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ON LORENTZIAN α-SASAKIAN MANIFOLDS ADMITTING A TYPE OF SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract The object of the present paper is to study a Lorentzian α-Sasakian manifold admitting a semi-symmetric non-metric connection due to Agashe and Chafle [1]. It is shown that if a Lorentzian α-Sasakian manifold is semisymmetric with respect to the semi-symmetric non-metric connections, then the manifold is an η-Einstein manifold provided $\alpha^2 \neq -1$. Among others we prove that a Lorentzian α-Sasakian manifold is Ricci semisymmetric with respect to the semi-symmetric non-metric connections and the Levi-Civita connections are equivalent. Moreover we deal with a three-dimensional Ricci semisymmetric on Lorentzian α-Sasakian manifold with respect to the semi-symmetric non-metric connection is a manifold of constant curvature. Finally, an illustrative example is given to verify our result.

1 Introduction

In [16], Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plain sections containing ξ is a constant, say c. He showed that they can be divided into three classes:

(1.1) homogeneous normal contact Riemannian manifolds with $c > 0$,

(1.2) global Riemannian products of a line or a circle with a Kaehlar manifold of constant holomorphic sectional curvature if $c = 0$ and

(1.3) a warped product space $\mathbb{R} \times_f \mathbb{C}$ if $c < 0$.

It is well known that the manifolds of class (1.1) are characterized by admitting a Sasakian structure. Kenmotsu [13] characterized the differential geometric properties of the manifolds of class (1.3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [13].

In the Gray-Hervella [8], classification of almost Hermitian manifolds there appears a class W_4, of Hermitian manifolds which are closely related to locally conformal Kaehlerian manifolds [6]. An almost contact metric structure on the manifold M is called a trans-Sasakian structure [14] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$ ([12], [13]) coincides with the class of trans-Sasakian structure of type (α, β). We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic, β-Kenmotsu [10] and α-Sasakian [10] respectively. Lorentzian α-Sasakian manifolds have been studied by Yildiz and Murathan [17], Yildiz, Turan and Murathan [19], Bagewadi and Ingalahalli [5], Barman [3], Yildiz, Turan, and Acet [18] and many others.

In 1924, Friedmann and Schouten [7] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\widetilde{\nabla}$ satisfies $T(X,Y) = u(Y)X - u(X)Y$, where u is a 1-form and ρ is a vector field defined by $u(X) = g(X,\rho)$, for all vector fields $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

In 1932, Hayden [9] introduced the idea of semi-symmetric metric connections on a Riemannian manifold (M, g). A semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric metric connection if $\widetilde{\nabla} g = 0$. A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of (M, g) was given by Yano [20]: $\widetilde{\nabla}_X Y = \nabla_X Y + u(Y)X - g(X,Y)\rho$, where $u(X) = g(X, \rho)$.

After a long gap the study of a semi-symmetric connection $\hat{\nabla}$ satisfying $\hat{\nabla} g \neq 0$, was initiated by Prvanović [15] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [2]. The semi-symmetric connection $\hat{\nabla}$ is said to be a semi-symmetric non-metric connection.

In 1992, Agashe and Chafle [1] studied a semi-symmetric non-metric connection $\hat{\nabla}$, whose torsion tensor T satisfies $T(X,Y) = u(Y)X - u(X)Y$ and $(\hat{\nabla}_X g)(Y,Z) = -u(Y)g(X,Z) - u(Z)g(X,Y)$. In 1992, Barua and Mukhopadhyay [4] studied a type of semi-symmetric connection $\hat{\nabla}$ which satisfies $(\hat{\nabla}_X g)(Y,Z) = 2u(X)g(Y,Z) - u(Y)g(X,Z) - u(Z)g(X,Y)$. Since $\hat{\nabla} g \neq 0$, this is another type of semi-symmetric non-metric connection. However, the authors preferred the name semi-symmetric semimetric connection. In 1994, Liang [11] studied another type of semi-symmetric non-metric connection $\hat{\nabla}$ for which we have $(\hat{\nabla}_X g)(Y,Z) = 2u(X)g(Y,Z)$, where u is a non-zero 1-form and he called this a semi-symmetric recurrent metric connection.

A Lorentzian α-Sasakian manifold is said to be a semi-symmetric manifold with respect to the semi-symmetric non-metric connection if it satisfies the relation

$$(\bar{R}(X,Y).\bar{R})(U,V)W = 0$$

holds for all $X, Y, U, V, W \in \chi(M)$, where $\bar{R}(X,Y)$ is the curvature operator.

In this paper we study Lorentzian α-Sasakian manifolds with respect to a semi-symmetric non-metric connection due to Agashe and Chafle [1]. The paper is organized as follows: After introduction in section 2, we give a brief account of the Lorentzian α-Sasakian manifolds. In section 3, we establish the relation of the curvature tensors between the Levi-Civita connection and the semi-symmetric non-metric connection of a Lorentzian α-Sasakian manifold. In the next section deals with a Lorentzian α-Sasakian manifold whose curvature tensor of manifold is covariant constant with respect to the semi-symmetric non-metric connection and manifold is recurrent with respect to the Levi-Civita connection. Section 5, we study semi symmetric on a Lorentzian α-Sasakian manifold with respect to the semi-symmetric non-metric connection, then the manifold is an η-Einstein manifold provided $\alpha^2 \neq -1$. The following statements for Lorentzian α-Sasakian manifolds with respect to a semi-symmetric non-metric connection are equivalent, the manifold is (i) M is an Einstein manifold with respect to the Levi-Civita connection, (ii) Locally Ricci symmetric admitting the Levi-Civita connection and (iii) Ricci semi symmetric on a Lorentzian α-Sasakian manifold with respect to the semi-symmetric non-metric connection have been studied in Section 6. Finally, we construct an example of a 3-dimensional Lorentzian α-Sasakian manifold admitting the semi-symmetric non-metric connection to support the results obtained in Section 3 and Section 6 respectively.

2 Lorentzian α-Sasakian manifolds

A $(2n+1)$-dimensional differentiable manifold M is called Lorentzian α-Sasakian manifolds if it admits a $(1,1)$ tensor field ϕ, a contravarient vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy [17]

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = -1, \quad g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y on M.

Also Lorentzian α-Sasakian manifolds is satisfy [17],

$$\nabla_X \xi = -\alpha\phi X, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y), \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and $\alpha \in \mathbb{R}$.

Further on a Lorentzian α-Sasakian manifold M the following relations holds [17]:

$$\eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.6}$$

$$R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X], \tag{2.7}$$

$$R(\xi, X)\xi = \alpha^2[\eta(X)\xi + X], \tag{2.8}$$

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$S(X, \xi) = 2n\alpha^2\eta(X), \tag{2.10}$$

$$(\nabla_X \phi)(Y) = \alpha^2[g(X, Y)\xi - \eta(Y)X], \tag{2.11}$$

where R and S are the curvature tensor and the Ricci tensor of the Levi-Civita connection respectively.

3 Curvature tensor of a Lorentzian α-Sasakian manifold with respect to the semi-symmetric non-metric connection

Proposition 3.1. *For a Lorentzian α-Sasakian manifold M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$*

(i) The curvature tensor $\bar{R}$ is given by $\bar{R}(X, Y)Z = R(X, Y)Z - \alpha g(X, \phi Z)Y - \eta(X)\eta(Z)Y + \alpha g(Y, \phi Z)X + \eta(Y)\eta(Z)X$,

(ii) The Ricci tensor $\bar{S}$ is given by $\bar{S}(Y, Z) = S(Y, Z) + 2n\alpha g(Y, \phi Z) + 2n\eta(Y)\eta(Z)$,

(iii)$\bar{R}(X, Y)Z = -\bar{R}(Y, X)Z$,

(iv) The Ricci tensor $\bar{S}$ is symmetric,

(v) $\bar{S}(Y, \xi) = 2n(\alpha^2 - 1)\eta(Y)$,

(vi) $\bar{R}(\xi, Y)Z = \alpha^2 g(Y, Z)\xi + (\alpha^2 + 1)\eta(Z)Y + \alpha g(Y, \phi Z)\xi + \eta(Y)\eta(Z)\xi$,

(vii)$\bar{R}(\xi, \xi)Z = 2\alpha^2\eta(Z)\xi$,

(viii)$\bar{R}(\xi, Y)\xi = (\alpha^2 - 1)\eta(Y)\xi - (\alpha^2 + 1)Y$,

(ix) $(\bar{\nabla}_W \eta)(X) = -\alpha g(X, \phi W) - \eta(X)\eta(W)$.

Proof. Let M be an $(2n+1)$-dimensional Riemannian manifold with Riemannian metric g. If $\bar{\nabla}$ is the semi-symmetric non-metric connection of a Riemannian manifold M, a linear connection $\bar{\nabla}$ is given by [1]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X. \tag{3.1}$$

Then $\bar{R}$ and R are related by [1]

$$\bar{R}(X,Y)Z = R(X,Y)Z + \gamma(X,Z)Y - \gamma(Y,Z)X, \tag{3.2}$$

for all vector fields X, Y, Z on M, where γ is a $(0,2)$ tensor field denoted by

$$\gamma(X,Z) = (\nabla_X \eta)(Z) - \eta(X)\eta(Z). \tag{3.3}$$

Taking inner product of (3.2) with respect to U, we get

$$\begin{aligned}\tilde{\bar{R}}(X,Y,Z,U) = \tilde{R}(X,Y,Z,U) + \gamma(X,Z)g(Y,U) - \\ \gamma(Y,Z)g(X,U),\end{aligned} \tag{3.4}$$

where

$$\tilde{\bar{R}}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U) \;\; and \;\; \tilde{R}(X,Y,Z,U) = g(R(X,Y)Z,U).$$

From (3.1) yields that

$$(\bar{\nabla}_W g)(X,Y) = -\eta(X)g(Y,W) - \eta(Y)g(X,W) \neq 0. \tag{3.5}$$

Using (2.5) in (3.3), we get

$$\gamma(X,Z) = -\alpha g(X,\phi Z) - \eta(X)\eta(Z). \tag{3.6}$$

By making use of (3.6) and (3.2), we derived that

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z - \alpha g(X,\phi Z)Y - \eta(X)\eta(Z)Y \; + \alpha g(Y,\phi Z)X \\ +\eta(Y)\eta(Z)X.\end{aligned} \tag{3.7}$$

From (3.7) yields that

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z. \tag{3.8}$$

We consider $X = \xi$ in (3.7) and using (2.1) and (2.7), we obtain

$$\begin{aligned}\bar{R}(\xi,Y)Z = \alpha^2 g(Y,Z)\xi + (\alpha^2+1)\eta(Z)Y + \alpha g(Y,\phi Z)\xi \\ +\eta(Y)\eta(Z)\xi.\end{aligned} \tag{3.9}$$

We select $Y = \xi$ in (3.9) and using (2.1), it follows that

$$\bar{R}(\xi,\xi)Z = 2\alpha^2\eta(Z)\xi. \tag{3.10}$$

Again putting $Z = \xi$ in (3.9) and using (2.1), we get

$$\bar{R}(\xi,Y)\xi = (\alpha^2-1)\eta(Y)\xi - (\alpha^2+1)Y. \tag{3.11}$$

Taking the inner product of (3.7) with U, it is obvious that

$$\begin{aligned}\tilde{\bar{R}}(X,Y,Z,U) = \tilde{R}(X,Y,Z,U) - \alpha g(X,\phi Z)g(Y,U) - \eta(X)\eta(Z)g(Y,U) \\ +\alpha g(Y,\phi Z)g(X,U) + \eta(Y)\eta(Z)g(X,U).\end{aligned} \tag{3.12}$$

Taking a frame field from (3.12), we obtain

$$\bar{S}(Y,Z) = S(Y,Z) + 2n\alpha g(Y,\phi Z) + 2n\eta(Y)\eta(Z). \tag{3.13}$$

From (3.13), implies that

$$\bar{S}(Y,Z) = \bar{S}(Z,Y). \tag{3.14}$$

We take $Z = \xi$ in (3.13) and using (2.1) and (2.10), it follows that

$$\bar{S}(Y,\xi) = 2n(\alpha^2 - 1)\eta(Y). \tag{3.15}$$

Combining (3.1) and (2.5), we get

$$(\bar{\nabla}_W \eta)(X) = -\alpha g(X,\phi W) - \eta(X)\eta(W).$$

This Proposition 3.1 completes the proof. □

4 The curvature tensor is covariant constant with respect to the semi-symmetric non-metric connection and M is recurrent with respect to the Levi-Civita connection

Definition 4.1. A Lorentzian α-Sasakian manifold M with respect to the Levi-Civita connection is called recurrent if its curvature tensor R satisfies the condition

$$(\nabla_W R)(X,Y)Z = \eta(W)R(X,Y)Z, \tag{4.1}$$

where η be the 1-form.

Definition 4.2. A Lorentzian α-Sasakian manifold M is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(Z,W) = ag(Z,W) + b\eta(Z)\eta(W), \tag{4.2}$$

where a and b are smooth functions on the manifold.

Theorem 4.3. *If in an $(2n+1)$-dimensional Lorentzian α-Sasakian manifold the curvature tensor of manifold is covariant constant with respect to the semi-symmetric non-metric connection and the manifold is recurrent with respect to the Levi-Civita connection, then the manifold is an η-Einstein manifold.*

Proof. From (3.1), implies that

$$(\bar{\nabla}_W R)(X,Y)Z = (\nabla_W R)(X,Y)Z + \eta(R(X,Y)Z)W. \tag{4.3}$$

Using (2.6) and (4.1) in (4.3), we can write

$$\begin{aligned}(\bar{\nabla}_W R)(X,Y)Z = \eta(W)R(X,Y)Z &+ \alpha^2 g(Y,Z)\eta(X)W \\ &-\alpha^2 g(X,Z)\eta(Y)W.\end{aligned} \tag{4.4}$$

Suppose $(\bar{\nabla}_W R)(X,Y)Z = 0$, then from (4.4), we have

$$\eta(W)R(X,Y)Z + \alpha^2 g(Y,Z)\eta(X)W - \alpha^2 g(X,Z)\eta(Y)W = 0. \tag{4.5}$$

Now contracting X in (4.5) and using (2.1), we obtain

$$\eta(W)S(Y,Z) + \alpha^2 g(Y,Z)\eta(W) - \alpha^2 g(W,Z)\eta(Y) = 0. \tag{4.6}$$

Putting $W = \xi$ in (4.6) and using (2.1), we derived that

$$S(Y,Z) = -\alpha^2 g(Y,Z) - \alpha^2\eta(Z)\eta(Y). \tag{4.7}$$

Therefore, $S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z)$, where $a = -\alpha^2$ and $b = -\alpha^2$. From which it follows that the manifold is an η-Einstein manifold. This completes the proof. □

5 Semisymmetric Lorentzian α-Sasakian manifolds with respect to the semi-symmetric non-metric connection

In this section we suppose that the manifold under consideration is semi-symmetric with respect to the semi-symmetric non-metric connection M^{2n+1}, that is,

$$(\bar{R}(U,V).\bar{R})(X,Y)Z = 0$$

Theorem 5.1. *If a $(2n+1)$-dimensional Lorentzian α-Sasakian manifold is semisymmetric with respect to the semi-symmetric non-metric connection then the manifold is an η-Einstein manifold provided $\alpha^2 \neq -1$.*

Proof. Then we have

$$\begin{aligned} \bar{R}(U,V)\bar{R}(X,Y)Z - \bar{R}(\bar{R}(U,V)X,Y)Z - \bar{R}(X,\bar{R}(U,V)Y)Z \\ -\bar{R}(X,Y)\bar{R}(U,V)Z = 0. \end{aligned} \tag{5.1}$$

Putting $U = \xi$ in (5.1), it follows that

$$\begin{aligned} \bar{R}(\xi,V)\bar{R}(X,Y)Z - \bar{R}(\bar{R}(\xi,V)X,Y)Z - \bar{R}(X,\bar{R}(\xi,V)Y)Z \\ -\bar{R}(X,Y)\bar{R}(\xi,V)Z = 0. \end{aligned} \tag{5.2}$$

By making use of (3.9) in (5.2), we obtain that

$$\begin{aligned} \bar{R}(\xi,V)\bar{R}(X,Y)Z - \alpha^2 g(X,V)\bar{R}(\xi,Y)Z - (\alpha^2+1)\eta(X)\bar{R}(V,Y)Z \\ -\alpha g(X,\phi V)\bar{R}(\xi,Y)Z - \eta(X)\eta(V)\bar{R}(\xi,Y)Z - \alpha^2 g(Y,V)\bar{R}(X,\xi)Z \\ -(\alpha^2+1)\eta(Y)\bar{R}(X,V)Z - \alpha g(Y,\phi V)\bar{R}(X,\xi)Z - \eta(Y)\eta(V)\bar{R}(X,\xi)Z \\ -\alpha^2 g(V,Z)\bar{R}(X,Y)\xi - (\alpha^2+1)\eta(Z)\bar{R}(X,Y)V - \alpha g(V,\phi Z)\bar{R}(X,Y)\xi \\ -\eta(V)\eta(Z)\bar{R}(X,Y)\xi = 0. \end{aligned} \tag{5.3}$$

We take $X = \xi$ in (5.3) and using (3.9), (3.10) and (3.11), we get

$$\begin{aligned} (\alpha^2+1)\bar{R}(V,Y)Z - \alpha^2(\alpha^2+1)g(Y,Z)V - 2\alpha^4\eta(Z)g(Y,V)\xi - 2\alpha^3\eta(Z)g(Y,\phi V)\xi \\ +\alpha(\alpha+1)\eta(Y)\eta(V)\eta(Z)\xi - \alpha(\alpha^2+1)g(Y,\phi Z)V - (\alpha^2+1)\eta(Y)\eta(Z)V \\ -\alpha^2(2\alpha^2+1)\eta(Z)\eta(V)Y - 2\alpha^4\eta(Y)g(V,Z)\xi - 2\alpha^3\eta(Y)g(Z,\phi V)\xi \\ +\alpha^2(\alpha^2+1)g(Z,V)Y \\ +\alpha(\alpha^2+1)g(Z,\phi V)Y = 0. \end{aligned} \tag{5.4}$$

Taking a frame field from (5.4) and using (2.1), we have

$$\begin{aligned} (\alpha^2+1)\bar{S}(Y,Z) - 2n\alpha^2(\alpha^2+1)g(Y,Z) \\ -[6\alpha^4 + (3+2n)\alpha^2 + \alpha + 2n + 1]\eta(Y)\eta(Z) \\ -2n\alpha(\alpha^2+1)g(Y,\phi Z) = 0. \end{aligned} \tag{5.5}$$

Using (3.13) in (5.5), we obtain

$$S(Y,Z) = [\frac{6\alpha^4 + 3\alpha^2 + \alpha + 1}{\alpha^2+1}]\eta(Y)\eta(Z) + 2n\alpha^2 g(Y,Z).$$

Therefore, $S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z)$, where $a = 2n\alpha^2$ and $b = \frac{6\alpha^4+3\alpha^2+\alpha+1}{\alpha^2+1}$.

This result shows that the manifold is an η-Einstein manifold. This Theorem 5.1 completes the proof. □

6 Ricci semisymmetric Lorentzian α-Sasakian manifolds admitting a semi-symmetric non-metric connection

In this section we characterize Ricci semisymmetric $\bar{R} \cdot \bar{S}$ on a Lorentzian α-Sasakian manifold admitting a special type of semi-symmetric non-metric connection $\bar{\nabla}$.

Definition 6.1. A Lorentzian α-Sasakian manifold is Ricci semisymmetric with respect to the Levi-Civita connection ∇, that is, $(R(X,Y) \cdot S)(U,V) = 0$.

Theorem 6.2. *A Lorentzian α-Sasakian manifold is Ricci semisymmetric with respect to a semi-symmetric non-metric connection iff the manifold is also Ricci semisymmetric with respect to the Levi-Civita connection.*

Proof. Then from the above equation, we can write

$$\bar{R} \cdot \bar{S} = \bar{S}(\bar{R}(X,Y)U, V) + \bar{S}(U, \bar{R}(X,Y)V) \tag{6.1}$$

Putting $U = \xi$ in (6.1) and using (2.1), (3.7) and (3.13), it follows that

$$\begin{aligned} \bar{R} \cdot \bar{S} = R \cdot S + 2n\alpha g(R(X,Y)\xi, \phi V) + 2n\eta(V)\eta(R(X,Y)\xi) + 2n\eta(R(X,Y)V) \\ -\eta(X)\bar{S}(Y,V) - \eta(Y)\bar{S}(X,V) + 2n(\alpha^2 - 1)\eta(Y)[g(X,\phi V) + \eta(X)\eta(V)] \\ -2n(\alpha^2 - 1)\eta(X)[g(Y,\phi V) + \eta(Y)\eta(V)]. \end{aligned} \tag{6.2}$$

We take $V = X = \xi$ in (6.2) and using (2.1), (2.9) and (3.7), we obtain

$$\bar{R} \cdot \bar{S} = R \cdot S.$$

This completes the proof. □

Lemma 6.3. [18] *A three-dimensional Ricci semisymmetric Lorentzian α-Sasakian manifold is a manifold of constant curvature.*

Therefore, from Theorem (6.2) and Lemma (6.3) we can state the following theorem:

Theorem 6.4. *A three-dimensional Ricci semisymmetric Lorentzian α-Sasakian manifold with respect to a semi-symmetric non-metric connection is a manifold of constant curvature.*

Lemma 6.5. [5] *The following statements for Lorentzian α-Sasakian manifolds are equivalent. The manifold is*
i) M is an Einstein manifold
ii) Locally Ricci symmetric
iii) Ricci semisymmetric that is $R(X,Y) \cdot S = 0$.

Hence, from Theorem (6.2) and Lemma (6.5) we can state the following theorem:

Theorem 6.6. *The following statements for Lorentzian α-Sasakian manifolds with respect to a semi-symmetric non-metric connection are equivalent. The manifold is*
i) M is an Einstein manifold with respect to the Levi-Civita connection
ii) Locally Ricci symmetric admitting the Levi-Civita connection
iii) $\bar{R}(X,Y) \cdot \bar{S} = 0$.

7 Example

Now, we give an example of a 3-dimensional Lorentzian α-Sasakian manifold admitting a semi-symmetric non-metric connection $\bar{\nabla}$, which verify the result of section 6.

We consider a 3-dimensional manifold $M = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinate in R^3. We choose the vector fields

$$e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad e_3 = \alpha \frac{\partial}{\partial z}$$

which are linearly independent at each point of M and α is constant.

Let g be the Lorentzian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$$

and

$$g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1,$$

that is, the form of the metric becomes

$$g = \frac{1}{(e^z)^2}(dy)^2 - \frac{1}{\alpha^2}(dz)^2,$$

which is a Lorentzian metric.

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_3)$$

for any $Z \in \chi(M)$.

Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi e_1 = -e_1, \ \phi e_2 = -e_2, \ \phi e_3 = 0$$

Using the linearity of ϕ and g, we have

$$\eta(e_3) = -1$$

$$\phi^2(Z) = Z + \eta(Z)e_3$$

and

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W)$$

for any $U, W \in \chi(M)$.

Then we have

$$[e_1, e_2] = 0, \ [e_1, e_3] = -\alpha e_1, \ [e_2, e_3] = -\alpha e_2$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$2g(\nabla_X Y, W) = Xg(Y, W) + Yg(X, W) - Wg(X, Y) - g(X, [Y, W])$$
$$-g(Y, [X, W]) + g(W, [X, Y]).$$

Using Koszul's formula we get the following

$$\nabla_{e_1} e_1 = -\alpha e_3, \ \nabla_{e_1} e_2 = 0, \ \nabla_{e_1} e_3 = -\alpha e_1,$$

$$\nabla_{e_2} e_1 = 0, \ \nabla_{e_2} e_2 = -\alpha e_3, \ \nabla_{e_2} e_3 = -\alpha e_2,$$

$$\nabla_{e_3} e_1 = 0, \ \nabla_{e_3} e_2 = 0, \ \nabla_{e_3} e_3 = 0.$$

In view of the above relations, we see that $\nabla_X \xi = -\alpha \phi X$, $(\nabla_X \eta)Y = -\alpha g(\phi X, Y)\xi$, for all $e_3 = \xi$. Therefore the manifold is a Lorentzian α-Sasakian manifold with the structure (ϕ, ξ, η, g).

Using (3.1) in above equation, we obtain

$$\bar{\nabla}_{e_1} e_1 = -\alpha e_3, \ \bar{\nabla}_{e_1} e_2 = 0, \ \bar{\nabla}_{e_1} e_3 = -(1 + \alpha)e_1,$$

$$\bar{\nabla}_{e_2}e_1 = 0,\ \bar{\nabla}_{e_2}e_2 = -\alpha e_3,\ \bar{\nabla}_{e_2}e_3 = -(1+\alpha)e_2,$$
$$\bar{\nabla}_{e_3}e_1 = 0,\ \bar{\nabla}_{e_3}e_2 = 0,\ \bar{\nabla}_{e_3}e_3 = -e_3.$$

Now, we can easily obtain the non-zero components of the curvature tensors as follows:

$$R(e_1,e_2)e_2 = -\alpha^2 e_2,\ \ R(e_1,e_3)e_3 = -\alpha^2 e_1,\ \ R(e_2,e_1)e_1 = \alpha^2 e_2,$$
$$R(e_2,e_3)e_3 = -\alpha^2 e_2,\ \ R(e_3,e_1)e_1 = \alpha^2 e_3,\ \ R(e_3,e_2)e_2 = \alpha^2 e_3,$$
$$R(e_1,e_2)e_3 = 0,\ \ R(e_2,e_3)e_2 = -\alpha^2 e_3,\ \ R(e_1,e_2)e_2 = \alpha^2 e_1,$$

and

$$\bar{R}(e_1,e_2)e_2 = \alpha(1+\alpha)(e_1-e_2),\ \bar{R}(e_1,e_3)e_1 = -\alpha(1+\alpha)e_3,$$
$$\bar{R}(e_2,e_3)e_2 = -\alpha(1+\alpha)e_3,\ \ \bar{R}(e_1,e_2)e_1 = -\alpha(1+\alpha)e_2,$$
$$\bar{R}(e_2,e_3)e_3 = (1-\alpha^2)e_2,$$

With the help of the above curvature tensors with respect to a semi-symmetric non-metric connection, we find the Ricci tensors as follows:

$$\bar{S}(e_1,e_1) = \bar{S}(e_2,e_2) = \alpha(1+\alpha), \bar{S}(e_3,e_3) = (1-\alpha^2),$$

Let X, Y, U and V be any four vector fields given by
$X = a_1e_1+a_2e_2+a_3e_3, Y = b_1e_1+b_2e_2+b_3e_3, U = c_1e_1+c_2e_2+c_3e_3$ and $V = d_1e_1+d_2e_2+d_3e_3$, where a_i, b_i, c_i, d_i, for all $i = 1,2,3$ are all non-zero real numbers.

Using the above curvature tensors admitting the semi-symmetric non-metric connection, we obtain

$$\bar{R}(X,Y)Z = -2(a_1b_2c_1e_2 + a_1b_3c_1e_3 + a_1b_4c_1e_4 + a_1b_5c_1e_5) = -\bar{R}(Y,X)Z.$$

Therefore, the curvature tensor of a Lorentzian α-Sasakian manifold admitting a semi-symmetric non-metric connection $\bar{\nabla}$ is satisfied the skew-symmetric property of the curvature tensors $\bar{R}$ of $\bar{\nabla}$. Now, we see that the Ricci Semisymmetric with respect to the semi-symmetric non-metric connections from the above relations as follow: $\bar{R}\cdot\bar{S} = 0$, if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

The above arguments tell us that the 3-dimensional Lorentzian α-Sasakian manifolds with respect to the semi-symmetric non-metric connections under consideration agrees with the Section 6. □

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