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Stability, Boundedness and Square Integrability Of Solutions To Certain Third Order Neutral Delay Differential Equations

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Abstract By constructing a Lyapunov functional, we obtain some sufficient conditions which guarantee the stability, boundedness and square integrability of solutions for some nonlinear neutral delay differential equations of third order. Our results improve and extend some well known results in the literature and one example is given for illustration of the subject.

1 Introduction

In this paper we consider a specific class of third order neutral delay differential equations of the following form

$$(x''(t) + \beta(t)x''(t-r))' + \Psi(x'(t))x''(t) + g(x'(t)) + f(x(t-\sigma)) = 0, \tag{1.1}$$

and

$$\begin{aligned}(x''(t) + \beta(t)x''(t-r))' + \Psi(x'(t))x''(t) + g(x'(t))\\ + f(x(t-\sigma)) = p(t, x, x(t-r), x'(t), x'(t-r), x''),\end{aligned} \tag{1.2}$$

for all $t \geq T \geq t_0 + \rho$, where $\rho = \sup\{r, \sigma\}$,and the functions $\Psi(x'(t)),\ g(x'(t-r))),\ f(x),\ \beta(t)$ and $p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t))$ are continuous in their respective arguments. Besides, it is supposed that the derivatives $f'(x), g'(y), \beta'(t)$ are continuous for all x, y with $f(0) = g(0) = 0, 0 \leq \beta(t)$ and $\alpha \leq \beta'(t) \leq 0$.

Neutral differential equations have many applications. For example, these equations arise in the study of two or more simple oscillatory systems with some interconnections between them and in modeling physical problems such as vibration of masses attached to an elastic bar. See [18] for reviews of this theory. In the qualitative analysis of such systems, the stability and asymptotic behavior of solutions play an important role. There is the permanent interest in obtaining new sufficient conditions for the stability and boundedness of the solutions of third order neutral differential equations. In many references the authors dealt with the problems by considering Lyapunov functions or functionals and obtained the criteria for the stability and boundedness see [1, 2, 8, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21].

By a solution of (1.1) (respectively (1.2)) we mean a continuous function $x : [t_x, \infty) \to \mathbb{R}$ such that $Z(t) = x''(t) + \beta(t)x''(t-r) \in C^1([t_x, \infty), \mathbb{R})$ and which satisfies equation (1.1) (resp. eq. (1.2)) on $[t_x, \infty)$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) (eq. (1.2)) under consideration here is continuable to the right and nontrivial, i.e $x(t)$ is defined on some ray $[t_x, \infty)$. Moreover, we tacitly assume that (1.1) (eq. (1.2)) possesses such solutions.

2 Stability

Suppose that there are positive constants $d_0,\ d_1,\ d,\ M,\ \delta,\ \Psi_0,\ \Psi_1$ and η such that the following conditions which will be used on the functions that appeared in equation (1.1) are satisfied:

i) $\Psi_0 < \Psi(y) < \Psi_1$.

ii) $\dfrac{f(x)}{x} \geq M > 0$ $(x \neq 0)$, and $|f'(x)| \leq \delta$ for all x.

iii) $d^2 < d_0 \leq \dfrac{g(y)}{y} \leq d_1$.

iv) $\dfrac{\delta}{2} < d < \Psi_0$.

v) $\displaystyle\int_T^t |\beta'(s)|\, ds < \eta$.

Remark 2.1. It's evident to see that $\beta(t) \leq \beta(T) = c$, for all $t \geq T$.

For the sake of simplicity, we introduce the following notation

$$Y(t) \quad = \quad x'(t) + \beta(t)x'(t-r).$$

Let us, for convenience, replace (1.1) by the equivalent differential system

$$\begin{cases} x' = y, \\ y' = z, \\ Z' = -\Psi(y)z - \; g(y) - f(x) + \displaystyle\int_{t-\sigma}^{t} \; f'(x(s))y(s)ds. \end{cases} \tag{2.1}$$

It's easy to see from (2.1) that

$$\begin{aligned} Y(t) &= y(t) + \beta(t)y(t-r), \\ Z(t) &= z(t) + \beta(t)z(t-r), \\ Y'(t) &= Z(t) + \beta'(t)y(t-r). \end{aligned}$$

For the case $p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)) \equiv 0$, the stability result of this paper is the following theorem.

Theorem 2.2. *If in addition to the hypotheses (i)-(v), suppose there exists a positive constant ε such that the following is also satisfied*

$$\sigma < \frac{2}{\delta} \min\left\{\frac{\varepsilon}{c}, \frac{A_1}{1+c+2d}, A_2\right\},$$

where

$$\begin{aligned} -A_1 &= -dd_0 + \delta + c\left(\frac{d_1^2}{2} + \delta\right) + (1+c)^2 + \frac{3\alpha}{2} < 0; \\ -A_2 &= -B_0 + (1+c)^2 + \frac{c}{2}(1+2B_1) + \varepsilon < 0, \\ B_0 &= \Psi_0 - d, \; \textit{and} \;\; B_1 = \Psi_1 - d. \end{aligned} \tag{2.2}$$

Then the null solution of (2.1) is asymptotically stable.

Proof. The proof of this theorem depends on properties of the continuously differentiable function $W = W(t, x_t, y_t, z_t)$ defined as

$$W(t) = Ve^{-\frac{1}{\omega}\int_T^t |\beta'(s)|\, ds}, \tag{2.3}$$

where

$$\begin{aligned} V &= V_1 + V_2 + \lambda \int_{-\sigma}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds, \\ V_1 &= dF(x) + f(x)Y + Y^2, && (2.4) \\ V_2 &= \frac{1}{2}Z^2 + dyZ + \int_0^y (g(u) + d\Psi(u)u)\, du, && (2.5) \\ &\quad + \int_{t-r}^{t} \left(\mu_1 y^2(s) + \mu_2 z^2(s)\right) ds, && (2.6) \end{aligned}$$

and

$$F(x) = \int_0^x f(u)du,$$

λ, μ_1 and μ_2 are positives constants which will be specified later in the proof.

Noting that

$$2\int_0^x f'(u)f(u)du = f^2(x),$$

and using (iv), we have

$$\begin{aligned} V_1 &= d\int_0^x f(u)du + \left(Y + \frac{1}{2}f(x)\right)^2 - \frac{1}{4}f^2(x) \\ &\geq d\int_0^x f(u)du - \frac{1}{2}\int_0^x f'(u)f(u)du \\ &\geq \int_0^x \left(d - \frac{\delta}{2}\right) f(u)du \\ &\geq \left(d - \frac{\delta}{2}\right) F(x). \end{aligned}$$

An application of condition (ii), give

$$F(x) = \int_0^x f(u)du \geq \frac{1}{2}Mx^2.$$

From conditions (i) and (iii), we have

$$\int_0^y \left(g(u) + d\Psi(u)u\right) du \geq \frac{1}{2}\left(d_0 + d\Psi_0\right) y^2.$$

Furthermore,

$$\begin{aligned} \frac{1}{2}Z^2 + dyZ + \frac{d_0}{2}y^2 &= \frac{1}{4}\left(dy + Z\right)^2 + \frac{d_0}{4}\left(y + \frac{d}{d_0}Z\right)^2 \\ &\quad + \frac{1}{4}\left(d_0 - d^2\right)y^2 + \frac{1}{4}\left(1 - \frac{d^2}{d_0}\right)Z^2. \end{aligned}$$

Since $\int_{t-r}^t \left(\mu_1 y^2(s) + \mu_2 z^2(s)\right) ds \geq 0$, we obtain

$$V_2 \geq k_0\left(y^2 + Z^2\right),$$

where

$$k_0 = \min\left\{\frac{d\Psi_0}{2} + \frac{1}{4}\left(d_0 - d^2\right), \frac{1}{4}\left(1 - \frac{d^2}{d_0}\right)\right\}.$$

Thus,

$$V \geq K_0(x^2 + y^2 + Z^2), \tag{2.7}$$

where

$$K_0 = \min\left\{\frac{1}{2}M\left(d - \frac{\delta}{2}\right), k_0\right\}.$$

From (2.3), (2.7) and condition (v) we have

$$W \geq K_1(x^2 + y^2 + Z^2), \tag{2.8}$$

with

$$K_1 = K_0 e^{-\frac{\eta}{\omega}}.$$

The time derivative of V gives

$$V' = U_1 + U_2 + U_3,$$

where

$$\begin{aligned}
U_1 &= [d - \Psi(y) + \mu_2]\, z^2 + [f'(x) + \mu_1 + \lambda\sigma]\, y^2 - \ dg(y)y \\
&\quad + [-\mu_1 + 2\beta'(t)\beta(t)]\, y^2(t-r) - \mu_2 z^2(t-r), \\
U_2 &= (\beta(t) f'(x) + 2\beta'(t))\, yy(t-r) + \beta'(t) f(x) y(t-r) \\
&\quad +\beta(t)\,(d - \Psi(y))\, z(t-r)z + 2yz \\
&\quad +2\beta(t) yz(t-r) + 2\beta(t) y(t-r)z + 2\beta^2(t) y(t-r)z(t-r) \\
&\quad -\beta(t)z(t-r)\ g(y),
\end{aligned}$$

and

$$U_3 = [Z + dy] \int_{t-\sigma}^{t} f'(x(s)y(s)ds - \lambda \int_{t-\sigma}^{t} y^2(s)ds.$$

By conditions (i)-(iii) and the fact that $\beta(t) \le c$ and $|f(x)| < \delta\,|x|$, we get

$$\begin{aligned}
U_1 &\le [-B_0 + \mu_2]\, z^2 + [-dd_0 + \delta + \mu_1 + \lambda\sigma]\, y^2 \\
&\quad + [-\mu_1 + 2\beta'(t)\beta(t)]\, y^2(t-r) - \mu_2 z^2(t-r).
\end{aligned}$$

Also by using Schwartz inequality we get

$$\begin{aligned}
U_2 &\le \frac{|\beta'(t)|\delta^2}{2} x^2 + \left(1 + \frac{c}{2}\left(2 + d_1^2 + \delta\right) + |\beta'(t)|\right) y^2 \\
&\quad + \left(1 + \frac{c}{2}\left(2 + B_1\right)\right) z^2 \\
&\quad + \left(c\left(1 + c + \frac{\delta}{2}\right) + \frac{3\alpha}{2}\right) y^2(t-r) \\
&\quad + \left(\frac{c}{2}\left(B_1 + 2c + 3\right)\right) z^2(t-r),
\end{aligned}$$

and

$$\begin{aligned}
U_3 &\le \frac{d\delta\sigma}{2} y^2 + \frac{\delta\sigma}{2} z^2 + \frac{c\delta\sigma}{2} z^2(t-r) \\
&\quad + \left[\frac{\delta}{2}\left(1 + c + d\right) - \lambda\right] \int_{t-\sigma}^{t} y^2(s)ds.
\end{aligned}$$

So, after rearrangement and using the fact that $\alpha \le \beta'(t) \le 0$, we have

$$\begin{aligned}
V' &\le \left[-dd_0 + \delta + 1 + \frac{c}{2}\left(2 + d_1^2 + \delta\right) + \mu_1 + \left(\frac{d\delta}{2} + \lambda\right)\sigma\right] y^2 \\
&\quad + \left[-B_0 + 1 + \frac{c}{2}\left(2 + B_1\right) + \mu_2 + \frac{\delta\sigma}{2}\right] z^2 \\
&\quad + \left[-\mu_1 + c\left(1 + c + \frac{\delta}{2}\right) + \frac{3\alpha}{2}\right] y^2(t-r) \\
&\quad + \left[-\mu_2 + \frac{c}{2}\left(3 + B_1 + 2c\right) + \frac{c\delta\sigma}{2}\right] z^2(t-r) \\
&\quad + \left[\frac{\delta^2}{2} x^2 + y^2\right] |\beta'(t)| \\
&\quad + \left[\frac{\delta}{2}\left(1 + c + d\right) - \lambda\right] \int_{t-\sigma}^{t} y^2(s)ds.
\end{aligned}$$

By taking

$$\begin{aligned}\frac{\delta}{2}(1+c+d) &= \lambda,\\ c\left(1+c+\frac{\delta}{2}\right)+\frac{3\alpha}{2} &= \mu_1,\\ \frac{c}{2}(3+B_1+2c)+\varepsilon &= \mu_2,\end{aligned}$$

we get

$$\begin{aligned}V' \leq\ & \left[-dd_0+\delta+c\left(\frac{d_1^2}{2}+\delta\right)+(1+c)^2+\frac{3\alpha}{2}+\delta(1+c+2d)\frac{\sigma}{2}\right]y^2\\ &+\left[-B_0+(1+c)^2+\frac{c}{2}(1+2B_1)+\varepsilon+\delta\frac{\sigma}{2}\right]z^2\\ &+\left[-\varepsilon+\frac{c\delta\sigma}{2}\right]z^2(t-r)\\ &+\left[\frac{\delta^2}{2}x^2+y^2\right]|\beta'(t)|.\end{aligned}$$

With the use of (2.2), we obtain

$$\begin{aligned}V' \leq\ & \left[-A_1+[\delta(1+c+2d)]\frac{\sigma}{2}\right]y^2(t)+\left[-A_2+\delta\frac{\sigma}{2}\right]z^2(t)\\ &+\left[-\varepsilon+\frac{c\delta\sigma}{2}\right]z^2(t-r)\\ &+|\beta'(t)|\left[\frac{\delta^2}{2}x^2(t)+y^2(t)\right].\end{aligned}$$

If

$$\sigma<\frac{2}{\delta}\min\left\{\frac{\varepsilon}{c},\frac{A_1}{1+c+2d},A_2\right\},$$

then

$$V'\leq -K_2\left(y^2(t)+z^2(t)\right)+|\beta'(t)|\left[\frac{\delta^2}{2}x^2(t)+y^2(t)\right],$$

where

$$K_2=\min\left\{-A_1+[\delta(1+c+2d)]\frac{\sigma}{2},-A_2+\delta\frac{\sigma}{2}\right\}.$$

From condition (v), one can see that

$$e^{\left(-\frac{\eta}{\omega}\right)}<e^{\left(-\frac{1}{\omega}\int_T^t|\beta'(s)|\,ds\right)}=E(t)<1.$$

Hence, by (2.3), we have

$$\begin{aligned}W'(t) &= \left(V'-\frac{|\beta'(t)|}{\omega}V\right)E(t)\\ &\leq \left(-K_2\left[y^2(t)+z^2(t)\right]+|\beta'(t)|\left[\frac{\delta^2}{2}x^2(t)+y^2(t)\right]-\frac{K_0|\beta'(t)|}{\omega}(x^2+y^2+Z^2)\right)E(t)\\ &\leq \left(-K_2\left[y^2(t)+z^2(t)\right]+K_3|\beta'(t)|(x^2+y^2+Z^2)-\frac{K_0|\beta'(t)|}{\omega}(x^2+y^2+Z^2)\right)E(t)\\ &\leq -K_4\left[y^2(t)+z^2(t)\right], \qquad (2.9)\end{aligned}$$

where

$$K_4=K_2e^{-\frac{\eta}{\omega}},K_3=\frac{1}{2}\max\{2,\delta^2\},\ \ \omega=\frac{K_0}{K_3}.$$

From (2.9), $W_3(\|X\|) = K_2 e^{-\dfrac{\eta}{\omega}}\left[y^2(t) + z^2(t)\right]$ is a positive definite function. The above discussion guarantees that the null solution of (2.1) is asymptotically stable and completes the proof of Theorem 2.2. □

3 Boundedness

For the case $p(t, x, y, x(t-r), y(t-r), z(t)) = p(\cdot) \neq 0$, equation (1.1) is equivalent to the system

$$\begin{cases} x' = y, \\ y' = z, \\ Z' = -\Psi(y)z - \ g(y) - f(x) + p(\cdot) + \displaystyle\int_{t-\sigma}^{t} \ f'(x(s))y(s)ds. \end{cases} \tag{3.1}$$

Theorem 3.1. *Assume that all the conditions of Theorem 2.2 are satisfied and there exist positive constants q_1 and q_2 such that*

I_1) $|p(t, x, y, x(t-r), y(t-r), z(t))| \leq q(t) < q_1$,

I_2) $\left|\displaystyle\int_0^t q(s)ds\right| < q_2$.

Then there exists a positive constant D such that any solution of (3.1) satisfies

$$|x(t)| \leq D, \ |y(t)| \leq D, \ |Z(t)| \leq D. \tag{3.2}$$

Proof. On differentiating (2.3) along the system (3.1), we obtain

$$\begin{aligned} W'_{(3.1)} &= W'_{(2.1)} + (Zp(\cdot) + dyp(\cdot))\, e^{-\dfrac{1}{\omega}\displaystyle\int_T^t |\beta'(s)|\, ds} \\ &\leq Zp(\cdot) + dyp(\cdot)\,. \end{aligned}$$

Using condition (I_1), we get

$$W'_{(3.1)} \leq q(t)|Z| + dq(t)|y|.$$

Now, the inequality $|u| \leq \ u^2 + 1$ lead

$$\begin{aligned} W'_{(3.1)} &\leq K_5 q(t)\left[y^2 + Z^2 + 2\right] \\ &\leq K_5 q(t)\left[x^2 + y^2 + Z^2 + 2\right], \end{aligned} \tag{3.3}$$

where $K_5 = \max\{1, d\}$.
In view of (2.8), the above estimates imply that

$$W'_{(3.1)} \leq \frac{K_5}{K_1} q(t)W + K_6 q(t), \tag{3.4}$$

with $K_6 = 2K_5$. Integrating both sides from t_1 to t, we easily obtain

$$W(t) - W(T) \leq K_6 \int_T^t q(s)ds + \frac{K_5}{K_1}\int_T^t W(s)q(s)ds.$$

Let

$$q_3 = W(T) + K_6 q_2. \tag{3.5}$$

Thus

$$W(t) \leq q_3 + \frac{K_5}{K_1}\int_T^t W(s)q(s)ds.$$

By using Gronwall inequality, it follows that

$$W(t) \leq q_3 \exp\left(\frac{K_5}{K_1}\int_T^t q(s)ds\right) \leq q_4, \tag{3.6}$$

where $q_4 = q_3 \exp\left(\frac{K_5}{K_1} q_2\right)$. This result implies that there exists a constant D such that

$$|x(t)| \leq D, \ |y(t)| \leq D, |Z(t)| \leq D.$$

This completes the proof of Theorem 3.1. □

4 Square Integrability

Our next result concerns the square integrability of solutions to equation (1.2).

Theorem 4.1. *If conditions (i)–(v), (I_1) and (I_2) hold, then any solution x of (1.2) satisfies*

$$\int_{t_0}^{\infty} \left(x''^2(s) + x'^2(s) + x^2(s)\right) ds < \infty.$$

Proof. Define $H(t)$ as

$$H(t) = W(t) + \varepsilon_0 \int_T^t (z^2(s) + y^2(s)) ds, \tag{4.1}$$

where $\varepsilon_0 > 0$, is a constant to be specified later. By differentiating $H(t)$ and using (3.4), we obtain

$$H'(t) \leq [\varepsilon_0 - K_4] \left(z^2(t) + y^2(t)\right) + \frac{K_5}{K_1} q(t) W + K_6 q(t).$$

If we Choose $\varepsilon_0 - K_4 < 0$, then from (3.6) we get

$$H'(t) \leq K_7 q(t), \tag{4.2}$$

where $K_7 = \frac{K_5}{K_1} q_4 + K_6$. Integrating (4.2) from T to t, and using condition (I_2) of Theorem 3.1, we obtain

$$H(t) - H(T) = \int_T^t H'(s) ds \leq K_7 q_2.$$

Using (3.5) and equality $H(T) = W(T)$, we get

$$H(t) \leq \frac{K_5}{K_1} q_4 q_2 + q_3.$$

We can conclude by (4.1) that

$$\int_T^t (z^2(s) + y^2(s)) ds < \frac{K_5 q_4 q_2 + K_1 q_3}{K_1 \varepsilon_0},$$

which imply the existence of positive constants σ_1 and σ_2, such that

$$\int_T^t y^2(s) ds \leq \sigma_1 \text{ and } \int_T^t z^2(s) ds \leq \sigma_2.$$

Hence

$$\int_T^t x'^2(s) ds \leq \sigma_1 \text{ and } \int_T^t x''^2(s) ds \leq \sigma_2. \tag{4.3}$$

Next multiplying (1.2) by $x(t-\sigma)$, we obtain

$$\begin{aligned} &\left[x''(t) + \beta x''(t-r)\right]' x(t-\sigma) + \Psi(x'(t)) x''(t) x(t-\sigma) + g(x'(t)) x(t-\sigma) \\ &+ f(x(t-\sigma)) x(t-\sigma) = p(t, x, x(t-\sigma), x'(t), x'(t-\sigma), x'') x(t-\sigma). \end{aligned} \tag{4.4}$$

Integrating (4.4) from T to t, we have

$$\int_T^t f(x(s-\sigma)) x(s-\sigma) ds = L_1(t) + L_2(t) + L_3(t) + L_4(t), \tag{4.5}$$

where

$$\begin{aligned}
L_1(t) &= -\int_T^t (x''(s)+\beta(s)x''(s-r))x(s-\sigma)ds,\\
L_2(t) &= -\int_T^t \Psi(x'(s))x''(s)x(s-r)ds,\\
L_3(t) &= -\int_T^t g(x'(s))x(s-r)ds,\\
L_4(t) &= \int_T^t p(t,x,x(s-r),x'(t),x'(s-r),x'')x(s-r)ds.
\end{aligned}$$

Integrating by parts and using (3.2) and (4.3), we obtain

$$\begin{aligned}
L_1(t) &= -x(t-\sigma)\left[x''(t)+\beta(t)x''(t-r)\right]+\int_T^t\Big(\left[x''(t)+\beta(s)x''(t-r)\right]x'(s-\sigma)\Big)ds+C_1\\
&\le C_1+D^2(1+c)+\frac{1}{2}\int_T^t\Big(\left[x''(s)+\beta(s)x''(s-r)\right]^2+x'^2(s-\sigma)\Big)ds\\
&\le C_1+D^2(1+c)+\frac{1}{2}\int_T^t\Big(x''^2(s)+c^2x''^2(s-r)+2cx''(s)x''(s-r)+x'^2(s-\sigma)\Big)ds\\
&\le l_1,
\end{aligned}$$

where

$$C_1=\left|x(T-\sigma)\left[x''(T)+\beta(T)x''(T-r)\right]\right|,$$

and

$$l_1=C_1+D^2(1+c)+\frac{1}{2}\left[\left(1+2c+c^2\right)\sigma_2+\sigma_1\right].$$

In the same way, after using (i)-(iii), (I_1), (I_2), (3.2) and (4.3), one arrives at

$$\begin{aligned}
L_2(t) &\le \int_T^t \left|\Psi(x'(s))x''(s)x(s-\sigma)\right|ds\\
&\le \left(\int_T^t \left[\Psi\left(x'(s)\right)x''(s)\right]^2ds\right)^{\frac{1}{2}}\left(\int_T^t x^2(s-\sigma)ds\right)^{\frac{1}{2}}\\
&\le l_2\left(\int_T^t x^2(s-\sigma)ds\right)^{\frac{1}{2}},\\
L_3(t) &\le \int_T^t \left|g'(x'(s))x(s-\sigma)\right|ds\\
&\le \left(\int_T^t \left[g'(x'(s))\right]^2ds\right)^{\frac{1}{2}}\left(\int_T^t x^2(s-\sigma)ds\right)^{\frac{1}{2}}\\
&\le \left(d_1^2\int_T^t x'^2(s)ds\right)^{\frac{1}{2}}\left(\int_T^t x^2(s-\sigma)ds\right)^{\frac{1}{2}}\\
&\le l_3\left(\int_{t_1}^t x^2(s-\sigma)ds\right)^{\frac{1}{2}},\\
L_4(t) &\le \int_T^t \left|p(s,x,y,x(s-\sigma),x'(s-\sigma),x''(s))x(s-\sigma)\right|ds\\
&\le D\int_T^t |q(s)|\,ds\le l_4,
\end{aligned}$$

where

$$l_2=\sqrt{\Psi_1^2\sigma_2},\ l_3=\sqrt{d_1^2\sigma_1},\ \text{and } l_4=Dq_2.$$

In the other hand from condition (ii), we have

$$\int_T^t x(s-\sigma)f(x(s-\sigma))ds \geq M\int_T^t x^2(s-\sigma)ds.$$

Hence, by (4.5) and condition I_3 of Theorem 4.1, we obtain

$$M\int_T^t x^2(s-\sigma)ds \leq l_1 + l_2\left(\int_T^t x^2(s-\sigma)ds\right)^{\frac{1}{2}} + l_3\left(\int_T^t x^2(s-\sigma)ds\right)^{\frac{1}{2}} + l_4. \tag{4.6}$$

If

$$\int_T^t x^2(s-\sigma)ds \to \infty \;\; as \;\; t\to\infty,$$

then dividing both sides of (4.6) by $\left(\int_T^t x^2(s-\sigma)ds\right)^{\frac{1}{2}}$, we immediately obtain a contradiction. Hence, we deduce that $\int_T^t x^2(s-\sigma)ds < \infty$, then $\int_T^{+\infty} x^2(s)ds < \infty$. This fact completes the proof of Theorem 4.1. □

5 Example

As a particular case of (1.2), consider the following third order neutral differential equation

$$\left(x''(t)+\frac{1}{10}e^{-\frac{10}{3}t}x''(t-r)\right)' + \left(\frac{11}{2}+\frac{1}{2}\sin x'(t)\right)x''(t) + (4.775)x'(t) + \frac{1}{200}x'(t)\cos(x'(t))$$
$$+\left[\frac{19}{10}x(t-r)+\frac{x(t-r)}{\sqrt{10}+|x(t-r)|}\right] = \frac{1}{1+t^2+|x|+|y|+|z|},$$

The appearing functions in the equation are as follows

$$5=\Psi_0 \leq \Psi(y) = \frac{11}{2}+\frac{1}{2}\sin y \leq \Psi_1 = 6,$$

$$4.41 = d^2 < d_0 = 4.77 \leq \frac{g(y)}{y} = 4.775 + \frac{1}{200}\cos y \leq d_1 = 4.87.$$

$$0 < \beta(t) = \frac{1}{10}e^{-\frac{10}{3}t} \leq \frac{1}{10} = c,$$

$$\beta'(t) = -\frac{1}{3}e^{-\frac{10}{3}t} < 0 \text{ and } |\beta'(t)| = \left|-\frac{1}{3}e^{-\frac{10}{3}t}\right| \leq \frac{1}{3} = \alpha,$$

$$1 = \frac{\delta}{2} < d = 2.1 < \Psi_0 = 5.$$

and the function

$$f(x) = \frac{19}{10}x + \frac{x}{\sqrt{10}+\mid x\mid}.$$

It is clear, from this relation, that $f(0)=0$. Also, since $0 \leq \frac{1}{\sqrt{10}+\mid x\mid} \leq 1$ for all x, we have that

$$\frac{f(x)}{x} \geq \frac{19}{10} = M,$$

for all $x \neq 0$. Moreover

$$|f'(x)| = \left|\frac{19}{10}+\frac{1}{(\sqrt{10}+\mid x\mid)^2}\right| \leq 2 = \delta.$$

We also have

$$\begin{aligned} -dd_0+\delta+c\left(\frac{d_1^2}{2}+\delta\right)+(1+c)^2+\frac{3\alpha}{2} &= -4.91=-A_1<0,\\ -B_0+(1+c)^2+\frac{c}{2}\left(3+2B_1\right)+\varepsilon &= -1.05=-A_2<0,\ \text{for } \varepsilon=\frac{1}{10}. \end{aligned}$$

The function

$$p(t,x,y,z)=\frac{1}{1+t^2+|x|+|y|+|z|}\leq\frac{1}{1+t^2}=q(t),$$

and

$$\int_0^{+\infty}|q(t)|dt<\infty,$$

for all t,x,y,z.
All assumptions of Theorem (4.1) hold true, thus, the conclusions also follow.

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