Palestine Journal of Mathematics

Vol. 9(2)(2020) , 858–865

Indicatrices of Curves in Affine 3-Space

Yılmaz Tunçer, Hüseyin Kocayiğit and Murat Kemal Karacan
Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53A15; Secondary 53A04.

Keywords and phrases: Affine Space, indicatrix curve, equi-affine frame.

Abstract In this study, we investigated the tangent, affine normal and binormal indicatrix curves of space curves in affine 3-space in both general case and in special case of space curve is constant curvature curve..

1 Introduction

In a set of points that corresponds a vector of the vector space constructed on a field is called an *affine space associated with the vector space*. We denote A_3 as affine 3-space associated with the vector space IR^3. In theory of curves in ordinary affine space developed by E. Salkowski and affine differential geometry investigates invariants with respect to the group of those affine transformations

$$x_j^* = \sum_{k=1}^{3} a_{jk}x_k + c_j, j = 1, 2, 3 \tag{1.1}$$

which are volume-preserving (det$(a_{jk}) = 1$). Such a transformation is said to be equiaffine, and affine differential geometry is also known as equiaffine differential geometry [5].

Let $\alpha : J \longrightarrow A_3$ be a curve in A_3, where $J = (t_1, t_2) \subset IR$ is a fixed open interval. We assume that $\alpha \in C^4(J)$ and $\begin{vmatrix} \dot{\alpha} & \ddot{\alpha} & \dddot{\alpha} \end{vmatrix} \neq 0$ on J, where $\dot{\alpha} = d\alpha/dt$, etc. Then with α, we may associate the invariant parameter

$$\sigma(t) = \int_{t_1}^{t} \begin{vmatrix} \dot{\alpha} & \ddot{\alpha} & \dddot{\alpha} \end{vmatrix}^{1/6} dt \tag{1.2}$$

which is called the affine arc length of α. The coordinates of a curve are given by three linearly independent solutions of the equations

$$\alpha^{(iv)}(s) + \kappa(s)\alpha''(s) + \tau_\alpha(s)\alpha'(s) = 0 \tag{1.3}$$

under the condition,

$$\begin{vmatrix} \alpha'(s) & \alpha''(s) & \alpha'''(s) \end{vmatrix} = 1 \tag{1.4}$$

where $\kappa(s)$ and $\tau_\alpha(s)$ denote the affine curvatures.

A trihedron of $\alpha(s)$ consist of the tangent vector $t(s) = \alpha'(s)$, the affine normal vector $n(s) = \alpha''(s)$, and the affine binormal vector $b(s) = \alpha'''(s)$, here $\alpha'(s) = d\alpha/ds$, etc. The vectors $n(s)$ and $b(s)$ span the affine normal plane. The equi-affine frame formulas are

$$\begin{aligned} t'(s) &= n(s) \\ n'(s) &= b(s) \\ b'(s) &= -\tau_\alpha(s)t(s) - \kappa(s)n(s). \end{aligned} \tag{1.5}$$

They involve the invariants

$$\kappa(s) = \begin{vmatrix} \alpha'(s) & \alpha'''(s) & \alpha^{iv}(s) \end{vmatrix}, \quad \tau_\alpha(s) = -\begin{vmatrix} \alpha''(s) & \alpha'''(s) & \alpha^{iv}(s) \end{vmatrix} \tag{1.6}$$

which are called the affine curvature and torsion of α. E. Kreyszig and A. Pendl gave spherical curves and their analogues in affine differential geometry and B. Su gave some classes of affine curves[1, 2, 5, 6]. N. Hu obtained the some curves with constant curvature and gave the following important theorem by using Shengjin's formulae[4].

Theorem 1.1. *Any nondegenerate equiaffine space curve $\alpha(s)$ with constant equiaffine curvature $\kappa(s) := \kappa$, $\tau(s) := \tau$ is equiaffinely equivalent to one of the following curves:*

i. $\alpha(s) = (s, \frac{1}{2}s^2, \frac{1}{2}s^3)$

ii. $\alpha(s) = (e^{\sigma s}, se^{\sigma s}, -\frac{1}{18\sigma^5}e^{-2\sigma s})$ *where* $\sigma = \left\{\frac{\kappa}{2}\right\}^{1/3}$,

iii. *If* $\kappa = 0$,

$$\alpha(s) = \tau(-\tau^{1/2}s, \sin(\tau^{1/2}s), \cos(\tau^{1/2}s))$$

if $\kappa \neq 0$,

$$\alpha(s) = (\frac{1}{2\sigma_1\sigma_2(9\sigma_1^2+\sigma_2^2)(\sigma_1^2+\sigma_2^2)}e^{-2\sigma_1 s}, e^{\sigma_1 s}\sin(\sigma_2 s), e^{\sigma_1 s}\cos(\sigma_2 s)),$$

where

$$\begin{aligned}
\sigma_1 &= \frac{1}{6}\left\{\sqrt[3]{\frac{3(9\kappa+\sqrt{12\tau^3+81\kappa^2})}{2}} + \sqrt[3]{\frac{3(9\kappa-\sqrt{12\tau^3+81\kappa^2})}{2}}\right\}, \\
\sigma_2 &= \frac{\sqrt{3}}{6}\left\{\sqrt[3]{\frac{3(9\kappa+\sqrt{12\tau^3+81\kappa^2})}{2}} - \sqrt[3]{\frac{3(9\kappa-\sqrt{12\tau^3+81\kappa^2})}{2}}\right\},
\end{aligned}$$

iv. *If* $\kappa = 0$,

$$\alpha(s) = -\tau^{1/2}(-\tau^{1/2}s, \sinh((-\tau)^{1/2}s), \cosh((-\tau)^{1/2}s)),$$

if $\kappa \neq 0$,

$$\alpha(s) = (\frac{1}{4\sigma_3\sigma_4(9\sigma_3^2-\sigma_4^2)(\sigma_3^2-\sigma_4^2)}e^{-2\sigma_3 s}, e^{(\sigma_3+\sigma_4)s}, e^{(\sigma_3-\sigma_4)s})$$

for which $\tau < 0$, $\frac{27}{2}\kappa(-3\tau)^{-3/2} \in (-1, 1)$, *where*

$$\begin{aligned}
\sigma_3 &= \frac{1}{3}\sqrt{-3\tau}\cos\left(\frac{1}{3}\arccos\left(\frac{27}{2}\kappa(-3\tau)^{-3/2}\right)\right), \\
\sigma_4 &= \sqrt{-\tau}\sin\left(\frac{1}{3}\arccos\left(\frac{27}{2}\kappa(-3\tau)^{-3/2}\right)\right).
\end{aligned}$$

2 Indicatrices of Affine Space Curves

Definition 2.1. Let $\alpha : J \longrightarrow A_3$ be a regular curve in affine 3-space with the affine frame defined $\{t_\alpha(s), n_\alpha(s), b_\alpha(s)\}$ at each point of $\alpha(s)$ and s be the affine arclength parameter. The curve $\beta(\widetilde{s}) = t_\alpha(s)$ with new arclength parameter $\widetilde{s}$, is called affine tangent indicatrix of $\alpha(s)$.

Let affine frame of $\beta(\widetilde{s})$ be $\{t_\beta(\widetilde{s}), n_\beta(\widetilde{s}), b_\beta(\widetilde{s})\}$ and the affine curvature and affine torsion be $\kappa_\beta(\widetilde{s})$ and $\tau_\beta(\widetilde{s})$, respectively. Thus affine frame vectors of $\beta(\widetilde{s})$ can be obtained as

$$\begin{aligned}
\frac{d\beta(\widetilde{s})}{d\widetilde{s}} &= \sigma n_\alpha(s) \\
\frac{d^2\beta(\widetilde{s})}{d\widetilde{s}^2} &= \sigma\sigma' n_\alpha(s) + \sigma^2 b_\alpha(s) \qquad (2.1) \\
\frac{d^3\beta(\widetilde{s})}{d\widetilde{s}^3} &= \left\{\begin{array}{c} -\sigma^3\tau_\alpha(s)t_\alpha(s) + \sigma\left\{\sigma\sigma'' + (\sigma')^2 - \sigma^2\kappa_\alpha(s)\right\}n_\alpha(s) \\ +3\sigma^2\sigma' b_\alpha(s) \end{array}\right\}
\end{aligned}$$

where the parametrization $\sigma = ds/d\widetilde{s}$ satisfies

$$\sigma^4(\sigma')^2\tau_\alpha(s) = -1. \tag{2.2}$$

For the curve $\beta(\widetilde{s})$, we have the relation

$$\frac{d^4\beta(\widetilde{s})}{d\widetilde{s}^4} = -\tau_\beta(\widetilde{s})\,t_\beta(\widetilde{s}) - \kappa_\beta(\widetilde{s})\,n_\beta(\widetilde{s}) \tag{2.3}$$

and its derivatives can be obtained as follows with respect to frame apparatus of $\alpha(s)$

$$\frac{d\beta(\widetilde{s})}{d\widetilde{s}} = \sigma n_\alpha(s)$$

$$\frac{d^2\beta(\widetilde{s})}{d\widetilde{s}^2} = \sigma\sigma' n_\alpha(s) + \sigma^2 b_\alpha(s)$$

$$\frac{d^3\beta(\widetilde{s})}{d\widetilde{s}^3} = -\sigma^3\tau_\alpha(s)\,t_\alpha(s) + \sigma\left\{\sigma\sigma'' + (\sigma')^2 - \sigma^2\kappa_\alpha(s)\right\}n_\alpha(s) + 3\sigma^2\sigma' b_\alpha(s)$$

and

$$\frac{d^4\beta(\widetilde{s})}{d\widetilde{s}^4} = \left\{\begin{array}{c} -\sigma\left\{3\sigma'\sigma^2\tau_\alpha(s) + \sigma^3\tau'_\alpha(s) + 3\sigma'\sigma\tau_\alpha(s)\right\}t_\alpha(s) \\ +\sigma\left\{\begin{array}{c} 4\sigma\sigma'\sigma'' - 6\sigma^2\sigma'\kappa_\alpha(s) + (\sigma')^3 \\ -\sigma^3\tau_\alpha(s) - \sigma^3\kappa'_\alpha(s) + \sigma^2\sigma''' \end{array}\right\}n_\alpha(s) \\ +\sigma\left\{7\sigma(\sigma')^2 + 4\sigma^2\sigma'' - \sigma^3\kappa_\alpha(s)\right\}b_\alpha(s) \end{array}\right\}. \tag{2.4}$$

From the equations (2.1) and (2.3) we can rewrite

$$\frac{d^4\beta(\widetilde{s})}{d\widetilde{s}^4} = -\sigma(\tau_\beta(\widetilde{s}) + \kappa_\beta(\widetilde{s})^2\sigma)n_\alpha(s) - \sigma^2\kappa_\beta(\widetilde{s})\,b_\alpha(s) \tag{2.5}$$

so, we obtain

$$\left\{\begin{array}{c} -\left\{3\sigma'\sigma^2\tau_\alpha(s) + \sigma^3\tau'_\alpha(s) + 3\sigma'\sigma\tau_\alpha(s)\right\}t_\alpha(s) \\ +\left\{\begin{array}{c} 4\sigma\sigma'\sigma'' - 6\sigma^2\sigma'\kappa_\alpha(s) + (\sigma')^3 - \sigma^3\tau_\alpha(s) \\ -\sigma^3\kappa'_\alpha(s) + \sigma^2\sigma''' + (\tau_\beta(\widetilde{s}) + \kappa_\beta(\widetilde{s})^2\sigma') \end{array}\right\}n_\alpha(s) \\ +\left\{7\sigma(\sigma')^2 + 4\sigma^2\sigma'' - \sigma^3\kappa_\alpha(s) + \sigma\kappa_\beta(\widetilde{s})\right\}b_\alpha(s) \end{array}\right\} = 0$$

by using (2.4) and so we have the following equations

$$7\sigma(\sigma')^2 + 4\sigma^2\sigma'' - \sigma^3\kappa_\alpha(s) + \sigma\kappa_\beta(\widetilde{s}) = 0 \tag{2.6}$$

$$\left\{\begin{array}{c} 4\sigma\sigma'\sigma'' - 6\sigma^2\sigma'\kappa_\alpha(s) + (\sigma')^3 - \sigma^3\tau_\alpha(s) - \sigma^3\kappa'_\alpha(s) \\ +\sigma^2\sigma''' + \tau_\beta(\widetilde{s}) + \kappa_\beta(\widetilde{s})^2\sigma' \end{array}\right\} = 0 \tag{2.7}$$

$$3\sigma'\sigma^2\tau_\alpha(s) + \sigma^3\tau'_\alpha(s) + 3\sigma'\sigma\tau_\alpha(s) = 0. \tag{2.8}$$

From (2.6) and (2.7) affine curvature and affine torsion of $\beta(\widetilde{s})$ are

$$\kappa_\beta(\widetilde{s}) = \sigma^2\kappa_\alpha(s) - 7(\sigma')^2 - 4\sigma\sigma''$$

$$\tau_\beta(\widetilde{s}) = \left\{\begin{array}{c} \sigma^3\tau_\alpha(s) + \sigma^3\kappa'_\alpha(s) + \sigma^2\sigma'\left\{6 + 14(\sigma')^2 + 8\sigma\sigma'' - \sigma^2\kappa_\alpha(s)\right\}\kappa_\alpha(s) \\ -(\sigma')^3 - \sigma^2\sigma''' - 49(\sigma')^5 - 16\sigma^2\sigma'(\sigma'')^2 - 56\sigma(\sigma')^3\sigma'' - 4\sigma\sigma'\sigma'' \end{array}\right\}.$$

Also from (2.2) and (2.8), the parametrization σ have satisfied the relation

$$0 = \left\{4\sigma^4 - 3\sigma - 3\right\}(\sigma')^2 + 2\sigma^5\sigma''.$$

Thus, we gave the following theorem.

Theorem 2.2. *Let $\alpha(s)$ be a regular curve and its tangent indicatrix be $\beta(\widetilde{s})$. Then affine frame is*

$$
\begin{aligned}
t_\beta(\widetilde{s}) &= \sigma n_\alpha(s) \\
n_\beta(\widetilde{s}) &= \sigma\sigma' n_\alpha(s) + \sigma^2 b_\alpha(s) \\
b_\beta(\widetilde{s}) &= \left\{ \begin{array}{c} -\sigma^3 \tau_\alpha(s) t_\alpha(s) + \sigma\left\{\sigma\sigma'' + (\sigma')^2 - \sigma^2 \kappa_\alpha(s)\right\} n_\alpha(s) \\ +3\sigma^2\sigma' b_\alpha(s) \end{array} \right\}
\end{aligned}
$$

and the affine curvatures are

$$
\begin{aligned}
\widetilde{\kappa}_\beta(\widetilde{s}) &= \sigma^2\kappa_\alpha(s) - 7(\sigma')^2 - 4\sigma\sigma'' \\
\widetilde{\tau}_\beta(\widetilde{s}) &= \left\{ \begin{array}{c} \sigma^3\tau_\alpha(s) + \sigma^3\kappa'_\alpha(s) + \sigma^2\sigma'\left\{6 + 14(\sigma')^2 + 8\sigma\sigma'' - \sigma^2\kappa_\alpha(s)\right\}\kappa_\alpha(s) \\ -(\sigma')^3 - \sigma^2\sigma''' - 49(\sigma')^5 - 16\sigma^2\sigma'(\sigma'')^2 - 56\sigma(\sigma')^3\sigma'' - 4\sigma\sigma'\sigma'' \end{array} \right\}.
\end{aligned}
$$

with the parametrization $\sigma = ds/d\widetilde{s}$ which is satisfy the equations

$$
\begin{aligned}
\sigma^4(\sigma')^2\tau_\alpha(s) &= -1 \\
\left\{4\sigma^4 - 3\sigma - 3\right\}(\sigma')^2 + 2\sigma^5\sigma'' &= 0.
\end{aligned}
$$

In the special case of $\alpha(s)$ with the constant curvatures,

Definition 2.3. Let $\alpha : J \longrightarrow A_3$ be a regular curve in affine 3-space with the affine frame defined $\{t_\alpha(s), n_\alpha(s), b_\alpha(s)\}$ at each point of $\alpha(s)$ and s be the affine arclength parameter. The curve $\gamma(\widetilde{s}) = n_\alpha(s)$ with new arclength parameter $\widetilde{s}$, is called affine normal indicatrix of $\alpha(s)$.

Let affine Frenet frame of $\gamma(\widetilde{s})$ be $\{t_\gamma(\widetilde{s}), n_\gamma(\widetilde{s}), b_\gamma(\widetilde{s})\}$ and the affine curvature and affine torsion be $\kappa_\gamma(\widetilde{s})$ and $\tau_\gamma(\widetilde{s})$, respectively.Thus affine frame vectors of $\gamma(\widetilde{s})$ can be obtained as

$$
\begin{aligned}
\frac{d\gamma(\widetilde{s})}{d\widetilde{s}} &= \sigma b_\alpha(s) \\
\frac{d^2\gamma(\widetilde{s})}{d\widetilde{s}^2} &= -\sigma^2\tau_\alpha(s) t_\alpha(s) - \sigma^2\kappa_\alpha(s) n_\alpha(s) + \sigma\sigma' b_\alpha(s) \qquad (2.9) \\
\frac{d^3\gamma(\widetilde{s})}{d\widetilde{s}^3} &= \left\{ \begin{array}{c} -\sigma\left\{3\sigma'\tau_\alpha(s) + \sigma\tau'_\alpha(s)\right\} t_\alpha(s) \\ -\sigma\left\{\sigma\tau_\alpha(s) + 3\sigma'\kappa_\alpha(s) + \sigma\kappa'_\alpha(s)\right\} n_\alpha(s) \\ +\left\{(\sigma')^2 + \sigma\sigma'' - \sigma^2\kappa_\alpha(s)\right\} b_\alpha(s) \end{array} \right\}
\end{aligned}
$$

where the parametrization $\sigma = ds/d\widetilde{s}$ satisfies

$$\sigma = \left\{(\tau_\alpha(s))^2 + \kappa'_\alpha(s)\tau_\alpha(s) - \kappa_\alpha(s)\tau'_\alpha(s)\right\}^{-1/5}. \qquad (2.10)$$

For the curve $\gamma(\widetilde{s})$, we have the relation

$$\frac{d^4\gamma(\widetilde{s})}{d\widetilde{s}^4} = -\tau_\gamma(\widetilde{s}) t_\gamma(\widetilde{s}) - \kappa_\gamma(\widetilde{s}) n_\gamma(\widetilde{s}) \qquad (2.11)$$

and its derivatives can be obtained as follows with respect to frame apparatus of $\alpha(s)$

$$\gamma'(\widetilde{s}) = \sigma b_\alpha(s) \qquad (2.12)$$

$$\gamma''(\widetilde{s}) = -\sigma^2\tau_\alpha(s) t_\alpha(s) - \sigma^2\kappa_\alpha(s) n_\alpha(s) + \sigma\sigma' b_\alpha(s) \qquad (2.13)$$

$$\gamma'''(\widetilde{s}) = \left\{ \begin{array}{c} -\sigma\left\{3\sigma'\tau_\alpha(s) + \sigma\tau'_\alpha(s)\right\} t_\alpha(s) \\ -\sigma\left\{\sigma\tau_\alpha(s) + 3\sigma'\kappa_\alpha(s) + \sigma\kappa'_\alpha(s)\right\} n_\alpha(s) \\ +\left\{(\sigma')^2 + \sigma\sigma'' - \sigma^2\kappa_\alpha(s)\right\} b_\alpha(s) \end{array} \right\} \qquad (2.14)$$

and

$$\frac{d^4\gamma(\widetilde{s})}{d\widetilde{s}^4} = Pt_\alpha(s) + Qn_\alpha(s) + Rb_\alpha(s) \tag{2.15}$$

where P, Q and R are

$$\begin{aligned} P &= -\left\{4(\sigma')^2\tau_\alpha(s) + 5\sigma\sigma'\tau'_\alpha(s) + 4\sigma\sigma''\tau_\alpha(s) + \sigma^2\tau''_\alpha(s) - \sigma^2\kappa_\alpha(s)\tau_\alpha(s)\right\} \\ Q &= -\left\{\begin{array}{c} 5\sigma\sigma'\tau_\alpha(s) + 2\sigma^2\tau'_\alpha(s) + 4(\sigma')^2\kappa_\alpha(s) + 5\sigma\sigma'\kappa'_\alpha(s) \\ +3\sigma\sigma''\kappa_\alpha(s) + \sigma^2\kappa''_\alpha(s) + \sigma\sigma''\kappa_\alpha(s) - \sigma^2(\kappa_\alpha(s))^2 \end{array}\right\} \\ R &= \left\{2\sigma'\sigma'' + \sigma'\sigma'' + \sigma\sigma''' - 5\sigma\sigma'\kappa_\alpha(s) - 2\sigma^2\kappa'_\alpha(s) - \sigma^2\tau_\alpha(s)\right\}. \end{aligned}$$

From the equations (2.9) and (2.11), we can rewrite

$$\frac{d^4\gamma(\widetilde{s})}{d\widetilde{s}^4} = \left\{\begin{array}{c} \sigma^2\kappa_\gamma(\widetilde{s})\tau_\alpha(s)t_\alpha(s) + \sigma^2\kappa_\alpha(s)\kappa_\gamma(\widetilde{s})n_\alpha(s) \\ -\left(\sigma\sigma'\kappa_\gamma(\widetilde{s}) + \tau_\gamma(\widetilde{s})\sigma\right)b_\alpha(s) \end{array}\right\}. \tag{2.16}$$

From (2.15) and (2.16), we obtain the equations

$$\sigma^2\kappa_\gamma(\widetilde{s})\tau_\alpha(s) - P = 0 \tag{2.17}$$

$$\sigma^2\kappa_\alpha(s)\kappa_\gamma(\widetilde{s}) - Q = 0 \tag{2.18}$$

$$\sigma\sigma'\kappa_\gamma(\widetilde{s}) + \tau_\gamma(\widetilde{s})\sigma + R = 0. \tag{2.19}$$

From(2.17) and (2.19), affine curvature and affine torsion of $\gamma(\widetilde{s})$ are

$$\begin{aligned} \kappa_\gamma(\widetilde{s}) &= \frac{P}{\sigma^2\tau_\alpha(s)} \\ \tau_\gamma(\widetilde{s}) &= \frac{-R - \sigma\sigma'\kappa_\gamma(\widetilde{s})}{\sigma} \end{aligned}$$

Also, from (2.10) and (2.18), the parametrization σ have satisfied the relation

$$\frac{5(\sigma - 1)\sigma'}{\sigma^7} = \tau_\alpha(s)\tau'_\alpha(s). \tag{2.20}$$

Theorem 2.4. *Let $\alpha(s)$ be a regular curve and its normal indicatrix be $\gamma(\widetilde{s})$. Then affine frame is*

$$\begin{aligned} t_\gamma(\widetilde{s}) &= \sigma b_\alpha(s) \\ n_\gamma(\widetilde{s}) &= -\sigma^2\widetilde{\tau}_\alpha(s)t_\alpha(s) - \sigma^2\widetilde{\kappa}_\alpha(s)n_\alpha(s) + \sigma\sigma' b_\alpha(s) \\ b_\gamma(\widetilde{s}) &= \left\{\begin{array}{c} -\sigma\left\{3\sigma'\widetilde{\tau}_\alpha(s) + \sigma\widetilde{\tau}'_\alpha(s)\right\}t_\alpha(s) \\ -\sigma\left\{\sigma\widetilde{\tau}_\alpha(s) + 3\sigma'\widetilde{\kappa}_\alpha(s) + \sigma\widetilde{\kappa}'_\alpha(s)\right\}n_\alpha(s) \\ +\left\{(\sigma')^2 + \sigma\sigma'' - \sigma^2\widetilde{\kappa}_\alpha(s)\right\}b_\alpha(s) \end{array}\right\} \end{aligned}$$

and the affine curvatures are

$$\begin{aligned} \widetilde{\kappa}_\gamma(\widetilde{s}) &= \frac{P}{\sigma^2\widetilde{\tau}_\alpha(s)} \\ \widetilde{\tau}_\gamma(\widetilde{s}) &= \frac{-R - \sigma\sigma'\widetilde{\kappa}_\gamma(\widetilde{s})}{\sigma} \end{aligned}$$

with the parametrization $\sigma = ds/d\widetilde{s}$ which is satisfy the equations

$$\begin{aligned} \sigma &= \left\{(\widetilde{\tau}_\alpha(s))^2 + \widetilde{\kappa}'_\alpha(s)\widetilde{\tau}_\alpha(s) - \widetilde{\kappa}_\alpha(s)\widetilde{\tau}'_\alpha(s)\right\}^{-1/5} \\ \frac{5(\sigma - 1)\sigma'}{\sigma^7} &= \widetilde{\tau}_\alpha(s)\widetilde{\tau}'_\alpha(s) \end{aligned}$$

where P, Q and R are

$$\begin{aligned} P &= -\left\{4(\sigma')^2\widetilde{\tau}_\alpha(s)+5\sigma\sigma'\widetilde{\tau}'_\alpha(s)+4\sigma\sigma''\widetilde{\tau}_\alpha(s)+\sigma^2\widetilde{\tau}''_\alpha(s)-\sigma^2\widetilde{\kappa}_\alpha(s)\widetilde{\tau}_\alpha(s)\right\} \\ Q &= -\left\{\begin{array}{c} 5\sigma\sigma'\widetilde{\tau}_\alpha(s)+2\sigma^2\widetilde{\tau}'_\alpha(s)+4(\sigma')^2\widetilde{\kappa}_\alpha(s)+5\sigma\sigma'\widetilde{\kappa}'_\alpha(s) \\ +3\sigma\sigma''\widetilde{\kappa}_\alpha(s)+\sigma^2\widetilde{\kappa}''_\alpha(s)+\sigma\sigma''\widetilde{\kappa}_\alpha(s)-\sigma^2(\widetilde{\kappa}_\alpha(s))^2 \end{array}\right\} \\ R &= \left\{2\sigma'\sigma''+\sigma'\sigma''+\sigma\sigma'''-5\sigma\sigma'\widetilde{\kappa}_\alpha(s)-2\sigma^2\widetilde{\kappa}'_\alpha(s)-\sigma^2\widetilde{\tau}_\alpha(s)\right\}. \end{aligned}$$

If the curve $\alpha(s)$ is a curve with constand affine curvature and torsion then $\sigma = c_0$ and so $P = \sigma^2\kappa_\alpha(s)\tau_\alpha(s)$, $Q = \sigma^2(\kappa_\alpha(s))^2$, $R = -\sigma^2\tau_\alpha(s)$. Thus, we obtain

$$\begin{aligned} \kappa_\gamma(\widetilde{s}) &= \kappa_\alpha(s) \\ \tau_\gamma(\widetilde{s}) &= \sigma\tau_\alpha(s) \end{aligned}$$

so normal indicatrix $\gamma(\widetilde{s})$ is the curve with constant affine curvatures.

Definition 2.5. Let $\alpha : J \longrightarrow A_3$ be a regular curve in affine 3-space with the affine frame defined $\{t_\alpha(s), n_\alpha(s), b_\alpha(s)\}$ at each point of $\alpha(s)$ and s be the affine arclength parameter. The curve $\eta(\widetilde{s}) = b_\alpha(s)$ with new arclength parameter $\widetilde{s}$, is called affine binormal indicatrix of $\alpha(s)$.

Let affine frame of $\eta(\widetilde{s})$ be $\{t_\eta(\widetilde{s}), n_\eta(\widetilde{s}), b_\eta(\widetilde{s})\}$ and the affine curvature and affine torsion be $\kappa_\eta(\widetilde{s})$ and $\tau_\eta(\widetilde{s})$, respectively.Thus affine frame vectors of $\eta(\widetilde{s})$ can be obtained as

$$\begin{aligned} \frac{d\eta(\widetilde{s})}{d\widetilde{s}} &= -qt_\alpha(s) - pn_\alpha(s) \\ \frac{d^2\eta(\widetilde{s})}{d\widetilde{s}^2} &= ut_\alpha(s) + vn_\alpha(s) + wb_\alpha(s) \\ \frac{d^3\eta(\widetilde{s})}{d\widetilde{s}^3} &= \left\{\begin{array}{c} (\sigma u' - wq)t_\alpha(s) + (\sigma v' + \sigma u - wp)n_\alpha(s) \\ +(\sigma w' + \sigma v)b_\alpha(s) \end{array}\right\} \end{aligned} \tag{2.21}$$

where the parametrization $\sigma = ds/d\widetilde{s}$ satisfies

$$\sigma q\left\{v^2(\frac{w}{v})' + v^2 - wu\right\} + \sigma p\left\{w^2(\frac{u}{w})' - uv\right\} = 1. \tag{2.22}$$

where

$$\begin{gathered} p = \sigma\kappa_\alpha(s), \quad q = \sigma\tau_\alpha(s) \\ u = -\sigma q', \quad v = -\sigma(p' + q), \quad w = -\sigma p. \end{gathered}$$

For the curve $\eta(\widetilde{s})$, we have the relation

$$\frac{d^4\eta(\widetilde{s})}{d\widetilde{s}^4} = -\tau_\eta(\widetilde{s})t_\eta(\widetilde{s}) - \kappa_\eta(\widetilde{s})n_\eta(\widetilde{s}) \tag{2.23}$$

and 4^{th} derivative of $(2.21)_3$ can obtained as follows with respect to frame apparatus of $\alpha(s)$

$$\frac{d^4\eta(\widetilde{s})}{d\widetilde{s}^4} = \left\{\begin{array}{c} (\sigma\sigma'u' + \sigma^2u'' - 2\sigma w'q - \sigma wq' - \sigma vq)t_\alpha(s) \\ +\left\{\begin{array}{c} \sigma\sigma'v' + \sigma\sigma'u - 2\sigma w'p + \sigma^2v'' + 2\sigma^2u' \\ -\sigma wp' - \sigma wq - \sigma vp \end{array}\right\}n_\alpha(s) \\ +(\sigma\sigma'w' + \sigma^2w'' + \sigma\sigma'v + 2\sigma^2v' + \sigma^2u - \sigma wp)b_\alpha(s) \end{array}\right\}. \tag{2.24}$$

From the equations (2.21) and (2.23), we can rewrite

$$\frac{d^4\eta(\widetilde{s})}{d\widetilde{s}^4} = \left\{\begin{array}{c} (q\tau_\eta(\widetilde{s}) - u\kappa_\eta(\widetilde{s}))t_\alpha(s) + (p\tau_\eta(\widetilde{s}) - v\kappa_\eta(\widetilde{s}))n_\alpha(s) \\ -w\kappa_\eta(\widetilde{s})b_\alpha(s) \end{array}\right\} \tag{2.25}$$

From (2.24) and (2.25), we obtain

$$\left\{\begin{array}{c}\left\{(\sigma\sigma' u' + \sigma^2 u'' - 2\sigma w' q - \sigma w q' - \sigma v q - \tau_\eta(\widetilde{s})\, q + \kappa_\eta(\widetilde{s})\, u\right\} t_\alpha \\ + \left\{\begin{array}{c}\sigma\sigma' v' + \sigma\sigma' u - 2\sigma w' p + \sigma^2 v'' + 2\sigma^2 u' - \sigma w p' \\ -\sigma w q - \sigma v p - \tau_\eta(\widetilde{s})\, p + \kappa_\eta(\widetilde{s})\, v\end{array}\right\} n_\alpha \\ + \left\{\sigma\sigma' w' + \sigma^2 w'' + \sigma\sigma' v + 2\sigma^2 v' + \sigma^2 u - \sigma w p + \kappa_\eta(\widetilde{s})\, w\right\} b_\alpha\end{array}\right\} = 0$$

and so we have the following equations

$$\sigma\sigma' u' + \sigma^2 u'' - 2\sigma w' q - \sigma w q' - \sigma v q - \tau_\eta(\widetilde{s})\, q + \kappa_\eta(\widetilde{s})\, u = 0 \tag{2.26}$$

$$\left\{\begin{array}{c}\sigma\sigma' v' + \sigma\sigma' u - 2\sigma w' p + \sigma^2 v'' + 2\sigma^2 u' - \sigma w p' \\ -\sigma w q - \sigma v p - \tau_\eta(\widetilde{s})\, p + \kappa_\eta(\widetilde{s})\, v\end{array}\right\} = 0 \tag{2.27}$$

$$\sigma\sigma' w' + \sigma^2 w'' + \sigma\sigma' v + 2\sigma^2 v' + \sigma^2 u - \sigma w p + \kappa_\eta(\widetilde{s})\, w = 0. \tag{2.28}$$

From (2.28) and (2.26), affine curvature and affine torsion of $\eta(\widetilde{s})$ are

$$\kappa_\eta(\widetilde{s}) = \frac{\sigma w p - \sigma\sigma'(w' + v) - \sigma^2(w'' + 2v' + u)}{w} \tag{2.29}$$

$$\tau_\eta(\widetilde{s}) = \frac{\sigma}{qw}\left\{\begin{array}{c}\sigma'(u'w - uw' - uv) + \sigma\left(u''w - uw'' - 2uv' - u^2\right) \\ + uwp - 2ww'q - w^2 q' - wvq\end{array}\right\}. \tag{2.30}$$

By using (2.22), (2.27), (2.29) and (2.30) the parametrization σ have satisfied the relation

$$\left\{\begin{array}{c}\left\{4\sigma\sigma'\sigma'' + (\sigma')^3 + \sigma^2\sigma''' - 2\sigma^3\kappa'_\alpha(s) - 6\sigma^2\sigma'\kappa_\alpha(s)\right\}(\kappa_\alpha(s) - \tau_\alpha(s)) \\ + \left\{7\sigma(\sigma')^2 + 4\sigma^2\sigma'' - 2\sigma^3\kappa_\alpha(s)\right\}(\kappa'_\alpha(s) - \tau'_\alpha(s) + \tau_\alpha(s)) \\ + \sigma^3(\kappa'''_\alpha(s) - \tau'''_\alpha(s) + \tau''_\alpha(s)) + 6\sigma^2\sigma'\{(\kappa''_\alpha(s) - \tau''_\alpha(s)) + \tau'_\alpha(s)\} \\ -2\sigma^3\kappa'_\alpha(s)\,\kappa_\alpha(s) + 3\sigma\sigma''(1-\sigma)\,\tau_\alpha(s) + 6\sigma\sigma'\tau'_\alpha(s) + \sigma^2(3-\sigma)\,\tau''_\alpha(s)\end{array}\right\} = 0.$$

Theorem 2.6. *Let $\alpha(s)$ be a regular curve and its affine binormal indicatrix be $\gamma(\widetilde{s})$. Then affine frame is*

$$\begin{array}{c} t_\eta(\widetilde{s}) = -q t_\alpha(s) - p n_\alpha(s) \\ n_\eta(\widetilde{s}) = u t_\alpha(s) + v n_\alpha(s) + w b_\alpha(s) \\ b_\eta(\widetilde{s}) = \left\{\begin{array}{c}(\sigma u' - wq) t_\alpha(s) + (\sigma v' + \sigma u - wp) n_\alpha(s) \\ + (\sigma w' + \sigma v) b_\alpha(s)\end{array}\right\}\end{array} \tag{2.31}$$

and the affine curvatures are

$$\begin{aligned}\widetilde{\kappa}_\eta(\widetilde{s}) &= \frac{\sigma w p - \sigma\sigma'(w' + v) - \sigma^2(w'' + 2v' + u)}{w} \\ \widetilde{\tau}_\eta(\widetilde{s}) &= \frac{\sigma}{qw}\left\{\begin{array}{c}\sigma'(u'w - uw' - uv) + \sigma\left(u''w - uw'' - 2uv' - u^2\right) \\ + uwp - 2ww'q - w^2 q' - wvq\end{array}\right\}\end{aligned}$$

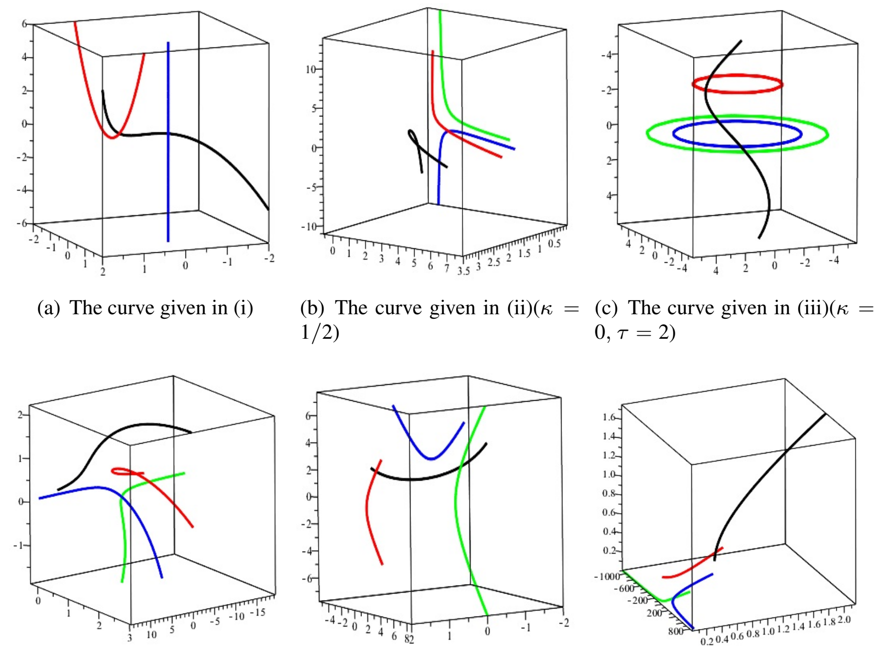
with the parametrization $\sigma = ds/d\widetilde{s}$ which is satisfy the equations

$$\sigma q\left\{v^2\left(\frac{w}{v}\right)' + v^2 - wu\right\} + \sigma p\left\{w^2\left(\frac{u}{w}\right)' - uv\right\} = 1,$$

$$\left\{\begin{array}{c}\left\{4\sigma\sigma'\sigma'' + (\sigma')^3 + \sigma^2\sigma''' - 2\sigma^3\widetilde{\kappa}'_\alpha(s) - 6\sigma^2\sigma'\widetilde{\kappa}_\alpha(s)\right\}(\widetilde{\kappa}_\alpha(s) - \widetilde{\tau}_\alpha(s)) \\ + \left\{7\sigma(\sigma')^2 + 4\sigma^2\sigma'' - 2\sigma^3\widetilde{\kappa}_\alpha(s)\right\}(\widetilde{\kappa}'_\alpha(s) - \widetilde{\tau}'_\alpha(s) + \widetilde{\tau}_\alpha(s)) \\ + \sigma^3(\widetilde{\kappa}'''_\alpha(s) - \widetilde{\tau}'''_\alpha(s) + \widetilde{\tau}''_\alpha(s)) + 6\sigma^2\sigma'\{(\widetilde{\kappa}''_\alpha(s) - \widetilde{\tau}''_\alpha(s)) + \widetilde{\tau}'_\alpha(s)\} \\ -2\sigma^3\widetilde{\kappa}'_\alpha(s)\,\widetilde{\kappa}_\alpha(s) + 3\sigma\sigma''(1-\sigma)\,\widetilde{\tau}_\alpha(s) + 6\sigma\sigma'\widetilde{\tau}'_\alpha(s) + \sigma^2(3-\sigma)\,\widetilde{\tau}''_\alpha(s)\end{array}\right\} = 0$$

where $p = \sigma\widetilde{\kappa}_\alpha(s)$, $q = \sigma\widetilde{\tau}_\alpha(s)$, $u = -\sigma q'$, $v = -\sigma(p' + q)$ and $w = -\sigma p$.

According to the classification in theorem 1.1, we obtain indicatrices of some space curves in affine 3-space. The black is the main curve, the red curve is tangent indicatrix, the blue curve is affine normal indicatrix and the green curve is binormal indicatrix of the main curve.



(a) The curve given in (i)

(b) The curve given in (ii)($\kappa = 1/2$)

(c) The curve given in (iii)($\kappa = 0, \tau = 2$)

(d) The curve given in (iii)($\kappa = 1, \tau = -1$)

(e) The curve given in (iv)($\kappa = 0, \tau = -2$)

(f) The curve given in (iv)($\kappa = 1/14, \tau = -1/3$)

Figure 1. Some curves given in theorem 1.1 and their indicatrices in affine 3-space.

Acknowledgment The authors are indebted to the referees for helpful suggestions and insights concerning the presentation of this paper.

References

[1] B. Su, Some Classes of Curves in The Affine space, *Tohoku Math. Journ.* **31**, 283–291, (1929).

[2] B. Su, *Affine Differential Geometry*, Science Press, Beijing, China, (1983).

[3] M. Nadjafikhah and A. Mahdipour-Shirayeh, Classification of curves in affine geometry, *Differential Geometry - Dynamical Systems* **13**, 191–200, (2011).

[4] N. Hu, *Affine Geometry of Space Curves and Homogeneous Surfaces*, phd thesis, Hokkaido University, August, (2012).

[5] E. Kreyszig and A. Pendl, Spherical curves and their analogues in a ne di erential geometry, *Proceedings of The American Math. Soc.* **48**, 423–428, (1975).

[6] E. Salkowski and W. Schells, *Allgemeine Theorie der Kurven doppelter Krümmung*, Leipzig und Berlin, (1914).

[7] W. Blaschke, *Differential Geometrie II*, Verlag von Julius springer, Berlin, (1923).

Author information

Yılmaz Tunçer, Department of Mathematics, Uşak University, Science and Art Faculty, Usak, TURKEY.
E-mail: `yilmaz.tuncer@usak.edu.tr`

Hüseyin Kocayiğit, Department of Mathematics, Celal Bayar University, Science and Art Faculty, Manisa, TURKEY.
E-mail: `huseyin.kocayigit@cbu.edu.tr`

Murat Kemal Karacan, Department of Mathematics, Uşak University, Science and Art Faculty, Usak, TURKEY.
E-mail: `murat.karacan@usak.edu.tr`

Received: August 3, 2018.

Accepted: November 29, 2018.