad\,\alpha) - grad\,\beta) - (X\beta - (\phi X)\alpha) + \psi(X\alpha))\xi] \\ &+ g(X,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2) - 4\alpha\beta\psi)\eta(Y)\xi \\ &+ \eta(Y)(\phi(grad\,\alpha) - \psi(grad\,\alpha) - grad\,\beta) - (Y\beta - (\phi Y) + \psi(Y\alpha))\xi] \\ &+ [(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2) - 4\alpha\beta\psi)\eta(Y)\eta(Z) \\ &+ \eta(Y)(-Z\beta + (\phi Z)\alpha - \psi(Z\alpha\psi)) - \eta(Z)(Y\beta - (\phi Y)\alpha + \psi(Y\alpha))]X \\ &- [(\frac{r}{2} + \xi\beta - 3(\alpha^2+\beta^2) - 4\alpha\beta\psi)\eta(X)\eta(Z)) \\ &+ \eta(X)(-Z\beta + (\phi Z)\alpha - \psi(Z\alpha\psi)) - \eta(Z)(X\beta - (\phi X)\alpha + \psi(X\alpha))]Y \\ &+ (2\alpha\beta - \xi\alpha)[g(\phi Y,Z)X - g(\phi X,Z)Y], \end{aligned} \tag{2.7}$$

where $\psi = \sum_{i=1}^{3} \epsilon_i g(\phi e_i, e_i)$. and $\epsilon_i = g(e_i, e_i), \epsilon_i = \pm 1$. r is the scalar curvature of the manifold M with respect to Levi-Civita connection.

Lemma 2.2. In a Lorentzian trans-Sasakian manifold [4], we have

$$\begin{aligned} R(X,Y)\xi =& (\alpha^2+\beta^2)(\eta(Y)X - \eta(X)Y) \\ &+ 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \\ &+ (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \tag{2.8}$$

where R is the curvature tensor.

Lemma 2.3. For a Lorentzian trans-Sasakian manifold [4], we have

$$R(\xi,Y)\xi = (\alpha^2+\beta^2 - \xi\beta)\phi^2 Y + (2\alpha\beta - \xi\alpha)\phi Y. \tag{2.9}$$

Lemma 2.4. In a $(2n+1)$-dimensional Lorentzian trans-Sasakian manifold [4], we have

$$\begin{aligned} S(X,\xi) =& (2n(\alpha^2+\beta^2) - \xi\beta)\eta(X) + (2n+1)(X\beta) \\ &- (\phi X)\alpha + \psi(2\alpha\beta\eta(X) + X\alpha), \end{aligned} \tag{2.10}$$

$$\begin{aligned} Q\xi =& (2n(\alpha^2+\beta^2) - \xi\beta)\xi + (2n-1)grad\,\beta \\ &- \phi(grad\,\alpha) + \psi(2\alpha\beta\xi + grad\,\alpha), \end{aligned} \tag{2.11}$$

where S is the Ricci curvature and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$, $\psi = \sum_{i=1}^{2n+1} \epsilon_i g(\phi e_i, e_i)$, and $\epsilon_i = g(e_i, e_i)$, $\epsilon_i = \pm 1$.

Let M be a 3-dimensional Riemannian manifold. Let $\gamma : I \to M$, I being an interval, be a curve in M which is parameterized by arc length, and let $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Levi-Civita connection on M. It is said to that γ is a Frenet curve if one of the following three cases holds:

(a) γ is of osculating order 1, if, $\nabla_T T = 0$ (geodesic), $T = \dot{\gamma}$. Here, . denotes differentiation with respect to the arc length.

(b) γ is of osculating order 2, if,there exist two orthonormal vector fields $T(= \dot{\gamma})$, N and a non-negative function κ (curvature) along γ such that $\nabla_T T = \kappa N$, $\nabla_T N = -\kappa T$.

(c) γ is of osculating order 3, if, there exist three orthonormal vectors $T = (\dot{\gamma}), N, B$ and two non-negative function κ(curvature) and τ(torsion) along γ such that

$$\nabla_T T = \kappa N, \tag{2.12}$$

$$\nabla_T N = -\kappa T + \tau B, \tag{2.13}$$

$$\nabla_T B = -\tau N, \tag{2.14}$$

Where $T = \dot{\gamma}$ and $\{T, N, B\}$is the Frenet frame κ and τ are the curvature and torsion of the curve. With respect to Levi-Civita connection, a Frenet curve of osculating order 3 is called a Geodesic if $\kappa = 0$. It is called a circle if κ is a positive constant and $\tau = 0$. The curve is called a helix in M if κ and τ both are positive constants and the curve is called a generalized Helix if $\frac{\kappa}{\tau} =$ constant.

A Frenet curve γ in an almost contact metric manifold is said to be a Legendre curve or almost contact curve if it is an integral curve of the contact distribution $D = \ker\eta$. Formally, it is said that a Frenet curve γ in an almost contact metric manifold is a Legendre curve if and only if $\eta(\dot{\gamma}) = 0$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$. For more details we refer [13].

3 SLANT CURVES IN LORENTZIAN TRANS-SASAKIAN MANIFOLDS

Definition 3.1. A unit speed curve γ in an almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be slant if its tangent vector field makes constant angle θ with ξ i.e., $\eta(\dot{\gamma}) = \cos\theta$ is constant alonge γ.

By definition, slant curves with constant angle $\frac{\pi}{2}$ are called almost Legendre curves or almost contact curves.

Consider a slant curve γ on a Lorentzian trans-Sasakian manifold. We get by definition

$$g(T, \xi) = \cos\theta,$$

where θ is a constant. Differentiating both side with respect to T we get

$$\nabla_T g(T, \xi) - g(\nabla_T T, \xi) - g(T, \nabla_T \xi) = 0. \tag{3.1}$$

Using (2.5) in the above equation we get,

$$-\kappa\eta(N) + \beta + \cos^2\theta = 0, \tag{3.2}$$

where $\{T, N, B\}$ is a Frenet frame with $T = \dot{\gamma}$. From above we get

$$\kappa\eta(N) = \cos^2\theta + \beta. \tag{3.3}$$

In particular, let $\theta = \frac{\pi}{2}$, i.e., the curve is Legendre curve, then we get $\beta = 0$. Therefore, we can conclude the following:

Theorem 3.1. If a three-dimensional Lorentzian trans-Sasakian manifold of type (α, β) admits a Legendre curve, then the manifold is not β-Kenmotsu manifolds.

In particular, let $\theta \neq 0$, then $\kappa = \frac{\cos^2\theta + \beta}{\kappa(N)}$. Thus we obtain the following:

Theorem 3.2. On a three-dimensional Lorentzian trans-Sasakian manifold of type $(\alpha, -1)$, the integral curve of Reeb vector field is a geodesic.

We also obtain the following:

Theorem 3.3. On a three-dimensional trans-Sasakian manifold of type $(\alpha, 0)$, a proper $(0 < \theta < \frac{\pi}{2})$ slant curve is of positive curvature.

Remark 3.4. The curvature of a slant curve on a Lorentzian trans-Sasakian manifold of type (α, β) is independent of α.

4 BIHARMONIC LEGENDRE CURVES ON LORENTZIAN TRANS-SASAKIAN MANIFOLDS

Definition 4.1. A unit speed smooth curve γ on a Lorentzian trans-Sasakian manifold is called a Legendre curve [2] if it satisfies $\eta(\dot{\gamma}) = 0$.

Definition 4.2. A Legendre curve γ on a three-dimensional Lorentzian trans-Sasakian manifold will be called biharmonic [13] if it satisfies

$$\nabla_T^3 T - \kappa R(N, T)T = 0, \tag{4.1}$$

where $T = \dot{\gamma}$.

Let us consider a Legendre curve γ. Let T be the unit tangent vector field of the Legendre curve. To maintain orientation let $T, \xi, \phi T$ be a orthonormal right handed system where $\phi T = -B, \phi B = T$. It is to be mentioned that such assumption is compatible with almost contact structure. We take $\{T, \xi, \phi T\}$ as Frenet frame.

Then the equation (4.1) reduces to the following:

$$\nabla_T^3 T - \kappa R(N, T)T = 0. \tag{4.2}$$

By Serret-Frenet formula we get

$$\nabla_T^3 T = -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\tau\kappa' + \kappa\tau')B. \tag{4.3}$$

For Legendre curve $\eta(T) = 0$, $\eta(N) = 0$, because we have considered the Frenet frame $T, N = \phi T, B = \phi T$. Using these facts in (2.7) we get, after simplification

$$R(\xi, T)T = (\alpha^2 + \beta^2)\xi + \psi(\xi\alpha)\xi - (-T\beta + (\phi T)\alpha - \psi(T\alpha))T. \tag{4.4}$$

Now in the view of (4.3) and (4.4), it follows that

$$\begin{aligned}\nabla_T^3 T - \kappa R(\xi, T)T =& \kappa(-3\kappa' - T\beta + (\phi T)\alpha - \psi(T\alpha)T \\ &+(\kappa'' - \kappa^3 - \kappa\tau^2 - \kappa(\alpha^2 + \beta^2) - \kappa\psi(\xi\alpha))\xi \\ &+(2\tau\kappa' + \kappa\tau')B.\end{aligned} \tag{4.5}$$

From the first component we get

$$\kappa = 0, \quad \text{or,} \quad -3\kappa' - T\beta + (\phi T)\alpha - \psi(T\alpha) = 0. \tag{4.6}$$

From second component we get

$$\kappa'' - \kappa^3 - \kappa\tau^2 - \kappa(\alpha^2 + \beta^2) - \kappa\psi(\xi\alpha) = 0. \tag{4.7}$$

And from the third component we get

$$2\tau\kappa' + \kappa\tau' = 0 \tag{4.8}$$

If $\kappa \neq 0$ and α, β are constants, then from (4.7) and (4.8) we get

$$\kappa = \pm\sqrt{2}\tau. \tag{4.9}$$

Hence we are in position to state the following:

Theorem 4.3. In a three-dimensional Lorentzian trans-Sasakian manifold of type (α, β) a biharmonic almost contact (Legendre) curve is a geodesic or a helix where α, β are constants.

5 C-LOXODROME AND C-PARALLEL IN LORENTZIAN TRANS-SASAKIAN MANIFOLDS

Definition 5.1. A unit speed curve γ in a Lorentzian trans-Sasakian manifold is said to be a C-loxodrome if it satisfies [6]

$$\nabla_T T = r\eta(T)\phi T. \tag{5.1}$$

Here r is a constant. In a Lorentzian trans-Sasakian manifold we have for C-loxodrome

$$\eta(T)' = \beta(1 - (\eta(T))^2) \tag{5.2}$$

Then we can state the following:

Theorem 5.2. In a three-dimensional Lorentzian trans-Sasakian manifold of type (α, β), the contact angle is not necessarily constant. It is so if $\beta = 0$.

Definition 5.3. Let γ be a unit speed curve in an almost contact metric 3-manifold. Then γ is said to have C-parallel mean curvature vector field if

$$g(\nabla_T H, X) = 0, \tag{5.3}$$

for all $X \in TM$ orthogonal to ξ.

Also we can say that γ has C-parallel mean curvature vector field if and only if there exist a differentiable function λ such that

$$\nabla_T H = \lambda\xi. \tag{5.4}$$

Putting $H = \nabla_T T$ and if $\{T, N, B\}$ is a Frenet frame then (5.4) implies

$$-\kappa^2 T + \kappa' N + \kappa\tau B = \lambda\xi. \tag{5.5}$$

Taking inner product of the above equation with T, N, B respectively we get

$$\eta(T) = -\frac{1}{\lambda}\kappa^2, \tag{5.6}$$

$$\eta(N) = \frac{1}{\lambda}\kappa', \tag{5.7}$$

$$\eta(B) = \frac{1}{\lambda}\kappa\tau. \tag{5.8}$$

But for a slant curves with constant slant angle θ, $\eta(T) = \cos\theta$, hence from (5.6) we get

$$\kappa^2 = -\lambda\cos\theta. \tag{5.9}$$

By virtue of $(2.5), (5.7), (5.9)$ and $(\nabla_T g)(T, \xi) = 0$ it follows after simplification that

$$\frac{\kappa'}{\kappa}\cos\theta - \beta + \beta\cos^2\theta = 0. \tag{5.10}$$

If $\beta = 0, \theta = 0$, it follows that $\kappa =$ constant, $(\kappa \neq 0)$. So, we state

Theorem 5.4. The curvature κ of a C-parallel Reeb flow in a three-dimensional trans-Sasakian manifold of type $(\alpha, 0)$ is a constant.

By virtue of (5.5) it follows that $\xi \in \text{span}\{T, N, B\}$. So we can write

$$\xi = \cos\theta T + \sin\theta(\cos\Psi N + \sin\Psi B), \tag{5.11}$$

where Ψ is the angle function between N and the orthogonal projection of ξ on to span$\{N, B\}$. Taking inner product of ξ with N and B respectively, and using (5.11), (5.9) and the Theorem 5.4. we find

$$\cos\Psi = 0, \quad \sin\Psi = -\frac{\tau\cot\theta}{\kappa}. \tag{5.12}$$

Hence from above we get

$$\tau^2 = \kappa^2\tan\theta. \tag{5.13}$$

For the Reeb flow $\theta = 0$. So by virtue of (5.13) and the Theorem 5.4. we state the following:

Theorem 5.5. The Reeb flow on a three-dimensional Lorentzian trans-Sasakian manifold of type $(\alpha, 0)$ is a circle.

6 Example

In this section we like to construct an example of a three-dimensional Lorentzian trans-Sasakian manifold and then Legendre curve on it. Let us consider a 3-dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3.

Let $e_1 = z\frac{\partial}{\partial x}$, $e_2 = z\frac{\partial}{\partial y}$ and $e_3 = z\frac{\partial}{\partial z}$, which are linearly independent vector fields at each point of M. Let g be a Riemannian metric define by

$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1.$

Let η be 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in TM$ and ϕ be the tensor field of type $(1, 1)$ defined by $\phi e_1 = -e_2$, $\phi e_2 = -e_1$, $\phi e_3 = 0$. Then by applying linearity of ϕ and g, we have

$\eta(e_3) = -1$, $\phi^2 Z = Z + \eta(Z)e_3$, $g(\phi Z, \phi U) = g(Z, U) + \eta(Z)\eta(U)$, for any $Z, U \in TM$.

Hence for $e_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian structure on M.

Let ∇ be the Levi-Civita connection with respect to g and R be the curvature tensor of type $(1, 3)$, then we have

$[e_1, e_2] = 0$, $[e_1, e_3] = -e_1$, $[e_2, e_3] = -e_2$.

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \end{aligned} \tag{6.1}$$

which known as Koszul′s formula. By using Koszul formula for Levi-Civita connection with respect to g, we obtain

$$\begin{array}{lll} \nabla_{e_1} e_3 = -e_1, & \nabla_{e_2} e_3 = -e_2, & \nabla_{e_3} e_3 = 0, \\ \nabla_{e_1} e_2 = 0, & \nabla_{e_2} e_2 = -e_3, & \nabla_{e_3} e_2 = 0, \\ \nabla_{e_1} e_1 = -e_3, & \nabla_{e_2} e_1 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

From the above we see that the manifold satisfies $\nabla_X \xi = -\alpha(\phi X) - \beta(X - \eta(X)\xi)$, for $\xi = e_3$, $\alpha = 0$ and $\beta = -1$. Hence the manifold $M(\phi, \xi, \eta, g)$ is a Lorentzian trans-Sasakian manifold of type $(0, -1)$.

With the help of the above results it can be verified that

$$\begin{array}{lll} R(e_1, e_2)e_3 = 0, & R(e_2, e_3)e_3 = -e_2, & R(e_1, e_3)e_3 = -e_1, \\ R(e_1, e_2)e_2 = -e_1, & R(e_2, e_3)e_2 = -e_3, & R(e_1, e_3)e_2 = 0, \\ R(e_1, e_2)e_1 = e_2, & R(e_2, e_3)e_1 = 0, & R(e_1, e_3)e_1 = -e_3. \end{array}$$

Hence the manifold is a Lorentzian trans-Sasakian manifold with constant curvature -1. Now we give an example of unit speed curves on the manifold.

Example 5.1. Consider a curve $\gamma : I \longrightarrow M$ defined by $\gamma(s) = (0, 0, -s)$. Hence $\dot{\gamma_1} = 0$, $\dot{\gamma_2} = 0$ and $\dot{\gamma_3} = -1$,

$\eta(\dot{\gamma}) = g(\dot{\gamma}, e_3) = g(\dot{\gamma_1} e_1 + \dot{\gamma_2} e_2 + \dot{\gamma_3} e_3, e_3) = 1.$

$$\begin{aligned} g(\dot{\gamma}, \dot{\gamma}) =& g(\dot{\gamma}, e_3) \\ =& g(\dot{\gamma_1} e_1 + \dot{\gamma_2} e_2 + \dot{\gamma_3} e_3, \dot{\gamma_1} e_1 + \dot{\gamma_2} e_2 + \dot{\gamma_3} e_3) \\ =& \dot{\gamma_1^2} + \dot{\gamma_2^2} + \dot{\gamma_3^2} \\ =& \dot{\gamma_3^2} \\ =& 1. \end{aligned} \tag{6.2}$$

Hence the curve is unit speed and it is the flow line of the Reeb vector field ξ. For this curve $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$. Hence the Reeb flow line is geodesic.

Example 5.2. Consider a curve $\gamma : I \longrightarrow M$ defined by $\gamma(s) = (\sqrt{\frac{2}{3}}s, \sqrt{\frac{1}{3}}s, 1)$. Hence $\dot{\gamma_1} = \sqrt{\frac{2}{3}}$, $\dot{\gamma_2} = \sqrt{\frac{1}{3}}$ and $\dot{\gamma_3} = 0$,

$\eta(\dot{\gamma}) = g(\dot{\gamma}, e_3) = g(\dot{\gamma_1} e_1 + \dot{\gamma_2} e_2 + \dot{\gamma_3} e_3, e_3) = 0.$

$$\begin{aligned} g(\dot{\gamma}, \dot{\gamma}) =& g(\dot{\gamma}, e_3) \\ =& g(\dot{\gamma_1} e_1 + \dot{\gamma_2} e_2 + \dot{\gamma_3} e_3, \dot{\gamma_1} e_1 + \dot{\gamma_2} e_2 + \dot{\gamma_3} e_3) \\ =& \dot{\gamma_1^2} + \dot{\gamma_2^2} + \dot{\gamma_3^2} \\ =& \dot{\gamma_1^2} + \dot{\gamma_2^2} \\ =& 1. \end{aligned} \tag{6.3}$$

Hence the curve is Legendre curve. For this curve $\nabla_{\dot{\gamma}} \dot{\gamma} = -e_3$. Hence the curve is not geodesic.

Note. We consider the dimension of the manifolds is three because the dimension of differentiable manifolds is odd, i.e., $(2n + 1)$.

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