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On an anti-Kaehler-Codazzi manifold

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Abstract The present paper contains the studied of curvature properties of anti-Kaehler-Codazzi manifolds equipped with a semi-symmetric metric connection.

1 Introduction

Let (M^n, g) $n > 2$ be 2n-dimensional Riemannian manifold with Riemannian metric g. A connection is said to be symmetric if the torsion tensor with respect to that connection be equal to zero otherwise it is called a non-symmetric connection. If the covariant derivative of a metric tensor with respect to a given connection be zero then the connection is called a metric connec-tion otherwise it is called a non-metric connection. The Riemannian manifold equipped with a semi-symmetric metric connection has been studied by O. C. Andonie [15], M. C. Chaki and A. Konar [14], K. Yano and M. Kon[12], K.Yano[13], B. B. Chaturvedi and P. N. Pandey [9, 10, 11] and B.B. Chaturvedi and B. K. Gupta [6, 7, 8]. The existence of semi-symmetric metric connec-tions on a (k, μ)- contact metric manifolds is studied by A. A. Shaikh and S. K. Jana [2] in 2006. In 2010, generalized pseudo-symmetric Ricci symmetric manifolds admitting semi-symmetric metric connection was discussed by A. Shaikh, C. $\ddot{O}zg\ddot{u}r$ and S. K. Jana [3].

Some class of Riemannian manifold is studied by A. A. Shaikh and T. Q. Binh [4] in 2008. In 2013, A. Salimov and S.Turanli [5] studied some curvature properties of anti-Kaehler-Codazzi manifolds with respect to the Riemannian connection. In consequences of these studies, we have inspired to study these curvature properties of anti-Kaehler-Codazzi manifolds with respect to semi-symmetric metric connection.

A. Friedman and J. A. Schouten [1] considered semi-symmetric metric connection ∇ and Riemannian connection D with coefficients $\Gamma^h_{i\,j}$ and $\{^{\,h}_{i\,j}\}$respectively. According to them if the torsion tensor T of the connection ∇ on M^n, $(n > 2)$ be

$$T^h_{i\,j} = \delta^h_i \omega_j - \delta^h_j \omega_i, \tag{1.1}$$

then

$$\Gamma^h_{i\,j} = \{^{\,h}_{i\,j}\} + \delta^h_i\, \omega_j - g_{i\,j}\, \omega^h, \tag{1.2}$$

where $\omega^h = \omega_t g^{t\,h}$, ω^h being the contravariant components of the generating vector w_h and

$$\nabla_j\, \omega_i = D_j\, \omega_i - \omega_i\, \omega_j + g_{i\,j}\, \omega, \quad \text{where } \omega = \omega^h\, \omega_h. \tag{1.3}$$

A. Friedman and J. A. Schouten [1] also obtained the relation between curvature tensor with respect to semi-symmetric metric connection and Riemannian connection i.e.

$$\overline{R}_{i\,j\,k\,h} = R_{i\,j\,k\,h} - g_{i\,h}\, \pi_{j\,k} + g_{j\,h}\, \pi_{i\,k} - g_{j\,k}\, \pi_{i\,h} + g_{i\,k}\, \pi_{j\,h}, \tag{1.4}$$

where

$$\pi_{j\,k} = \nabla_j\, \omega_k - \omega_j\, \omega_k + \frac{1}{2}\, g_{j\,k}\, \omega. \tag{1.5}$$

Equation (1.4) satisfies

$$
\begin{aligned}
&(a)\overline{R}_{(i\,j)\,k\,h} = 0,\\
&(b)\overline{R}_{i\,j\,(k\,h)} = 0,\\
&(c)\overline{R}_{i\,j\,k\,h} = \overline{R}_{i\,k\,j\,h} \quad if \quad (g_{i\,k}\pi_{j\,h} = g_{i\,j}\pi_{k\,h} \ and\ \pi_{j\,k} = \pi_{k\,j}),\\
&(d)\overline{R}_{i\,j\,k\,h} = \overline{R}_{h\,k\,j\,i} \quad if \quad (g_{i\,k}\pi_{j\,h} = g_{i\,j}\pi_{k\,h} \quad and \quad \pi_{j\,k} = \pi_{k\,j}).
\end{aligned}
\tag{1.6}
$$

Taking covariant derivative of F_i^h with respect to Riemannian connection D and semi-symmetric metric connection ∇, we have

$$D_k\, F_i^h = \partial_k\, F_i^h + F_i^r\, \{{}^{\,h}_{r\,k}\} - F_r^h \{{}^{\,r}_{i\,k}\}, \tag{1.7}$$

and

$$\nabla_k\, F_i^h = \partial_k\, F_i^h + F_i^r\, \Gamma^h_{r\,k} - F_r^h \Gamma^r_{i\,k}. \tag{1.8}$$

Subtracting (1.7) from (1.8), we have

$$\nabla_k\, F_i^h - D_k\, F_i^h = F_i^r\, (\Gamma^h_{r\,k} - \{{}^{\,h}_{r\,k}\}) - F_r^h (\Gamma^r_{i\,k} - \{{}^{\,r}_{i\,k}\}), \tag{1.9}$$

using (1.2) in (1.9), we get

$$\nabla_k\, F_i^h = D_k\, F_i^h. \tag{1.10}$$

Therefore, we can say that the covariant derivative of F_i^h with respect to Riemannian connection D and semi-symmetric metric connection ∇ are equal.

Again taking covariant derivative of (1.8) with respect to semi-symmetric metric connection ∇, we have

$$
\begin{aligned}
\nabla_j \nabla_k\, F_i^h =& \partial_j \partial_k\, F_i^h - \partial_r\, F_i^h \Gamma^r_{j\,k} - \partial_k\, F_r^h \Gamma^r_{i\,j}\\
&+ \partial_k\, F_i^r \Gamma^h_{r\,j} + (\partial_j\, F_i^r + F_i^m \Gamma^r_{m\,j} - F_m^r \Gamma^m_{i\,j}\,)\Gamma^h_{r\,k}\\
&+ F_i^r \nabla_j \Gamma^h_{r\,k} - (\partial_j\, F_r^h + F_r^m \Gamma^r_{m\,j} - F_m^h \Gamma^m_{r\,j}\,)\Gamma^r_{i\,k} - F_r^h \nabla_j \Gamma^r_{i\,k}.
\end{aligned}
\tag{1.11}
$$

Interchanging j and k in equation (1.11), we get

$$
\begin{aligned}
\nabla_k \nabla_j\, F_i^h =& \partial_k \partial_j\, F_i^h - \partial_r\, F_i^h \Gamma^r_{j\,k} - \partial_j\, F_r^h \Gamma^r_{i\,k}\\
&+ \partial_j\, F_i^r \Gamma^h_{r\,k} + (\partial_k\, F_i^r + F_i^m \Gamma^r_{m\,k} - F_m^r \Gamma^m_{i\,k}\,)\Gamma^h_{r\,j}\\
&+ F_i^r \nabla_k \Gamma^h_{r\,j} - (\partial_k\, F_r^h + F_r^m \Gamma^r_{m\,k} - F_m^h \Gamma^m_{r\,k}\,)\Gamma^r_{i\,j} - F_r^h \nabla_k \Gamma^r_{i\,j}.
\end{aligned}
\tag{1.12}
$$

Subtracting (1.11) from (1.12), we get

$$
\begin{aligned}
\nabla_k \nabla_j\, F_i^h - \nabla_j \nabla_k\, F_i^h =& F_i^m (\Gamma^r_{m\,k} \Gamma^h_{r\,j} - \Gamma^r_{m\,j} \Gamma^h_{r\,k} + \nabla_k \Gamma^h_{m\,j} - \nabla_j \Gamma^h_{m\,k})\\
&- F_r^h (\Gamma^r_{m\,k}\, \Gamma^m_{i\,j} - \Gamma^r_{m\,j}\, \Gamma^m_{i\,k} + \nabla_j \Gamma^r_{i\,k} - \nabla_k \Gamma^r_{i\,j}).
\end{aligned}
\tag{1.13}
$$

Equation (1.13) implies

$$\nabla_k \nabla_j\, F_i^h - \nabla_j \nabla_k\, F_i^h = \overline{R}^h_{k\,j\,m}\, F_i^m - \overline{R}^r_{k\,j\,i}\, F_r^h. \tag{1.14}$$

2 Anti-Kaehler-Codazzi manifold

An even n-dimensional almost complex manifold M^n is said to be an almost complex manifold with almost complex structure F if

$$F^2 + I = 0. \tag{2.1}$$

A semi Riemannian metric g having signature (n,n) is said to be an anti-Hermitian if the metric g satisfies

$$g(FX, Y) = g(X, FY), \tag{2.2}$$

for any vector fields X and Y. An almost complex manifold M^n with an anti-Hermitian metric define by (2.2) is called an almost anti-Hermitian manifold. An anti-Hermitian manifold is said to be an anti-Kaehler manifold if $D_X F = 0$, where D is a Riemannian connection. We know that the 2-dimensional anti-Kaehler manifold is flat, therefore, throughout this paper we have considered the dimension $n \geq 4$.

Now, we define a fundamental 2-form ω satisfies

$$\omega(X, Y) = g(FX, Y), \tag{2.3}$$

where $\omega(Y, X) + \omega(X, Y) = 0$, this skew-symmetric tensor ω is said to be Killing-Yano tensor if

$$(D_X \omega)(Y, Z) + (D_Y \omega)(X, Z) = 0. \tag{2.4}$$

An almost complex manifold is said to be nearly Kaehler manifold if the almost complex structure F satisfies

$$(D_X\, F)Y + (D_Y\, F)X = 0, \tag{2.5}$$

for any vector fields X and Y.

The twin anti-Hermitian metric G is defined by

$$G(Y, Z) = g(FY, Z), \tag{2.6}$$

where $G(Y, Z) = G(Z, Y)$, since G is symmetric but 2-form ω is not symmetric so the Killing-Yano equation (2.4) has no immediate meaning. Therefore we can change the Killing-Yano equation by Codazzi equation

$$(D_X\, G)(Y, Z) - (D_Y\, G)(X, Z) = 0. \tag{2.7}$$

Equation (2.7) is equivalent to

$$(D_X\, F)Y - (D_Y\, F)X = 0. \tag{2.8}$$

If almost complex structure of almost anti-Hermitian manifold satisfies (2.8), then the triplet (M^n, F, g) is called an anti-Kaehler-Codazzi manifold.
By straight forward calculation we can also show that the Nijenhuis tensor with respect to Riemannian connection is equal to Nijenhuis tensor with respect to semi-symmetric metric connection i.e.

$$\overline{N}(X, Y) = N(X, Y). \tag{2.9}$$

In 2013, A. Salimov and S.Turanli [5] proved that

Theorem 2.1. *Anti-Kaehler-Codazzi manifolds have integrable almost anti-Hermitian structure.*

Now we propose:

Theorem 2.2. *Anti-Kaehler-Codazzi manifolds equipped with semi-symmetric metric connection have an integrable almost anti-Hermitian structure with respect to the semi-symmetric metric connection.*

Proof. From (2.9) we see that the Nijenhuis tensor with respect to Riemannian connection is equal to Nijenhuis tensor with respect to the semi-symmetric metric connection.

From theorem (2.1), we get

$$N(X, Y) = 0, \tag{2.10}$$

using this in (2.9), we have

$$\overline{N}(X, Y) = 0. \tag{2.11}$$

This means if Nijenhuis tensor in anti-Kaehler-Codazzi manifolds has an integrable almost anti-Hermitian structure with respect to Riemannian connection then it is also has an integrable almost anti-Hermitian structure with respect to the semi-symmetric metric connection. □

3 Some curvature properties with respect to a semi-symmetric metric connection

Applying the Ricci identity to the tensor field F, we get

$$D_k D_j F_i^h - D_j D_k F_i^h = R^h_{kjm} F_i^m - R^r_{kji} F_r^h. \tag{3.1}$$

Now, subtracting (1.14) from (3.1), we get

$$(D_k D_j F_i^h - \nabla_k \nabla_j F_i^h) - (D_j D_k F_i^h - \nabla_j \nabla_k F_i^h) = R^h_{kjm} F_i^m - R^r_{kji} F_r^h - \overline{R}^h_{kjm} F_i^m + \overline{R}^r_{kji} F_r^h. \tag{3.2}$$

Contracting (3.2) by h and k and using (1.10), we have

$$(D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = S_{jm} F_i^m - R^r_{hji} F_r^h - \overline{S}_{jm} F_i^m + \overline{R}^r_{hji} F_r^h. \tag{3.3}$$

Equation (3.3) implies

$$\begin{aligned}(D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) =& S_{jm} F_i^m - R_{hjil} g^{rl} F_r^h - \overline{S}_{jm} F_i^m + \overline{R}_{hjil} g^{rl} F_r^h \\ =& S_{jm} F_i^m - R_{hjil} G^{hl} - \overline{S}_{jm} F_i^m + \overline{R}_{hjil} G^{hl}.\end{aligned} \tag{3.4}$$

In 2013, A. Solimov and S. Turanli [5] considered

$$H_{ji} = R_{hjil} G^{hl}. \tag{3.5}$$

Similarly, we can take

$$\overline{H}_{ji} = \overline{R}_{hjil} G^{hl}, \tag{3.6}$$

using (3.5) and (3.6) in (3.4), we have

$$(D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = S_{jm} F_i^m - H_{ji} - \overline{S}_{jm} F_i^m + \overline{H}_{ji}, \tag{3.7}$$

where S_{jm}, $\overline{S}_{jm}$ and G^{hl} are Ricci tensor with respect to Riemannian connection, semi-symmetric metric connection and twin anti-Hermitian metric G respectively.
From (1.6) and (3.6), $\overline{H}_{ji}$ can be written as

$$\overline{H}_{ji} = \frac{1}{2}(\overline{R}_{hjil} + \overline{R}_{ljih}) G^{lh} = \frac{1}{2}(\overline{R}_{hjil} + \overline{R}_{ihlj}) G^{lh}. \tag{3.8}$$

Interchanging i and j in (3.8), we get

$$\overline{H}_{ij} = \frac{1}{2}(\overline{R}_{hijl} + \overline{R}_{lijh}) G^{lh} = \frac{1}{2}(\overline{R}_{hijl} + \overline{R}_{jhli}) G^{lh}. \tag{3.9}$$

Subtracting (3.8) from (3.9), we get

$$\overline{H}_{ij} - \overline{H}_{ji} = \frac{1}{2}(\overline{R}_{hijl} + \overline{R}_{lijh} - \overline{R}_{hjil} - \overline{R}_{ljih}) G^{lh}, \tag{3.10}$$

now using (1.6)in (3.10), we get

$$\overline{H}_{ij} = \overline{H}_{ji}. \tag{3.11}$$

Hence from (3.11), we conclude

Theorem 3.1. *In an anti-Kaehler-Codazzi manifold equipped with a semi-symmetric metric connection. The tensor $\overline{H}_{ij}$ defined by (3.6) is symmetric.*

Now equation (3.7) can be written as

$$\begin{aligned}D_h(D_j F_i^h - D_i F_j^h) - \nabla_h(\nabla_j F_i^h - \nabla_i F_j^h) =& (S_{jm} F_i^m - H_{ji}) - (S_{im} F_j^m - H_{ij}) \\ &+ (\overline{S}_{im} F_j^m - \overline{H}_{ij}) - (\overline{S}_{jm} F_i^m - \overline{H}_{ji}),\end{aligned} \tag{3.12}$$

In 2013, A. Solimov and S. Turanli [5] shown that

$$H_{i\,j} = H_{j\,i}. \tag{3.13}$$

using (1.10), (2.8) and (3.13) in equation (3.12), we have

$$S_{j\,m}\,F_i^m - S_{i\,m}\,F_j^m - \overline{S}_{j\,m}\,F_i^m + \overline{S}_{i\,m}\,F_j^m = 0. \tag{3.14}$$

In 2013, A. Solimov and S. Turanli [5] proved that

Theorem 3.2. *In an anti-Kaehler-Codazzi manifold, the Ricci tensor is pure with respect to Riemannian connection D i.e.*

$$S_{j\,m}\,F_i^m = S_{i\,m}\,F_j^m. \tag{3.15}$$

Using (3.15) in (3.14), we get

$$\overline{S}_{j\,m}\,F_i^m = \overline{S}_{i\,m}\,F_j^m. \tag{3.16}$$

Thus we conclude:

Theorem 3.3. *If M^n be an anti-Kaehler-Codazzi manifold equipped with semi-symmetric metric connection ∇ then the Ricci tensor with respect to a semi-symmetric metric connection ∇ is pure if the Ricci tensor with respect to Riemannian connection D is pure.*

In 2013, A. Salimov and S. Turanli [5] considered *Ricci tensor with respect to Riemannian connection D which is given by

$$S^*_{j\,i} = -H_{j\,r}\,F_i^r = -R_{h\,j\,r\,l}\,G^{l\,h}F_i^r, \tag{3.17}$$

Now, we are taking *Ricci tensor with respect to semi-symmetric metric connection ∇

$$\overline{S^*}_{j\,i} = -\overline{H}_{j\,r}\,F_i^r = -\overline{R}_{h\,j\,r\,l}\,G^{l\,h}\,F_i^r, \tag{3.18}$$

In 2013 A. Salimov and S. Turanli [5] proved that

Theorem 3.4. *Let (M, g, F) be an anti-Kaehler-codazzi manifold then*

$$S_{j\,m} = S^*_{j\,m}, \tag{3.19}$$

if only if

$$D_h D_j F_i^h = 0, \tag{3.20}$$

where $S^*_{j\,m}$ and $S_{j\,m}$ are *Ricci tensor with respect to Riemannian connection and Ricci tensor with respect to Riemannian connection.

Now, we propose:

Theorem 3.5. *If $\overline{S}^*_{j\,m}$ and $\overline{S}_{j\,m}$ be *Ricci tensor with respect to a semi-symmetric metric connection and Ricci tensor with respect to semi-symmetric metric connection in an anti-Kaehler-Codazzi manifold equipped with a semi-symmetric metric connection and $D_h D_j F_i^h = 0$ then $\overline{S}^*_{j\,m} = \overline{S}_{j\,m}$ if and only if $\nabla_h \nabla_j\,F_i^h = 0$.*

Proof. Equation (3.17) and (3.18) can be written as

$$S^*_{j\,r}\,F_i^r = H_{j\,i} \quad and \quad \overline{S^*}_{j\,r}F_i^r = \overline{H}_{j\,i}, \tag{3.21}$$

from (3.7) and (3.21), we have

$$(D_h D_j\,F_i^h - \nabla_h \nabla_j\,F_i^h) = (S^*_{j\,r}\,F_i^r - S_{j\,r}\,F_i^r) - (\overline{S}^*_{j\,m}\,F_i^m - \overline{S}_{j\,m}\,F_i^m), \tag{3.22}$$

using (3.19) and (3.20) in (3.22), we get

$$\nabla_h \nabla_j\,F_i^h = \overline{S}^*_{j\,m}\,F_i^m - \overline{S}_{j\,m}\,F_i^m, \tag{3.23}$$

equation (3.23) implies that

$\nabla_h \nabla_j\,F_i^h = 0,$ if only if $\overline{S}^*_{j\,m}\,F_i^m = \overline{S}_{j\,m}\,F_i^m.$ □

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