ON LORENTZIAN $\alpha$-SASAKIAN MANIFOLDS ADMITTING A TYPE OF SEMI-SYMMETRIC NON-METRIC CONNECTION

Ajit Barman

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Abstract The object of the present paper is to study a Lorentzian $\alpha$-Sasakian manifold admitting a semi-symmetric non-metric connection due to Agashe and Chafle [1]. It is shown that if a Lorentzian $\alpha$-Sasakian manifold is semisymmetric with respect to the semi-symmetric non-metric connections, then the manifold is an $\eta$-Einstein manifold provided $\alpha^2 \neq -1$. Among others we prove that a Lorentzian $\alpha$-Sasakian manifold is Ricci semisymmetric with respect to the semi-symmetric non-metric connections and the Levi-Civita connections are equivalent. Moreover we deal with a three-dimensional Ricci semisymmetric on Lorentzian $\alpha$-Sasakian manifold with respect to the semi-symmetric non-metric connection is a manifold of constant curvature. Finally, an illustrative example is given to verify our result.

1 Introduction

In [16], Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plain sections containing $\xi$ is a constant, say $c$. He showed that they can be divided into three classes:

(1.1) homogeneous normal contact Riemannian manifolds with $c > 0$,

(1.2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and

(1.3) a warped product space $\mathbb{R} \times f \mathbb{C}$ if $c < 0$.

It is well known that the manifolds of class (1.1) are characterized by admitting a Sasakian structure. Kenmotsu [13] characterized the differential geometric properties of the manifolds of class (1.3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [13].

In the Gray-Hervella [8], classification of almost Hermitian manifolds there appears a class $W_4$, of Hermitian manifolds which are closely related to locally conformal Kaehlerian manifolds [6]. An almost contact metric structure on the manifold $M$ is called a trans-Sasakian structure [14] if the product manifold $M \times \mathbb{R}$ belongs to the class $W_4$. The class $C_6 \oplus C_5$ ([12], [13]) coincides with the class of trans-Sasakian structure of type $(\alpha, \beta)$. We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic, $\beta$-Kenmotsu [10] and $\alpha$-Sasakian [10] respectively. Lorentzian $\alpha$-Sasakian manifolds have been studied by Yildiz and Murathan [17], Yildiz, Turan and Murathan [19], Bagewadi and Ingalahalli [5], Barman [3], Yildiz, Turan, and Acet [18] and many others.

In 1924, Friedmann and Schouten [7] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\tilde{\nabla}$ on a differentiable manifold $M$ is said to be
a semi-symmetric connection if the torsion tensor $T$ of the connection $\tilde{\nabla}$ satisfies $T(X,Y) = u(Y)X - u(X)Y$, where $u$ is a 1-form and $\rho$ is a vector field defined by $u(X) = g(X,\rho)$, for all vector fields $X,Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on $M$.

In 1932, Hayden [9] introduced the idea of semi-symmetric metric connections on a Riemannian manifold $(M,g)$. A semi-symmetric connection $\tilde{\nabla}$ is said to be a semi-symmetric metric connection if $\tilde{\nabla}g = 0$. A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ of $(M,g)$ was given by Yano [20]: $\tilde{\nabla}X Y = \nabla X Y + u(X)X - g(X,Y)\rho$, where $u(X) = g(X,\rho)$.

After a long gap the study of a semi-symmetric connection $\tilde{\nabla}$ satisfying $\tilde{\nabla}g \neq 0$, was initiated by Prvanović [15] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [2]. The semi-symmetric connection $\tilde{\nabla}$ is said to be a semi-symmetric non-metric connection.

In 1992, Agashe and Chafle [1] studied a semi-symmetric non-metric connection $\tilde{\nabla}$, whose torsion tensor $T$ satisfies $T(X,Y) = u(Y)X - u(X)Y$ and $(\tilde{\nabla}X g)(Y,Z) = -u(Y)g(X,Z) - u(Z)g(X,Y)$. In 1992, Barua and Mukhopadhyay [4] studied a type of semi-symmetric connection $\tilde{\nabla}$ which satisfies $(\tilde{\nabla}X g)(Y,Z) = 2u(X)g(Y,Z) - u(Y)g(X,Z) - u(Z)g(X,Y)$. Since $\tilde{\nabla}g \neq 0$, this is another type of semi-symmetric non-metric connection. However, the authors preferred the name semi-symmetric semimetric connection. In 1994, Liang [11] studied another type of semi-symmetric non-metric connection $\tilde{\nabla}$ for which we have $(\tilde{\nabla}X g)(Y,Z) = 2u(X)g(Y,Z)$, where $u$ is a non-zero 1-form and he called this a semi-symmetric recurrent metric connection.

A Lorentzian $\alpha$-Sasakian manifold is said to be a semi-symmetric manifold with respect to the semi-symmetric non-metric connection if it satisfies the relation

$$(\tilde{R}(X,Y)).\tilde{R}(U,V)W = 0$$

holds for all $X,Y,U,V,W \in \chi(M)$, where $\tilde{R}(X,Y)$ is the curvature operator.

In this paper we study Lorentzian $\alpha$-Sasakian manifolds with respect to a semi-symmetric non-metric connection due to Agashe and Chafle [1]. The paper is organized as follows: After introduction in section 2, we give a brief account of the Lorentzian $\alpha$-Sasakian manifolds. In section 3, we establish the relation of the curvature tensors between the Levi-Civita connection and the semi-symmetric non-metric connection of a Lorentzian $\alpha$-Sasakian manifold. In the next section deals with a Lorentzian $\alpha$-Sasakian manifold whose curvature tensor of manifold is covariant constant with respect to the semi-symmetric non-metric connection and manifold is recurrent with respect to the Levi-Civita connection. Section 5, we study semi symmetric on a Lorentzian $\alpha$-Sasakian manifold with respect to the semi-symmetric non-metric connection, then the manifold is an $\eta$-Einstein manifold provided $\alpha^2 \neq -1$. The following statements for Lorentzian $\alpha$-Sasakian manifolds with respect to a semi-symmetric non-metric connection are equivalent, the manifold is (i) $M$ is an Einstein manifold with respect to the Levi-Civita connection, (ii) Locally Ricci symmetric admitting the Levi-Civita connection and (iii) Ricci semi symmetric on a Lorentzian $\alpha$-Sasakian manifold with respect to the semi-symmetric non-metric connection have been studied in Section 6. Finally, we construct an example of a 3-dimensional Lorentzian $\alpha$-Sasakian manifold admitting the semi-symmetric non-metric connection to support the results obtained in Section 3 and Section 6 respectively.

## 2 Lorentzian $\alpha$-Sasakian manifolds

A $(2n+1)$-dimensional differentiable manifold $M$ is called Lorentzian $\alpha$-Sasakian manifolds if it admits a $(1,1)$ tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ which satisfy [17]

$$\phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = -1, \quad g(X,\xi) = \eta(X),$$

(2.1)
\[ \phi^2(X) = X + \eta(X)\xi, \] (2.2)
\[ g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \] (2.3)
for any vector fields \( X, Y \) on \( M \).

Also Lorentzian \( \alpha \)-Sasakian manifolds is satisfy [17],

\[ \nabla_X \xi = -\alpha \phi X, \] (2.4)
\[ (\nabla_X \eta)(Y) = -\alpha g(\phi X, Y), \] (2.5)

where \( \nabla \) denotes the operator of covariant differentiation with respect to the Lorentzian metric \( g \) and \( \alpha \in \mathbb{R} \).

Further on a Lorentzian \( \alpha \)-Sasakian manifold \( M \) the following relations holds [17]:

\[ \eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \] (2.6)
\[ R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X], \] (2.7)
\[ R(\xi, X)\xi = \alpha^2[\eta(X)\xi + X], \] (2.8)
\[ R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \] (2.9)
\[ S(X, \xi) = 2n\alpha^2\eta(X), \] (2.10)
\[ (\nabla_X \phi)(Y) = \alpha^2[g(X, Y)\xi - \eta(Y)X], \] (2.11)

where \( R \) and \( S \) are the curvature tensor and the Ricci tensor of the Levi-Civita connection respectively.

3 Curvature tensor of a Lorentzian \( \alpha \)-Sasakian manifold with respect to the semi-symmetric non-metric connection

**Proposition 3.1.** For a Lorentzian \( \alpha \)-Sasakian manifold \( M \) with respect to the semi-symmetric non-metric connection \( \nabla \)

(i) The curvature tensor \( \tilde{R} \) is given by \( \tilde{R}(X, Y)Z = R(X, Y)Z - \alpha g(X, \phi Z)Y - \eta(X)\eta(Z)Y - \alpha g(Y, \phi Z)X + \eta(Y)\eta(Z)X \)

(ii) The Ricci tensor \( \tilde{S} \) is given by \( \tilde{S}(Y, Z) = S(Y, Z) + 2n\alpha g(Y, \phi Z) + 2n\eta(Y)\eta(Z) \)

(iii) \( \tilde{R}(X, Y)Z = -\tilde{R}(Y, X)Z \)

(iv) The Ricci tensor \( \tilde{S} \) is symmetric.

(v) \( \tilde{S}(Y, \xi) = 2n(\alpha^2 - 1)\eta(Y) \)

(vi) \( \tilde{R}(\xi, X)Y = \alpha^2 g(Y, Z)\xi + (\alpha^2 + 1)\eta(Z)Y + \alpha g(Y, \phi Z)\xi + \eta(Y)\eta(Z)\xi \)

(vii) \( \tilde{R}(\xi, \xi)Z = 2\alpha^2\eta(Z)\xi \)

(viii) \( \tilde{R}(\xi, Y)\xi = (\alpha^2 - 1)\eta(Y)\xi - (\alpha^2 + 1)Y \)

(ix) \( (\nabla_W \eta)(X) = -\alpha g(X, \phi W) - \eta(X)\eta(W) \).
Proof. Let $M$ be an $(2n + 1)$-dimensional Riemannian manifold with Riemannian metric $g$. If $\tilde{\nabla}$ is the semi-symmetric non-metric connection of a Riemannian manifold $M$, a linear connection $\bar{\nabla}$ is given by

$$\bar{\nabla} X Y = \nabla X Y + \eta(Y) X.$$ (3.1)

Then $\bar{R}$ and $R$ are related by [1]

$$\bar{R}(X, Y) Z = R(X, Y) Z + \gamma(X, Z) Y - \gamma(Y, Z) X,$$ (3.2)

for all vector fields $X, Y, Z$ on $M$, where $\gamma$ is a $(0, 2)$ tensor field denoted by

$$\gamma(X, Z) = (\nabla_X \eta)(Z) - \eta(X) \eta(Z).$$ (3.3)

Taking inner product of (3.2) with respect to $U$, we get

$$\bar{\tilde{R}}(X, Y, Z, U) = \tilde{R}(X, Y, Z, U) + \gamma(X, Z) g(Y, U) - \gamma(Y, Z) g(X, U).$$ (3.4)

where

$$\bar{R}(X, Y, Z, U) = g(\bar{R}(X, Y) Z, U) \text{ and } \tilde{R}(X, Y, Z, U) = g(R(X, Y) Z, U).$$

From (3.1) yields that

$$\langle \tilde{\nabla}_w g \rangle(X, Y) = -\eta(X) g(Y, W) - \eta(Y) g(X, W) \neq 0.$$ (3.5)

Using (2.5) in (3.3), we get

$$\gamma(X, Z) = -\alpha g(X, \phi Z) - \eta(X) \eta(Z).$$ (3.6)

By making use of (3.6) and (3.2), we derived that

$$\bar{R}(X, Y) Z = R(X, Y) Z - \alpha g(X, \phi Z) Y - \eta(X) \eta(Z) Y + \alpha g(Y, \phi Z) X + \eta(Y) \eta(Z) X.$$ (3.7)

From (3.7) yields that

$$\bar{R}(X, Y) Z = -\bar{R}(Y, X) Z.$$ (3.8)

We consider $X = \xi$ in (3.7) and using (2.1) and (2.7), we obtain

$$\bar{R}(\xi, Y) Z = \alpha^2 g(Y, Z) \xi + (\alpha^2 + 1) \eta(Z) Y + \alpha g(Y, \phi Z) \xi + \eta(Y) \eta(Z) \xi.$$ (3.9)

We select $Y = \xi$ in (3.9) and using (2.1), it follows that

$$\bar{R}(\xi, \xi) Z = 2\alpha^2 \eta(Z) \xi.$$ (3.10)

Again putting $Z = \xi$ in (3.9) and using (2.1), we get

$$\bar{R}(\xi, Y) \xi = (\alpha^2 - 1) \eta(Y) \xi - (\alpha^2 + 1) Y.$$ (3.11)

Taking the inner product of (3.7) with $U$, it is obvious that

$$\bar{R}(X, Y, Z, U) = \bar{R}(X, Y, Z, U) - \alpha g(X, \phi Z) g(Y, U) - \eta(X) \eta(Z) g(Y, U) + \alpha g(Y, \phi Z) g(X, U) + \eta(Y) \eta(Z) g(X, U).$$ (3.12)
Taking a frame field from (3.12), we obtain
\[
\tilde{S}(Y, Z) = S(Y, Z) + 2n\alpha g(Y, \phi Z) + 2n\eta(Y)\eta(Z).
\]  
(3.13)

From (3.13), implies that
\[
\tilde{S}(Y, Z) = S(Z, Y).
\]  
(3.14)

We take \( Z = \xi \) in (3.13) and using (2.1) and (2.10), it follows that
\[
\tilde{S}(Y, \xi) = 2n(\alpha^2 - 1)\eta(Y).
\]  
(3.15)

Combining (3.1) and (2.5), we get
\[
(\tilde{\nabla}_W \eta)(X) = -\alpha g(X, \phi W) - \eta(X)\eta(W).
\]  
(4.3)

Using (2.6) and (4.1) in (4.3), we can write
\[
(\tilde{\nabla}_W \eta)(X, Y, Z) = \eta(W)R(X, Y, Z) + \alpha^2 g(Y, Z)\eta(X)W - \alpha^2 g(X, Z)\eta(Y)W.
\]  
(4.4)

Suppose \( (\tilde{\nabla}_W \eta)(X, Y, Z) = 0 \), then from (4.4), we have
\[
\eta(W)R(X, Y, Z) + \alpha^2 g(Y, Z)\eta(X)W - \alpha^2 g(X, Z)\eta(Y)W = 0.
\]  
(4.5)

Now contracting \( X \) in (4.5) and using (2.1), we obtain
\[
\eta(W)S(Y, Z) + \alpha^2 g(Y, Z)\eta(W) - \alpha^2 g(W, Z)\eta(Y) = 0.
\]  
(4.6)

Putting \( W = \xi \) in (4.6) and using (2.1), we derived that
\[
S(Y, Z) = -\alpha^2 g(Y, Z) - \alpha^2 \eta(Z)\eta(Y).
\]  
(4.7)

Therefore, \( S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z) \), where \( a = -\alpha^2 \) and \( b = -\alpha^2 \).

From which it follows that the manifold is an \( \eta \)-Einstein manifold. This completes the proof. \( \Box \)

4 The curvature tensor is covariant constant with respect to the semi-symmetric non-metric connection and \( M \) is recurrent with respect to the Levi-Civita connection

Definition 4.1. A Lorentzian \( \alpha \)-Sasakian manifold \( M \) with respect to the Levi-Civita connection is called recurrent if its curvature tensor \( R \) satisfies the condition
\[
(\nabla_W R)(X, Y)Z = \eta(W)R(X, Y)Z,
\]  
(4.1)

where \( \eta \) be the 1-form.

Definition 4.2. A Lorentzian \( \alpha \)-Sasakian manifold \( M \) is said to be an \( \eta \)-Einstein manifold if its Ricci tensor \( S \) of the Levi-Civita connection is of the form
\[
S(Z, W) = ag(Z, W) + b\eta(Z)\eta(W),
\]  
(4.2)

where \( a \) and \( b \) are smooth functions on the manifold.

Theorem 4.3. If in an \((2n+1)\)-dimensional Lorentzian \( \alpha \)-Sasakian manifold the curvature tensor of manifold is covariant constant with respect to the semi-symmetric non-metric connection and the manifold is recurrent with respect to the Levi-Civita connection, then the manifold is an \( \eta \)-Einstein manifold.

Proof. From (3.1), implies that
\[
(\tilde{\nabla}_W R)(X, Y)Z = (\nabla_W R)(X, Y)Z + \eta(R(X, Y)Z)W.
\]  
(4.3)

Using (2.6) and (4.1) in (4.3), we can write
\[
(\tilde{\nabla}_W R)(X, Y)Z = \eta(W)R(X, Y)Z + \alpha^2 g(Y, Z)\eta(X)W - \alpha^2 g(X, Z)\eta(Y)W.
\]  
(4.4)

Suppose \( (\tilde{\nabla}_W R)(X, Y)Z = 0 \), then from (4.4), we have
\[
\eta(W)R(X, Y)Z + \alpha^2 g(Y, Z)\eta(X)W - \alpha^2 g(X, Z)\eta(Y)W = 0.
\]  
(4.5)

Now contracting \( X \) in (4.5) and using (2.1), we obtain
\[
\eta(W)S(Y, Z) + \alpha^2 g(Y, Z)\eta(W) - \alpha^2 g(W, Z)\eta(Y) = 0.
\]  
(4.6)

Putting \( W = \xi \) in (4.6) and using (2.1), we derived that
\[
S(Y, Z) = -\alpha^2 g(Y, Z) - \alpha^2 \eta(Z)\eta(Y).
\]  
(4.7)

Therefore, \( S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z) \), where \( a = -\alpha^2 \) and \( b = -\alpha^2 \).

From which it follows that the manifold is an \( \eta \)-Einstein manifold. This completes the proof. \( \Box \)
5 Semisymmetric Lorentzian $\alpha$-Sasakian manifolds with respect to the semi-symmetric non-metric connection

In this section we suppose that the manifold under consideration is semi-symmetric with respect to the semi-symmetric non-metric connection $M^{2n+1}$, that is,

$$\langle \tilde{R}(U, V) \tilde{R} \rangle (X, Y) Z = 0$$

**Theorem 5.1.** If a $(2n+1)$-dimensional Lorentzian $\alpha$-Sasakian manifold is semi-symmetric with respect to the semi-symmetric non-metric connection then the manifold is an $\eta$-Einstein manifold provided $\alpha^2 \neq -1$.

**Proof.** Then we have

$$\tilde{R}(U, V) \tilde{R}(X, Y) Z = \tilde{R}(\tilde{R}(U, V) X, Y) Z - \tilde{R}(X, \tilde{R}(U, V) Y) Z$$

Putting $U = \xi$ in (5.1), it follows that

$$\tilde{R}(\xi, V) \tilde{R}(X, Y) Z - \tilde{R}(\tilde{R}(\xi, V) X, Y) Z - \tilde{R}(X, \tilde{R}(\xi, V) Y) Z$$

By making use of (3.9) in (5.2), we obtain that

$$\tilde{R}(\xi, V) \tilde{R}(X, Y) Z - \alpha^2 g(X, V) \tilde{R}(\xi, Y) Z - (\alpha^2 + 1) \eta(X) \tilde{R}(V, Y) Z$$

Taking a frame field from (5.4) and using (2.1), we have

$$\tilde{R}(\xi, V) \tilde{R}(X, Y) Z = \alpha^2 g(X, V) \tilde{R}(\xi, Y) Z - (\alpha^2 + 1) \eta(X) \tilde{R}(V, Y) Z - \alpha^2 g(V, Z) \tilde{R}(\xi, Y) Z - (\alpha^2 + 1) \eta(Z) \tilde{R}(V, Y) Z$$

We take $X = \xi$ in (5.3) and using (3.9), (3.10) and (3.11), we get

$$\tilde{R}(\xi, V) \tilde{R}(X, Y) Z - \alpha^2 (\alpha^2 + 1) g(Y, Z) V - 2\alpha^4 \eta(Z) g(Y, V) \xi - 2\alpha^3 \eta(Z) g(Y, \phi V) \xi$$

Using (3.13) in (5.5), we obtain

$$S(Y, Z) = \frac{6\alpha^4 + 3\alpha^2 + \alpha + 1}{\alpha^2 + 1} \eta(Y) \eta(Z) + 2na^2 g(Y, Z).$$

Therefore, $S(Y, Z) = a g(Y, Z) + b \eta(Y) \eta(Z)$, where $a = 2n\alpha^2$ and $b = \frac{6\alpha^4 + 3\alpha^2 + \alpha + 1}{\alpha^2 + 1}$.

This result shows that the manifold is an $\eta$-Einstein manifold. This Theorem 5.1 completes the proof. □
6 Ricci semisymmetric Lorentzian $\alpha$-Sasakian manifolds admitting a semi-symmetric non-metric connection

In this section we characterize Ricci semisymmetric $\tilde{R} \cdot \tilde{S}$ on a Lorentzian $\alpha$-Sasakian manifold admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$.

Definition 6.1. A Lorentzian $\alpha$-Sasakian manifold is Ricci semisymmetric with respect to the Levi-Civita connection $\nabla$, that is, $(R(X,Y) \cdot S)(U,V) = 0$.

Theorem 6.2. A Lorentzian $\alpha$-Sasakian manifold is Ricci semisymmetric with respect to a semi-symmetric non-metric connection iff the manifold is also Ricci semisymmetric with respect to the Levi-Civita connection.

Proof. Then from the above equation, we can write

$$\tilde{R} \cdot \tilde{S} = \tilde{S}(\tilde{R}(X,Y)U,V) + \tilde{S}(U,\tilde{R}(X,Y)V) \quad (6.1)$$

Putting $U = \xi$ in (6.1) and using (2.1), (3.7) and (3.13), it follows that

$$\tilde{R} \cdot \tilde{S} = R \cdot S + 2n\alpha g(R(X,Y)\xi,\phi V) + 2n\eta(V)\eta(R(X,Y)\xi) + 2n\eta(R(X,Y)V)$$

$$-\eta(X)S(Y,V) - \eta(Y)S(X,V) + 2n(\alpha^2 - 1)\eta(\eta(V)) [g(X,\phi V) + \eta(X)\eta(V)]$$

$$- 2n(\alpha^2 - 1)\eta(X) [g(Y,\phi V) + \eta(Y)\eta(V)]. \quad (6.2)$$

We take $V = X = \xi$ in (6.2) and using (2.1), (2.9) and (3.7), we obtain

$$\tilde{R} \cdot \tilde{S} = R \cdot S.$$ 

This completes the proof. □

Lemma 6.3. [18] A three-dimensional Ricci semisymmetric Lorentzian $\alpha$-Sasakian manifold is a manifold of constant curvature.

Therefore, from Theorem (6.2) and Lemma (6.3) we can state the following theorem:

Theorem 6.4. A three-dimensional Ricci semisymmetric Lorentzian $\alpha$-Sasakian manifold with respect to a semi-symmetric non-metric connection is a manifold of constant curvature.

Lemma 6.5. [5] The following statements for Lorentzian $\alpha$-Sasakian manifolds are equivalent. The manifold is

i) $M$ is an Einstein manifold

ii) Locally Ricci symmetric

iii) Ricci semisymmetric that is $R(X,Y) \cdot S = 0$.

Hence, from Theorem (6.2) and Lemma (6.5) we can state the following theorem:

Theorem 6.6. The following statements for Lorentzian $\alpha$-Sasakian manifolds with respect to a semi-symmetric non-metric connection are equivalent. The manifold is

i) $M$ is an Einstein manifold with respect to the Levi-Civita connection

ii) Locally Ricci symmetric admitting the Levi-Civita connection

iii) $\tilde{R}(X,Y) \cdot \tilde{S} = 0$.

7 Example

Now, we give an example of a 3-dimensional Lorentzian $\alpha$-Sasakian manifold admitting a semi-symmetric non-metric connection $\tilde{\nabla}$, which verify the result of section 6.
We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where $(x, y, z)$ are the standard coordinate in $\mathbb{R}^3$. We choose the vector fields
\[ e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad e_3 = \alpha \frac{\partial}{\partial z}, \]
which are linearly independent at each point of $M$ and $\alpha$ is constant.

Let $g$ be the Lorentzian metric defined by
\[ g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0 \]
and
\[ g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1, \]
that is, the form of the metric becomes
\[ g = \frac{1}{(e^z)^2} (dy)^2 - \frac{1}{\alpha^2} (dz)^2, \]
which is a Lorentzian metric.

Let $\eta$ be the 1-form defined by
\[ \eta(Z) = g(Z, e_3) \]
for any $Z \in \chi(M)$.

Let $\phi$ be the $(1, 1)$-tensor field defined by
\[ \phi e_1 = -e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0 \]
Using the linearity of $\phi$ and $g$, we have
\[ \eta(e_3) = -1 \]
\[ \phi^2(Z) = Z + \eta(Z)e_3 \]
and
\[ g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W) \]
for any $U, W \in \chi(M)$.

Then we have
\[ [e_1, e_2] = 0, \quad [e_1, e_3] = -\alpha e_1, \quad [e_2, e_3] = -\alpha e_2 \]
The Riemannian connection $\nabla$ of the metric tensor $g$ is given by Koszul’s formula which is given by
\[ 2g(\nabla_X Y, W) = Xg(Y, W) + Yg(X, W) - Wg(X, Y) - g(X, [Y, W]) \]
\[ -g(Y, [X, W]) + g(W, [X, Y]). \]
Using Koszul’s formula we get the following
\[ \nabla_{e_1} e_1 = -\alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\alpha e_1, \]
\[ \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -\alpha e_3, \quad \nabla_{e_2} e_3 = -\alpha e_2, \]
\[ \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \]
In view of the above relations, we see that $\nabla_X \xi = -\alpha \phi X$, $\nabla_X \eta = -\alpha g(\phi X, Y)\xi$, for all $e_3 = \xi$. Therefore the manifold is a Lorentzian $\alpha$-Sasakian manifold with the structure $(\phi, \xi, \eta, g)$.

Using (3.1) in above equation, we obtain
\[ \tilde{\nabla}_{e_1} e_1 = -\alpha e_3, \quad \tilde{\nabla}_{e_1} e_2 = 0, \quad \tilde{\nabla}_{e_1} e_3 = -(1 + \alpha)e_1, \]
Now, we can easily obtain the non-zero components of the curvature tensors as follows:

\[ R(e_1, e_2)e_2 = -\alpha^2 e_2, \quad R(e_1, e_3)e_3 = -\alpha^2 e_1, \quad R(e_2, e_1)e_1 = \alpha^2 e_2, \]
\[ R(e_2, e_3)e_3 = -\alpha^2 e_2, \quad R(e_3, e_1)e_1 = \alpha^2 e_3, \quad R(e_3, e_2)e_2 = \alpha^2 e_3, \]
\[ R(e_1, e_2)e_3 = 0, \quad R(e_1, e_3)e_2 = -\alpha e_3, \quad R(e_2, e_1)e_3 = -(1 + \alpha)e_2, \]
\[ R(e_2, e_3)e_1 = 0, \quad R(e_3, e_1)e_2 = 0, \quad R(e_3, e_2)e_1 = -e_3. \]

With the help of the above curvature tensors with respect to a semi-symmetric non-metric connection, we find the Ricci tensors as follows:

\[ S(e_1, e_1) = S(e_2, e_2) = \alpha(1 + \alpha), \quad S(e_3, e_3) = (1 - \alpha)^2, \]

Let \( X, Y, U \) and \( V \) be any four vector fields given by \( X = a_1 e_1 + a_2 e_2 + a_3 e_3, Y = b_1 e_1 + b_2 e_2 + b_3 e_3, U = c_1 e_1 + c_2 e_2 + c_3 e_3 \) and \( V = d_1 e_1 + d_2 e_2 + d_3 e_3 \), where \( a_i, b_i, c_i, d_i \), for all \( i = 1, 2, 3 \) are all non-zero real numbers.

Using the above curvature tensors admitting the semi-symmetric non-metric connection, we obtain

\[ R(X, Y)Z = -2(a_1 b_2 c_1 e_2 + a_1 b_3 c_1 e_3 + a_1 b_4 c_1 e_4 + a_1 b_5 c_1 e_5) = -\bar{R}(Y, X)Z. \]

Therefore, the curvature tensor of a Lorentzian \( \alpha \)-Sasakian manifold admitting a semi-symmetric non-metric connection \( \bar{\nabla} \) is satisfied the skew-symmetric property of the curvature tensors \( \bar{R} \) of \( \bar{\nabla} \). Now, we see that the Ricci Semisymmetric with respect to the semi-symmetric non-metric connections from the above relations as follow: \( \bar{R} \cdot \bar{S} = 0 \), if \( \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \).

The above arguments tell us that the 3-dimensional Lorentzian \( \alpha \)-Sasakian manifolds with respect to the semi-symmetric non-metric connections under consideration agrees with the Section 6. \( \square \)

References


**Author information**

Ajit Barman, Department of Mathematics, Ramthakur College, Agartala-799003, Dist- West Tripura, Tripura, India.

E-mail: ajitbarmanav@yahoo.in

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