Palestine Journal of Mathematics

Vol. 9(2) (2020) , 891–902

ON COMPLETENESS OF SOME BICOMPLEX SEQUENCE SPACES

Nilay Sager and Birsen Sağır

Communicated by S.P. Goyal

MSC 2010 Classifications: Primary 46A45, 46B45; Secondary 30G35, 30L99.

Keywords and phrases: Bicomplex sequence spaces, bicomplex Hölder' s inequality, completeness.

Abstract In this paper, we present Hölder' s and Minkowski's inequalities with Euclidean norm in the set of bicomplex numbers and define bicomplex sequence spaces. We also investigate the completeness property of our bicomplex sequence spaces using these Hölder' s and Minkowski' s inequalities.

1 Introduction

Bicomplex numbers have been studied for quite a long time and a lot of work has been done in this area. In 1892, Corrado Segre published a paper [12] in which he defined an infinite set of algebras and gave the concept of bicomplex numbers. The most comprehensive study of analysis in the bicomplex setting is available in the book of Price [11]. Alpay et al. [1] developed a general theory of functional analysis with bicomplex scalars. In recent years, many important results were obtained in this area. Few of them, which pertain and lead to our work, are [6], [7], [8], [9], [10], [13], [14], [15]. Goyal and Goyal [7] developed bicomplex Hurwitz zeta function and discussed the zeros and analytic continuation of this function. Goyal et al. in [8] defined bicomplex Gamma and Beta functions and studied various properties connected with these functions. Goyal [6] developed bicomplex Polygamma function and investigated integral representation, recurrence relation, multiplication formula and reflection formula for this function. Srivastava [13] initiated the systematic study of topological aspects of $\mathbb{BC}$. He defined three topologies τ_1, τ_2 and τ_3 on $\mathbb{BC}$. Kumar and Saini in [9] developed topological modules over the ring of bicomplex numbers and discussed bicomplex convexivity, hyperbolic valued semi-norms and hyperbolic valued Minkowski functionals in bicomplex modules. They also gave the conditions under which topological bicomplex modules and locally bicomplex convex modules become hyperbolic normable and hyperbolic metrizable, respectively.

Sequence spaces play an important role in functional analysis. These Banach spaces and their structure has been studied by many authors [2], [3], [5]. A function f of a bicomplex variable is said to be an entire function if it is holomorphic in the entire bicomplex space $\mathbb{BC}$. If $f(\zeta) = \sum\limits_{k=1}^{\infty} \alpha_k (\zeta - \eta)^k$ represents an entire function, the series $\sum\limits_{k=1}^{\infty} \alpha_k$ is called entire bicomplex series and the sequence (α_k) is called entire bicomplex sequence [11]. Srivastava and Srivastava [14] defined and studied a class of entire bicomplex sequences, and also showed that this class is a bicomplex module. Nigam [10] and Wagh [15] studied the subclasses of this class.

In this paper, we examine the validity of the bicomplex version of the well - known Hölder' s and Minkowski' s inequalities for sums, introduce bicomplex sequence spaces with Euclidean norm in the set of bicomplex numbers and study completeness property of the spaces.

Now, we give definition and algebraic operations of bicomplex nıumbers and summarize the notion of sequence and series in the set of bicomplex numbers. For further details we refer the reader to [1], [11].

Definition 1.1. [11] Let i and j be independent imaginary units such that $i^2 = j^2 = -1$, $ij = ji$ and $\mathbb{C}(i)$ be the set of complex numbers with the imaginary unit i. The set of bicomplex numbers

$\mathbb{BC}$ is defined by
$$\mathbb{BC}=\{\zeta=\zeta_1+j\zeta_2:\zeta_1,\zeta_2\in\mathbb{C}(i)\}.$$

Lemma 1.2. *[11] The set $\mathbb{BC}$ forms a ring with respect to the addition and multiplication defined as*

$$\begin{aligned}\zeta+\eta &= (\zeta_1+j\zeta_2)+(\eta_1+j\eta_2)=(\zeta_1+\eta_1)+j(\zeta_2+\eta_2),\\ \zeta.\eta &= \zeta\eta=(\zeta_1+j\zeta_2).(\eta_1+j\eta_2)=(\zeta_1\eta_1-\zeta_2\eta_2)+j(\zeta_1\eta_2+\zeta_2\eta_1).\end{aligned}$$

Lemma 1.3. *[11] For every $\zeta,\eta\in\mathbb{BC}$ we have*

$$|\|\zeta\|_{\mathbb{BC}}-\|\eta\|_{\mathbb{BC}}|\leq\|\zeta-\eta\|_{\mathbb{BC}} \tag{1.1}$$

and

$$\|\zeta\eta\|_{\mathbb{BC}}\leq\sqrt{2}\,\|\zeta\|_{\mathbb{BC}}\,\|\eta\|_{\mathbb{BC}}, \tag{1.2}$$

where $\|\|_{\mathbb{BC}}$ is Euclidean norm in $\mathbb{BC}$ defined by

$$\|\|_{\mathbb{BC}}:\mathbb{BC}\to[0,\infty),\ \zeta\to\|\zeta\|_{\mathbb{BC}}=\|\zeta_1+j\zeta_2\|_{\mathbb{BC}}=\left(|\zeta_1|^2+|\zeta_2|^2\right)^{\frac{1}{2}}. \tag{1.3}$$

Definition 1.4. [11] A sequence in $\mathbb{BC}$ (a bicomplex sequence) is a function defined by $\zeta:\mathbb{N}\to\mathbb{BC}, n\to\zeta_n,\ \zeta_n\in\mathbb{BC}$. This sequence converges to a point $\zeta^*\in\mathbb{BC}$ if and only if to each $\varepsilon>0$ there corresponds an $n_0(\varepsilon)\in\mathbb{N}$ such that $\|\zeta_n-\zeta^*\|_{\mathbb{BC}}<\varepsilon$ for all $n\geq n_0(\varepsilon)$. The sequence $\zeta=(\zeta_n)$ is a Cauchy sequence in $\mathbb{BC}$ (a bicomplex Cauchy sequence) if and only if to each $\varepsilon>0$ there corresponds an $n_0(\varepsilon)\in\mathbb{N}$ such that $\|\zeta_n-\zeta_m\|_{\mathbb{BC}}<\varepsilon$ for all $n,m\geq n_0(\varepsilon)$. Also, $\zeta=(\zeta_n)$ converges to a point in $\mathbb{BC}$ if and only if it is a bicomplex Cauchy sequence.

Lemma 1.5. *[11] If $\zeta:\mathbb{N}\to\mathbb{BC}, n\to\zeta_n=\zeta_{1n}+j\zeta_{2n}$ is a bicomplex sequence and $\lim\limits_{n\to\infty}\zeta_n=\zeta_1^*+j\zeta_2^*=\zeta^*$ then the following limits exist and have the values shown:*

$$\lim_{n\to\infty}\zeta_{1n}=\zeta_1^*,\ \lim_{n\to\infty}\zeta_{2n}=\zeta_2^*. \tag{1.4}$$

Furthermore, if the limits exist as indicated in (1.4)*, then $\lim\limits_{n\to\infty}\zeta_n$ exists and $\lim\limits_{n\to\infty}\zeta_n=\zeta^*$.*

Definition 1.6. [11] Let $(\zeta_k)_{k\in\mathbb{N}}$ be a bicomplex sequence. Then, the infinite sum

$$\sum_{k=1}^{\infty}\zeta_k=\sum_{k=1}^{\infty}(\zeta_{1k}+j\zeta_{2k})=\zeta_1+\zeta_2+...+\zeta_n+... \tag{1.5}$$

is called an infinite series in $\mathbb{BC}$. Define the sequence $s:\mathbb{N}\to\mathbb{BC}, n\to s_n$ by setting $s_n=\sum\limits_{k=1}^{n}\zeta_k$ for all $n\in\mathbb{N}$. The infinite series (1.5) converges if and only if

$$\lim_{n\to\infty}s_n \tag{1.6}$$

exists; if the limit (1.6) does not exist, the series diverges. If $\lim\limits_{n\to\infty}s_n=\zeta^*$ then, ζ^* is called the sum of series, and we write $\sum\limits_{k=1}^{\infty}\zeta_k=\zeta^*$.

Lemma 1.7. *[11] The infinite series* (1.5) *converges and has the sum $\zeta^*=\zeta_1^*+j\zeta_2^*$ if and only if the following infinite series converge and have the sums shown:*

$$\sum_{k=1}^{\infty}\zeta_{1k}=\zeta_1^*,\ \sum_{k=1}^{\infty}\zeta_{2k}=\zeta_2^*.$$

Lemma 1.8. *[11] By definition of absolutely convergence, we know that $\sum\limits_{k=1}^{\infty}\zeta_k$ converges absolutely if and only if $\sum\limits_{k=1}^{\infty}\|\zeta_k\|_{\mathbb{BC}}$ converges. Then a series in $\mathbb{BC}$ converges if it converges absolutely in $\mathbb{BC}$.*

Lemma 1.9 (Young' s Inequality). *[4] Let* $1 < p < q < \infty$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $a.b \leq \frac{a^p}{p} + \frac{b^q}{q}$ *for* a *and* b *positive real numbers. The equality holds if* $a^p = b^q$.

Lemma 1.10. *[16] Let* $p \in (0, 1)$. *Then for* $a \geq 0$ *and* $b \geq 0$ *we have* $(a + b)^p \leq a^p + b^p$. *The equality holds if and only if at least one of* a *and* b *is equal to* 0.

Lemma 1.11 (Hölder' s Inequality). *[16] Let* $(\alpha_n : n \in \mathbb{N})$ *and* $(\beta_n : n \in \mathbb{N})$ *be two sequences of complex numbers. Let* $p, q \in (1, \infty)$ *be conjugates, that is,* $\frac{1}{p} + \frac{1}{q} = 1$. *Then, we have Hölder' s inequality for series, that is,*

$$\sum_{n=1}^{\infty} |\alpha_n \beta_n| \leq \left[\sum_{n=1}^{\infty} |\alpha_n|^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} |\beta_n|^p\right]^{\frac{1}{p}}.$$

2 Some Inequalities with Euclidean Norm in the Set of Bicomplex Numbers

In this section, we introduce three inequalities with Euclidean norm in the set of bicomplex numbers which will be required in the subsequent sections.

Theorem 2.1. *Let* $\zeta, \eta \in \mathbb{BC}$. *Then, the following inequality holds :*

$$\frac{\|\zeta + \eta\|_{\mathbb{BC}}}{1 + \|\zeta + \eta\|_{\mathbb{BC}}} \leq \frac{\|\zeta\|_{\mathbb{BC}}}{1 + \|\zeta\|_{\mathbb{BC}}} + \frac{\|\eta\|_{\mathbb{BC}}}{1 + \|\eta\|_{\mathbb{BC}}}.$$

Proof. Define the function $f : \mathbb{R} - \{-1\} \to \mathbb{R}, f(t) = \frac{t}{t+1}$. Since $f'(t) = \frac{1}{(1+t)^2}$ for all $t \in \mathbb{R} - \{-1\}$, the function is monotone increasing. Thus, since $\|\zeta + \eta\|_{\mathbb{BC}} \neq -1$ and $\|\zeta + \eta\|_{\mathbb{BC}} \leq \|\zeta\|_{\mathbb{BC}} + \|\eta\|_{\mathbb{BC}}$ for all $\zeta, \eta \in \mathbb{BC}$, we have $f(\|\zeta + \eta\|_{\mathbb{BC}}) \leq f(\|\zeta\|_{\mathbb{BC}} + \|\eta\|_{\mathbb{BC}})$ and hence

$$\begin{aligned} \frac{\|\zeta + \eta\|_{\mathbb{BC}}}{1 + \|\zeta + \eta\|_{\mathbb{BC}}} &\leq \frac{\|\zeta\|_{\mathbb{BC}} + \|\eta\|_{\mathbb{BC}}}{1 + \|\zeta\|_{\mathbb{BC}} + \|\eta\|_{\mathbb{BC}}} \\ &= \frac{\|\zeta\|_{\mathbb{BC}}}{1 + \|\zeta\|_{\mathbb{BC}} + \|\eta\|_{\mathbb{BC}}} + \frac{\|\eta\|_{\mathbb{BC}}}{1 + \|\zeta\|_{\mathbb{BC}} + \|\eta\|_{\mathbb{BC}}} \\ &\leq \frac{\|\zeta\|_{\mathbb{BC}}}{1 + \|\zeta\|_{\mathbb{BC}}} + \frac{\|\eta\|_{\mathbb{BC}}}{1 + \|\eta\|_{\mathbb{BC}}} \end{aligned}$$

holds. This is what we wished to show. □

Theorem 2.2 (Bicomplex Hölder' s Inequality). *Let* p *and* q *be real numbers with* $1 < p < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\zeta_k, \eta_k \in \mathbb{BC}$ *for* $k \in \{1, 2, ..., n\}$. *Then*

$$\sum_{k=1}^{n} \|\zeta_k \eta_k\|_{\mathbb{BC}} \leq \sqrt{2} \left(\sum_{k=1}^{n} \|\zeta_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \|\eta_k\|_{\mathbb{BC}}^q\right)^{\frac{1}{q}}.$$

Proof. Let us take

$$\alpha = \frac{\|\zeta_k\|_{\mathbb{BC}}}{\left(\sum\limits_{k=1}^{n} \|\zeta_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}}, \beta = \frac{\|\eta_k\|_{\mathbb{BC}}}{\left(\sum\limits_{k=1}^{n} \|\eta_k\|_{\mathbb{BC}}^q\right)^{\frac{1}{q}}}.$$

By Young' s inequality, we get

$$\alpha.\beta = \frac{\|\zeta_k\|_{\mathbb{BC}} \|\eta_k\|_{\mathbb{BC}}}{\left(\sum\limits_{k=1}^{n} \|\zeta_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} \left(\sum\limits_{k=1}^{n} \|\eta_k\|_{\mathbb{BC}}^q\right)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{\|\zeta_k\|_{\mathbb{BC}}^p}{\sum\limits_{k=1}^{n} \|\zeta_k\|_{\mathbb{BC}}^p} + \frac{1}{q} \frac{\|\eta_k\|_{\mathbb{BC}}^q}{\sum\limits_{k=1}^{n} \|\eta_k\|_{\mathbb{BC}}^q}.$$

Termwise summation gives

$$\frac{\sum\limits_{k=1}^{n} \|\zeta_k\|_{\mathbb{BC}} \|\eta_k\|_{\mathbb{BC}}}{\left(\sum\limits_{k=1}^{n} \|\zeta_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} \left(\sum\limits_{k=1}^{n} \|\eta_k\|_{\mathbb{BC}}^q\right)^{\frac{1}{q}}} \leq \frac{1}{p} + \frac{1}{q} = 1$$

and from this

$$\begin{aligned}
\sum_{k=1}^{n}\|\zeta_k\eta_k\|_{\mathbb{BC}} &\leq \sum_{k=1}^{n}\sqrt{2}\,\|\zeta_k\|_{\mathbb{BC}}\,\|\eta_k\|_{\mathbb{BC}} \\
&= \sqrt{2}\sum_{k=1}^{n}\|\zeta_k\|_{\mathbb{BC}}\,\|\eta_k\|_{\mathbb{BC}} \\
&\leq \sqrt{2}\left(\sum_{k=1}^{n}\|\zeta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}\|\eta_k\|_{\mathbb{BC}}^{q}\right)^{\frac{1}{q}}
\end{aligned}$$

by the inequality (1.2) in Lemma 1.3. This completes the proof. □

Theorem 2.3 (Bicomplex Minkowski' s Inequality). *Let p be a real number with $1 < p < \infty$ and $\zeta_k, \eta_k \in \mathbb{BC}$ for $k \in \{1, 2, ..., n\}$. Then*

$$\left(\sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}} \leq \left[\left(\sum_{k=1}^{n}\|\zeta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}\|\eta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}\right].$$

Proof. We have

$$\begin{aligned}
\sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p} &= \sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p-1}\,\|\zeta_k+\eta_k\|_{\mathbb{BC}} \\
&\leq \sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p-1}\left(\|\zeta_k\|_{\mathbb{BC}}+\|\eta_k\|_{\mathbb{BC}}\right) \\
&= \sum_{k=1}^{n}\|\zeta_k\|_{\mathbb{BC}}\,\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p-1}+\sum_{k=1}^{n}\|\eta_k\|_{\mathbb{BC}}\,\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p-1}.
\end{aligned}$$

Set $q = \frac{p}{p-1}$. Then $\frac{1}{p}+\frac{1}{q} = 1$, so by Lemma 1.11 we write

$$\begin{aligned}
\sum_{k=1}^{n}\|\zeta_k\|_{\mathbb{C}_2}\,\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p-1} &\leq \left(\sum_{k=1}^{n}\|\zeta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{(p-1)q}\right)^{\frac{1}{q}} \\
\sum_{k=1}^{n}\|\eta_k\|_{\mathbb{BC}}\,\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p-1} &\leq \left(\sum_{k=1}^{n}\|\eta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{(p-1)q}\right)^{\frac{1}{q}}.
\end{aligned}$$

Adding these two inequalities, we obtain that

$$\sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p} \leq \left[\left(\sum_{k=1}^{n}\|\zeta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}\|\eta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}\right]\left(\sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{(p-1)q}\right)^{\frac{1}{q}}.$$

Observing that $(p-1)\,q = p$ by definition, we have

$$\sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p} \leq \left[\left(\sum_{k=1}^{n}\|\zeta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}\|\eta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}\right]\left(\sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{q}}$$

and so

$$\left(\sum_{k=1}^{n}\|\zeta_k+\eta_k\|_{\mathbb{BC}}^{p}\right)^{1-\frac{1}{q}} \leq \left[\left(\sum_{k=1}^{n}\|\zeta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}\|\eta_k\|_{\mathbb{BC}}^{p}\right)^{\frac{1}{p}}\right].$$

Finally, observe that $\frac{1}{p} = 1-\frac{1}{q}$, and the result follows as required. □

3 Some Sequence Spaces over the Set of Bicomplex Numbers

In this section, we define the sets $w(\mathbb{BC}), l_\infty(\mathbb{BC}), c(\mathbb{BC}), c_0(\mathbb{BC})$ and $l_p(\mathbb{BC})$ of all, bounded, convergent, null and absolutely $p-$ summable bicomplex sequences, we show that these sets are metric spaces and we also give completeness property of these metric spaces. Then we say,

$$\begin{aligned}
w(\mathbb{BC}) &: = \{\zeta = (\zeta_k) : \zeta_k \in \mathbb{BC} \text{ for all } k \in \mathbb{N}\},\\
l_\infty(\mathbb{BC}) &: = \left\{\zeta = (\zeta_k) \in w(\mathbb{BC}) : \sup_{k\in\mathbb{N}} \|\zeta_k\|_{\mathbb{BC}} < \infty\right\},\\
c(\mathbb{BC}) &: = \left\{\zeta = (\zeta_k) \in w(\mathbb{BC}) : \text{there exists } l^* \in \mathbb{BC} \text{ such that } \lim_{k\to\infty} \zeta_k = l^*\right\},\\
c_0(\mathbb{BC}) &: = \left\{\zeta = (\zeta_k) \in w(\mathbb{BC}) : \lim_{k\to\infty} \zeta_k = 0\right\},\\
l_p(\mathbb{BC}) &: = \left\{\zeta = (\zeta_k) \in w(\mathbb{BC}) : \sum_{k=1}^{\infty} \|\zeta_k\|_{\mathbb{BC}}^p < \infty\right\} \text{ for } 0 < p < \infty.
\end{aligned}$$

Definition 3.1. The algebraic operations addition $\oplus$, scalar multiplication $\odot$ and multiplication $\otimes$ defined on $w(\mathbb{BC})$ as follows, respectively :

$$\begin{aligned}
\oplus &: w(\mathbb{BC}) \times w(\mathbb{BC}) \to w(\mathbb{BC}), (s,t) \to s \oplus t = (s_k + t_k),\\
\odot &: \mathbb{R} \times w(\mathbb{BC}) \to w(\mathbb{BC}), (\alpha, s) \to \alpha \odot s = (\alpha s_k),\\
\otimes &: w(\mathbb{BC}) \times w(\mathbb{BC}) \to w(\mathbb{BC}), (s,t) \to s \otimes t = (s_k t_k),
\end{aligned}$$

where $s = (s_k), t = (t_k) \in w(\mathbb{BC})$ *and* $\alpha \in \mathbb{R}$.

Theorem 3.2. *The set* $w(\mathbb{BC})$ *forms a linear space over* $\mathbb{R}$ *with respect to addition* $\oplus$ *and scalar multiplication* $\odot$.

Proof. The proof of this theorem is direct applications of definitions. □

Theorem 3.3. *Define the function* $d_{w(\mathbb{BC})}$ *on the space* $w(\mathbb{BC})$ *of all bicomplex sequences by*

$$d_{w(\mathbb{BC})} : w(\mathbb{BC}) \times w(\mathbb{BC}) \to [0,\infty), (s,t) \to d_{w(\mathbb{BC})}(s,t) = \sum_{k=1}^{\infty} \mu_k . \frac{\|s_k - t_k\|_{\mathbb{BC}}}{1 + \|s_k - t_k\|_{\mathbb{BC}}},$$

where $s = (s_k), t = (t_k) \in w(\mathbb{BC})$ *and* $(\mu_k) \subset [0,\infty)$ *such that* $\sum_{k=1}^{\infty} \mu_k$ *is convergent with* $\mu_k > 0$ *for all* $k \in \mathbb{N}$. *Then,* $\left(w(\mathbb{BC}), d_{w(\mathbb{BC})}\right)$ *is a metric space.*

Proof. We show that $d_{w(\mathbb{BC})}$ satisfies the metric axioms on the space $w(\mathbb{BC})$ of all bicomplex sequences. We have $\|s_k - t_k\|_{\mathbb{BC}} \geq 0$ and hence $\mu_k . \frac{\|s_k - t_k\|_{\mathbb{BC}}}{1+\|s_k - t_k\|_{\mathbb{BC}}} \geq 0$ for all $s_k, t_k \in \mathbb{BC}$ and $\mu_k \in [0,\infty)$. This means that $d_{w(\mathbb{BC})}(s,t) = \sum_{k=1}^{\infty} \mu_k . \frac{\|s_k - t_k\|_{\mathbb{BC}}}{1+\|s_k - t_k\|_{\mathbb{BC}}} \geq 0$ for all $s, t \in w(\mathbb{BC})$. It is easy to see that

$$\begin{aligned}
d_{w(\mathbb{BC})}(s,t) &= \sum_{k=1}^{\infty} \mu_k . \frac{\|s_k - t_k\|_{\mathbb{BC}}}{1 + \|s_k - t_k\|_{\mathbb{BC}}} = 0 \Longleftrightarrow \mu_k . \frac{\|s_k - t_k\|_{\mathbb{BC}}}{1 + \|s_k - t_k\|_{\mathbb{BC}}} = 0, \forall k \in \mathbb{N}\\
&\Longleftrightarrow \frac{\|s_k - t_k\|_{\mathbb{BC}}}{1 + \|s_k - t_k\|_{\mathbb{BC}}} = 0, \forall k \in \mathbb{N}\\
&\Longleftrightarrow \|s_k - t_k\|_{\mathbb{BC}} = 0, \forall k \in \mathbb{N}\\
&\Longleftrightarrow s_k = t_k, \forall k \in \mathbb{N}\\
&\Longleftrightarrow s = t,
\end{aligned}$$

and also

$$d_{w(\mathbb{BC})}(s,t)=\sum_{k=1}^{\infty}\mu_k.\frac{\|s_k-t_k\|_{\mathbb{BC}}}{1+\|s_k-t_k\|_{\mathbb{BC}}}=\sum_{k=1}^{\infty}\mu_k.\frac{\|t_k-s_k\|_{\mathbb{BC}}}{1+\|t_k-s_k\|_{\mathbb{BC}}}=d_{w(\mathbb{BC})}(t,s)$$

for all $s=(s_k), t=(t_k)\in w(\mathbb{BC})$.

Now, we show that $d_{w(\mathbb{BC})}(s,u)\leq d_{w(\mathbb{BC})}(s,t)+d_{w(\mathbb{BC})}(t,u)$ for all $s=(s_k), t=(t_k), u=(u_k)\in w(\mathbb{BC})$ .We know that by Theorem 2.1

$$\begin{aligned}\sum_{k=1}^{n}\mu_k.\frac{\|s_k-u_k\|_{\mathbb{BC}}}{1+\|s_k-u_k\|_{\mathbb{BC}}} &= \sum_{k=1}^{n}\mu_k.\frac{\|(s_k-t_k)+(t_k-u_k)\|_{\mathbb{BC}}}{1+\|(s_k-t_k)+(t_k-u_k)\|_{\mathbb{BC}}} \\ &\leq \sum_{k=1}^{n}\mu_k.\left(\frac{\|s_k-t_k\|_{\mathbb{BC}}}{1+\|s_k-t_k\|_{\mathbb{BC}}}+\frac{\|t_k-u_k\|_{\mathbb{BC}}}{1+\|t_k-u_k\|_{\mathbb{BC}}}\right) \\ &= \sum_{k=1}^{n}\mu_k.\frac{\|s_k-t_k\|_{\mathbb{BC}}}{1+\|s_k-t_k\|_{\mathbb{BC}}}+\sum_{k=1}^{n}\mu_k.\frac{\|t_k-u_k\|_{\mathbb{BC}}}{1+\|t_k-u_k\|_{\mathbb{BC}}}\end{aligned} \tag{3.1}$$

holds for all $n\in\mathbb{N}$ and inequalities

$$\begin{aligned}\mu_k.\frac{\|s_k-u_k\|_{\mathbb{BC}}}{1+\|s_k-u_k\|_{\mathbb{BC}}} &\leq \mu_k \\ \mu_k.\frac{\|s_k-t_k\|_{\mathbb{BC}}}{1+\|s_k-t_k\|_{\mathbb{BC}}} &\leq \mu_k \\ \mu_k.\frac{\|t_k-u_k\|_{\mathbb{BC}}}{1+\|t_k-u_k\|_{\mathbb{BC}}} &\leq \mu_k\end{aligned}$$

hold for all $k\in\mathbb{N}$. Then, the comparison test implies the convergence of the series

$$\sum_{k=1}^{\infty}\mu_k.\frac{\|s_k-u_k\|_{\mathbb{BC}}}{1+\|s_k-u_k\|_{\mathbb{BC}}},\sum_{k=1}^{\infty}\mu_k.\frac{\|s_k-t_k\|_{\mathbb{BC}}}{1+\|s_k-t_k\|_{\mathbb{BC}}},\sum_{k=1}^{\infty}\mu_k.\frac{\|t_k-u_k\|_{\mathbb{BC}}}{1+\|t_k-u_k\|_{\mathbb{BC}}}.$$

Therefore, by letting $n\to\infty$ in (3.1), $d_{w(\mathbb{BC})}(s,u)\leq d_{w(\mathbb{BC})}(s,t)+d_{w(\mathbb{BC})}(t,u)$, as required. □

Theorem 3.4. *The set of* $l_\infty(\mathbb{BC})$ *is a sequence space.*

Proof. It is obvious that the inclusion $l_\infty(\mathbb{BC})\subset w(\mathbb{BC})$ holds.

(i) Let $s=(s_k), t=(t_k)\in l_\infty(\mathbb{BC})$. In this situation, combining the hypothesis $\sup_{k\in\mathbb{N}}\|s_k\|_{\mathbb{BC}}<\infty$, $\sup_{k\in\mathbb{N}}\|t_k\|_{\mathbb{BC}}<\infty$ with the fact $\|s_k+t_k\|_{\mathbb{BC}}\leq\|s_k\|_{\mathbb{BC}}+\|t_k\|_{\mathbb{BC}}$, we can see that

$$\sup_{k\in\mathbb{N}}\|s_k+t_k\|_{\mathbb{BC}}\leq\sup_{k\in\mathbb{N}}\|s_k\|_{\mathbb{BC}}+\sup_{k\in\mathbb{N}}\|t_k\|_{\mathbb{BC}}<\infty$$

which means that $s\oplus t\in l_\infty(\mathbb{BC})$.

(ii) Let $\alpha\in\mathbb{R}$ and $s=(s_k)\in l_\infty(\mathbb{BC})$. Since $\|\alpha s_k\|_{\mathbb{BC}}=|\alpha|\,.\,\|s_k\|_{\mathbb{BC}}$ and $\sup_{k\in\mathbb{N}}\|s_k\|_{\mathbb{BC}}<\infty$, we can easily derive that

$$\sup_{k\in\mathbb{N}}\|\alpha s_k\|_{\mathbb{BC}}=|\alpha|\,.\sup_{k\in\mathbb{N}}\|s_k\|_{\mathbb{BC}}<\infty.$$

Hence, $\alpha\odot s\in l_\infty(\mathbb{BC})$. That is to say that $l_\infty(\mathbb{BC})$ is a subspace of the space $w(\mathbb{BC})$. □

Theorem 3.5. *The norm function* $\|\|_{\mathbb{BC}}$ *defined by* (1.3) *is continuous.*

Proof. The proof depends on the inequality (1.1) in Lemma 1.3. □

Theorem 3.6. *Define the function $d_{l_\infty(\mathbb{BC})}$ by*

$$d_{l_\infty(\mathbb{BC})} : l_\infty(\mathbb{BC}) \times l_\infty(\mathbb{BC}) \to [0,\infty), (s,t) \to d_{l_\infty(\mathbb{BC})}(s,t) = \sup_{k\in\mathbb{N}} \|s_k - t_k\|_{\mathbb{BC}},$$

where $s = (s_k), t = (t_k) \in w(\mathbb{BC})$. Then $\left(l_\infty(\mathbb{BC}), d_{l_\infty(\mathbb{BC})}\right)$ is a complete metric space.

Proof. It is not hard to show that $d_{l_\infty(\mathbb{BC})}$ satisfies the metric axioms on the space $l_\infty(\mathbb{BC})$. So, we omit the details.

Now, we show that $l_\infty(\mathbb{BC})$ is complete. Let (s_m) be an arbitrary Cauchy sequence in $l_\infty(\mathbb{BC})$, where $s_m = (s_k^m)_{k\in\mathbb{N}}$. Then, there exists an $n_0(\varepsilon) \in \mathbb{N}$ such that $d_{l_\infty(\mathbb{BC})}(s_m, s_r) = \sup_{k\in\mathbb{N}} \|s_k^m - s_k^r\|_{\mathbb{BC}} < \varepsilon$ for all $m, r \geq n_0(\varepsilon)$. Then, for any fixed k,

$$\|s_k^m - s_k^r\|_{\mathbb{BC}} < \varepsilon \tag{3.2}$$

for all $m, r \geq n_0(\varepsilon)$. In this case, for any fixed k, $\left(s_k^1, s_k^2, ..., s_k^m, ...\right)$ is a bicomplex Cauchy sequence and so, it converges to a point say $s_k^* \in \mathbb{BC}$ by Definition 1.4. Define the sequence $s^* = (s_k^*) = (s_1^*, s_2^*, ...)$ with infinitely many limits $s_1^*, s_2^*, ...$ and show $s^* \in l_\infty(\mathbb{BC})$ and $s_m \to s^*$, as $m \to \infty$.

Indeed, in (3.2), by letting $r \to \infty$ for any fixed k and using the continuity of Euclidean norm function $\|\|_{\mathbb{BC}}$ by Theorem 3.5, for all $m \geq n_0(\varepsilon)$ we obtain that

$$\|s_k^m - s_k^*\|_{\mathbb{BC}} \leq \varepsilon \tag{3.3}$$

and so, $d_{l_\infty(\mathbb{BC})}(s_m, s^*) = \sup_{k\in\mathbb{N}} \|s_k^m - s_k^*\|_{\mathbb{BC}} \leq \varepsilon$. This shows that the sequence $(s_m) \subset l_\infty(\mathbb{BC})$ converges to $s^* = (s_k^*) \in w(\mathbb{BC})$.

On the other hand, since $s_m = (s_k^m)_{k\in\mathbb{N}} \in l_\infty(\mathbb{BC})$ for each $n \in \mathbb{N}$, there exists $t_m \in (0,\infty)$ such that $\|s_k^m\|_{\mathbb{BC}} \leq t_m$ for all $k \in \mathbb{N}$. Therefore, by (3.3), the inequality

$$\|s_k^*\|_{\mathbb{BC}} \leq \|s_k^* - s_k^m\|_{\mathbb{BC}} + \|s_k^m\|_{\mathbb{BC}} \leq \varepsilon + t_m$$

holds for all $k \in \mathbb{N}$ and for all $m \geq n_0(\varepsilon)$, which is independent of k. Hence $s^* = (s_k^*) \in l_\infty(\mathbb{BC})$ which means that $l_\infty(\mathbb{BC})$ is complete. The proof is completed. □

Corollary 3.7. *$l_\infty(\mathbb{BC})$ is a Banach space with the norm $\|\|_{l_\infty(\mathbb{BC})}$ defined by*

$$\|s\|_{l_\infty(\mathbb{BC})} = \sup_{k\in\mathbb{N}} \|s_k\|_{\mathbb{BC}}; s = (s_k) \in l_\infty(\mathbb{BC}). \tag{3.4}$$

Proof. Since it is known by Theorem 3.6 that $l_\infty(\mathbb{BC})$ is a complete metric space with the metric $d_{l_\infty(\mathbb{BC})}$ induced by the norm $\|\|_{l_\infty(\mathbb{BC})}$ defined by (3.4). Then, the proof is clear. □

Theorem 3.8. *The sets $c(\mathbb{BC}), c_0(\mathbb{BC})$ and $l_p(\mathbb{BC})$ for $0 < p < \infty$ are sequence spaces.*

Proof. It is trivial that the inclusions $c(\mathbb{BC}) \subset w(\mathbb{BC}), c_0(\mathbb{BC}) \subset w(\mathbb{BC})$ and $l_p(\mathbb{BC}) \subset w(\mathbb{BC})$ for $0 < p < \infty$ hold. Firstly, we consider the set $c(\mathbb{BC})$.

(i) Let $s = (s_k), t = (t_k) \in c(\mathbb{BC})$. Then, there exist $l_1^*, l_2^* \in \mathbb{BC}$ such that $\lim_{k\to\infty} s_k = l_1^*$ and $\lim_{k\to\infty} t_k = l_2^*$, and so for every $\varepsilon > 0$ there exist $k_1(\varepsilon), k_2(\varepsilon) \in \mathbb{N}$ such that $\|s_k - l_1^*\|_{\mathbb{BC}} < \frac{\varepsilon}{2}$ for all $k \geq k_1(\varepsilon)$ and $\|t_k - l_2^*\|_{\mathbb{BC}} < \frac{\varepsilon}{2}$ for all $k \geq k_2(\varepsilon)$. Therefore, taking $k_0(\varepsilon) = \max\{k_1(\varepsilon), k_2(\varepsilon)\}$, we obtain that

$$\begin{aligned} \|(s_k + t_k) - (l_1^* + l_2^*)\|_{\mathbb{BC}} &= \|(s_k - l_1^*) + (t_k - l_2^*)\|_{\mathbb{BC}} \\ &\leq \|s_k - l_1^*\|_{\mathbb{BC}} + \|t_k - l_2^*\|_{\mathbb{BC}} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $k \geq k_0(\varepsilon)$ which means that

$$\lim_{k\to\infty}(s_k + t_k) = l_1^* + l_2^* = \lim_{k\to\infty} s_k + \lim_{k\to\infty} t_k$$

and from this $s \oplus t \in c(\mathbb{BC})$.

(ii) Let $s = (s_k) \in c(\mathbb{BC})$ and $\alpha \in \mathbb{R} - \{0\}$. Since $s \in c(\mathbb{BC})$, there exists an $l^* \in \mathbb{BC}$ such that $\lim\limits_{k\to\infty} s_k = l^*$ and so for every $\varepsilon > 0$ there exists an $k_0(\varepsilon) \in \mathbb{N}$ such that $\|s_k - l^*\|_{\mathbb{BC}} < \frac{\varepsilon}{|\alpha|}$ for all $k \geq k_0(\varepsilon)$. Thus, we obtain for all $k \geq k_0(\varepsilon)$ that

$$\|(\alpha s_k) - (\alpha l^*)\|_{\mathbb{BC}} = \|\alpha (s_k - l^*)\|_{\mathbb{BC}} = |\alpha| \, \|s_k - l^*\|_{\mathbb{BC}} \leq |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon$$

which implies that

$$\lim_{k\to\infty} (\alpha s_k) = \alpha l^* = \alpha \lim_{k\to\infty} s_k,$$

and so $\alpha \odot s \in c(\mathbb{BC})$.

The proof is clear for $\alpha = 0$. Therefore, we have proved that $c(\mathbb{BC})$ is a subspace of the space $w(\mathbb{BC})$. Also, taking $l_1^* = l_2^* = l^* = 0$ above, by a routine verification, we can easily show that $c_0(\mathbb{BC})$ is the sequence space.

Now, we show that $l_p(\mathbb{BC})$ is sequence space, where $0 < p < \infty$.

(i) Let $s = (s_k), t = (t_k) \in l_p(\mathbb{BC})$. Then $\sum\limits_{k=1}^{\infty} \|s_k\|_{\mathbb{BC}}^p < \infty$ and $\sum\limits_{k=1}^{\infty} \|t_k\|_{\mathbb{BC}}^p < \infty$. We know by bicomplex Minkowski' s inequality for $1 < p < \infty$ that

$$\sum_{k=1}^{n} \|s_k + t_k\|_{\mathbb{BC}}^p \leq \left[\left(\sum_{k=1}^{n} \|s_k\|_{\mathbb{BC}}^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} \|t_k\|_{\mathbb{BC}}^p \right)^{\frac{1}{p}} \right]^p$$

holds for all $n \in \mathbb{N}$ and the comparison test implies the convergence of the series $\sum\limits_{k=1}^{\infty} \|s_k + t_k\|_{\mathbb{BC}}^p$. Therefore, $s \oplus t \in l_p(\mathbb{BC})$ for $1 < p < \infty$, as required.

For $0 < p \leq 1$, by Lemma 1.10,

$$\begin{aligned} \sum_{k=1}^{n} \|s_k + t_k\|_{\mathbb{BC}}^p &\leq \sum_{k=1}^{n} \left(\|s_k\|_{\mathbb{BC}} + \|t_k\|_{\mathbb{BC}}\right)^p \\ &\leq \sum_{k=1}^{n} \left(\|s_k\|_{\mathbb{BC}}^p + \|t_k\|_{\mathbb{BC}}^p\right) \\ &= \sum_{k=1}^{n} \|s_k\|_{\mathbb{BC}}^p + \sum_{k=1}^{n} \|t_k\|_{\mathbb{BC}}^p \end{aligned}$$

holds for all $n \in \mathbb{N}$ and the comparison test implies the convergence of the series $\sum\limits_{k=1}^{\infty} \|s_k + t_k\|_{\mathbb{BC}}^p$. Therefore, $s \oplus t \in l_p(\mathbb{BC})$ for $0 < p \leq 1$, as required.

(ii) Let $s = (s_k) \in l_p(\mathbb{BC})$ and $\alpha \in \mathbb{R} - \{0\}$. Since $s \in l_p(\mathbb{BC})$, we can write $\sum\limits_{k=1}^{\infty} \|s_k\|_{\mathbb{BC}}^p < \infty$. Thus, we have

$$\sum_{k=1}^{\infty} \|\alpha s_k\|_{\mathbb{BC}}^p = \sum_{k=1}^{\infty} |\alpha|^p \|s_k\|_{\mathbb{BC}}^p = |\alpha|^p \sum_{k=1}^{\infty} \|s_k\|_{\mathbb{BC}}^p < \infty$$

which implies that $\alpha \odot s \in l_p(\mathbb{BC})$. The proof is clear for $\alpha = 0$. That is to say $l_p(\mathbb{BC})$ is a subspace of $w(\mathbb{BC})$. □

Theorem 3.9. *$\left(c(\mathbb{BC}), d_{l_\infty(\mathbb{BC})}\right)$ and $\left(c_0(\mathbb{BC}), d_{l_\infty(\mathbb{BC})}\right)$ are complete metric spaces.*

Proof. We show that the metric space $\left(c(\mathbb{BC}), d_{l_\infty(\mathbb{BC})}\right)$ is complete. Let (s_m) be an arbitrary Cauchy sequence in the space $c(\mathbb{BC})$, where $s_m = \left(s_k^m\right)_{k\in\mathbb{N}}$. Then, for every $\varepsilon > 0$ there exists an $n_0(\varepsilon) \in \mathbb{N}$ such that $d_{l_\infty(\mathbb{BC})}(s_m, s_r) = \sup\limits_{k\in\mathbb{N}} \|s_k^m - s_k^r\|_{\mathbb{BC}} < \frac{\varepsilon}{3}$ for all $m, r \geq n_0(\varepsilon)$. Hence, for any fixed k,

$$\|s_k^m - s_k^r\|_{\mathbb{BC}} < \frac{\varepsilon}{3} \tag{3.5}$$

for all $m, r \geq n_0(\varepsilon)$. In this case, for any fixed k, $\left(s_k^1, s_k^2, ..., s_k^m, ...\right)$ is a bicomplex Cauchy sequence and so, it converges to a point say $s_k^* \in \mathbb{BC}$. Define the sequence $s^* = (s_k^*) = (s_1^*, s_2^*, ...)$ with these limits and show that $s^* \in c(\mathbb{BC})$ and $s_m \to s^*$, as $m \to \infty$. Indeed, by (3.5), by letting $r \to \infty$, we obtain $d_{l_\infty(\mathbb{BC})}(s_m, s^*) = \sup_{k\in\mathbb{N}} \|s_k^m - s_k^*\|_{\mathbb{BC}} \leq \frac{\varepsilon}{3}$ for all $m \geq n_0(\varepsilon)$. Therefore, the sequence $(s_m) \subset c(\mathbb{BC})$ converges to $s^* = (s_k^*) \in w(\mathbb{BC})$. On the other hand, since $(s_k^{n_0}) \in c(\mathbb{BC})$ is a bicomplex Cauchy sequence, for every $\varepsilon > 0$ there exists an $k_0(\varepsilon) \in \mathbb{N}$ such that $\|s_k^{n_0} - s_l^{n_0}\|_{\mathbb{BC}} < \frac{\varepsilon}{3}$ for all $k, l \geq k_0(\varepsilon)$. In this situation, for every $\varepsilon > 0$

$$\begin{aligned}
\|s_k^* - s_l^*\|_{\mathbb{BC}} &= \|s_k^* - s_k^{n_0} + s_k^{n_0} - s_l^{n_0} + s_l^{n_0} - s_l^*\|_{\mathbb{BC}} \\
&\leq \|s_k^* - s_k^{n_0}\|_{\mathbb{BC}} + \|s_k^{n_0} - s_l^{n_0}\|_{\mathbb{BC}} + \|s_l^{n_0} - s_l^*\|_{\mathbb{BC}} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

for all $k, l \geq k_0(\varepsilon)$, and so the sequence $s^* = (s_k^*)$ is a bicomplex Cauchy sequence. Since $\mathbb{BC}$ is complete, $s^* = (s_k^*)$ is convergent in $\mathbb{BC}$. Finally, obtain that $s^* = (s_k^*) \in c(\mathbb{BC})$ and the result follows as required.

Also, we can similarly show completeness of $c_0(\mathbb{BC})$ with completeness of $c(\mathbb{BC})$. □

Corollary 3.10. *$c(\mathbb{BC})$ and $c_0(\mathbb{BC})$ are Banach spaces with the norm $\|\|_{l_\infty(\mathbb{BC})}$ defined by* (3.4).

Proof. The proof depends on Theorem 3.9. □

Theorem 3.11. *$\left(l_p(\mathbb{BC}), d_{l_p(\mathbb{BC})}\right)$ is a complete metric space for $0 < p < \infty$, where $d_{l_p(\mathbb{BC})}$ is defined as follows :*

$$\begin{aligned}
d_{l_p(\mathbb{BC})}(s,t) &: \quad l_p(\mathbb{BC}) \times l_p(\mathbb{BC}) \to [0,\infty), \\
(s,t) &\to \quad d_{lp(\mathbb{BC})}(s,t) = \begin{cases} \sum\limits_{k=1}^{\infty} \|s_k - t_k\|_{\mathbb{BC}}^p, & 0 < p \leq 1 \\ \\ \left(\sum\limits_{k=1}^{\infty} \|s_k - t_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}, & 1 < p < \infty \end{cases},
\end{aligned}$$

where $s = (s_k), t = (t_k) \in l_p(\mathbb{BC})$.

Proof. Firstly, we consider the space $l_p(\mathbb{BC})$ with $1 < p < \infty$. We know that $d_{lp(\mathbb{BC})}(s,t) \geq 0$ for all $s, t \in l_p(\mathbb{BC})$ since $\|s_k - t_k\|_{\mathbb{BC}} \geq 0$ for all $s_k, t_k \in \mathbb{BC}$. Also,

$$\begin{aligned}
d_{lp(\mathbb{BC})}(s,t) &= \left(\sum_{k=1}^{\infty} \|s_k - t_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} = 0 \Longleftrightarrow \sum_{k=1}^{\infty} \|s_k - t_k\|_{\mathbb{BC}}^p = 0 \\
&\Longleftrightarrow \|s_k - t_k\|_{\mathbb{BC}}^p = 0, \forall k \in \mathbb{N} \\
&\Longleftrightarrow \|s_k - t_k\|_{\mathbb{BC}} = 0, \forall k \in \mathbb{N} \\
&\Longleftrightarrow s_k = t_k, \forall k \in \mathbb{N} \\
&\Longleftrightarrow s = t
\end{aligned}$$

and

$$d_{lp(\mathbb{BC})}(s,t) = \left(\sum_{k=1}^{\infty} \|s_k - t_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{\infty} \|t_k - s_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} = d_{lp(\mathbb{BC})}(t,s)$$

for all $s, t \in l_p(\mathbb{BC})$. On the other hand, by bicomplex Minkowski' s inequality we have for $s = (s_k), t = (t_k), u = (u_k) \in l_p(\mathbb{BC})$ that

$$\begin{aligned}
d_{lp(\mathbb{BC})}(s,t) &= \left(\sum_{k=1}^{\infty} \|s_k - t_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} = \left[\sum_{k=1}^{\infty} \|(s_k - u_k) + (u_k - t_k)\|_{\mathbb{BC}}^p\right]^{\frac{1}{p}} \\
&\leq \left[\left(\sum_{k=1}^{\infty} \|s_k - u_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} \|u_k - t_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}\right] \\
&= d_{lp(\mathbb{BC})}(s,u) + d_{lp(\mathbb{BC})}(u,t).
\end{aligned}$$

Therefore, the function $d_{lp(\mathbb{BC})}$ is a metric over the space $l_p(\mathbb{BC})$ with $1<p<\infty$.

Now, we show that the metric space $\left(l_p(\mathbb{BC}), d_{l_p(\mathbb{BC})}\right)$ with $1<p<\infty$ is complete. Let (s_m) be an arbitrary Cauchy sequence in the space $l_p(\mathbb{BC})$, where $s_m=(s_k^m)_{k\in\mathbb{N}}$. Then, for every $\varepsilon>0$ there exists an $n_0(\varepsilon)\in\mathbb{N}$ such that

$$d_{l_p(\mathbb{BC})}(s_m,s_r)=\left(\sum_{k=1}^{\infty}\|s_k^m-s_k^r\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}<\varepsilon \tag{3.6}$$

for all $m,r\geq n_0(\varepsilon)$. Then, for any fixed k,

$$\|s_k^m-s_k^r\|_{\mathbb{BC}}<\varepsilon \tag{3.7}$$

for all $m,r\geq n_0(\varepsilon)$. In this case, for any fixed k, $\left(s_k^1,s_k^2,...,s_k^m,...\right)$ is a bicomplex Cauchy sequence and so, it converges to a point say s_k^*. Let us define the sequence $s^*=(s_k^*)=(s_1^*,s_2^*,...)$ with infinitely many limits $s_1^*,s_2^*,...$ and show $s^*=(s_k^*)\in l_p(\mathbb{BC})$ and $s_m\to s^*$, as $m\to\infty$. By (3.7), we can write $\|s_k^m-s_k^*\|_{\mathbb{BC}}\leq\varepsilon$ for all $m\geq n_0(\varepsilon)$ which means that $s_k^m\to s_k^*$ as $m\to\infty$. Also, from (3.6), we know that $\left(\sum\limits_{k=1}^{n}\|s_k^m-s_k^r\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}<\varepsilon$ for all $m,r\geq n_0(\varepsilon)$, and by letting $r\to\infty$, we have $\left(\sum\limits_{k=1}^{n}\|s_k^m-s_k^*\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}<\varepsilon$ for all $n\in\mathbb{N}$. Then, by letting $n\to\infty$, we obtain that $d_{l_p(\mathbb{BC})}(s_m,s^*)=\left(\sum\limits_{k=1}^{\infty}\|s_k^m-s_k^*\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}\leq\varepsilon$ for all $m\geq n_0(\varepsilon)$. Thus, the sequence $(s_m)\subset l_p(\mathbb{BC})$ converges to $s^*=(s_k^*)\in w(\mathbb{BC})$.

On the other hand, since $s_m=(s_k^m)\in l_p(\mathbb{BC})$, by bicomplex Minkowski' s inequality and convergence of the series $\sum\limits_{k=1}^{\infty}\|s_k^*-s_k^m\|_{\mathbb{BC}}^p$,

$$\begin{aligned}\left(\sum_{k=1}^{\infty}\|s_k^*\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}} &= \left(\sum_{k=1}^{\infty}\|s_k^m+(s_k^*-s_k^m)\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}\\ &\leq \left(\sum_{k=1}^{\infty}\|s_k^m\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{\infty}\|s_k^*-s_k^m\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}\\ &< \infty\end{aligned}$$

which means that $s^*=(s_k^*)\in l_p(\mathbb{BC})$. Therefore, $l_p(\mathbb{BC})$ with $1<p<\infty$ is complete. This completes the proof.

Now, we consider the space $l_p(\mathbb{BC})$ with $0<p\leq 1$. It can be shown that the function $d_{lp(\mathbb{BC})}$ is a metric over the space $l_p(\mathbb{BC})$ with $0<p\leq 1$ in the similar way to $1<p<\infty$ by using Lemma 1.10.

Now, we show that $l_p(\mathbb{BC})$ with $0<p\leq 1$ is complete. Let (s_m) be an arbitrary Cauchy sequence in the space $l_p(\mathbb{BC})$, where $s_m=(s_k^m)_{k\in\mathbb{N}}$. Then, for every $\varepsilon>0$ there exists an $n_0(\varepsilon)\in\mathbb{N}$ such that

$$d_{l_p(\mathbb{BC})}(s_m,s_r)=\sum_{k=1}^{\infty}\|s_k^m-s_k^r\|_{\mathbb{BC}}^p<\varepsilon^p \tag{3.8}$$

for all $m,r\geq n_0(\varepsilon)$. Therefore, $\|s_k^m-s_k^r\|_{\mathbb{BC}}<\varepsilon$ for any fixed $k\in\mathbb{N}$ and for all $m,r\geq n_0(\varepsilon)$. Thus, for any fixed $k\in\mathbb{N}$, $(s_k^m)=\left(s_k^1,s_k^2,...,s_k^n,...\right)$ is a bicomplex Cauchy sequence and from this, it converges, say $s_k^m\to s_k^*$ as $m\to\infty$. Define the sequence $s^*=(s_k^*)=(s_1^*,s_2^*,...)$ and show that $s_m\to s^*$, as $m\to\infty$ and $s^*=(s_k^*)\in l_p(\mathbb{BC})$. From (3.8), we obtain the inequalities for all $m,r\geq n_0(\varepsilon)$ that $\sum\limits_{k=1}^{n}\|s_k^m-s_k^r\|_{\mathbb{BC}}^p<\varepsilon^p$ and so, by letting $r\to\infty$, for all $m\geq n_0(\varepsilon)$ that $\sum\limits_{k=1}^{n}\|s_k^m-s_k^*\|_{\mathbb{BC}}^p<\varepsilon^p$ for all $n\in\mathbb{N}$ which means that as $n\to\infty$ and for all $m\geq n_0(\varepsilon)$, $d_{l_p(\mathbb{BC})}(s_m,s^*)=\sum\limits_{k=1}^{\infty}\|s_k^m-s_k^*\|_{\mathbb{BC}}^p<\varepsilon^p$. Thus, the sequence $(s_m)\subset l_p(\mathbb{BC})$ converges to

$s^{*}=(s_k^{*})\in w(\mathbb{BC})$. Since $s_m=(s_k^m)\in l_p(\mathbb{BC})$, by Lemma 1.10 and convergence of the series $\sum\limits_{k=1}^{\infty}\|s_k^{*}-s_k^m\|_{\mathbb{BC}}^p$,

$$\begin{aligned}\sum_{k=1}^{\infty}\|s_k^{*}\|_{\mathbb{BC}}^p &= \sum_{k=1}^{\infty}\|s_k^m+(s_k^{*}-s_k^m)\|_{\mathbb{BC}}^p\\ &\leq \sum_{k=1}^{\infty}(\|s_k^m\|_{\mathbb{BC}}+\|s_k^{*}-s_k^m\|_{\mathbb{BC}})^p\\ &\leq \sum_{k=1}^{\infty}\|s_k^m\|_{\mathbb{BC}}^p+\sum_{k=1}^{\infty}\|s_k^{*}-s_k^m\|_{\mathbb{BC}}^p\\ &< \infty\end{aligned}$$

which implies that $s^{*}=(s_k^{*})\in l_p(\mathbb{BC})$. That is to say that $l_p(\mathbb{BC})$ with $0<p\leq 1$ is a complete metric space. □

Corollary 3.12. *The space $l_p(\mathbb{BC})$ is a Banach space with the norm $\|\|_{l_p(\mathbb{BC})}$ defined by*

$$\|s\|_{l_p(\mathbb{BC})}=\begin{cases}\sum\limits_{k=1}^{\infty}\|s_k\|_{\mathbb{BC}}^p, & 0<p\leq 1\\ & \\ \left(\sum\limits_{k=1}^{\infty}\|s_k\|_{\mathbb{BC}}^p\right)^{\frac{1}{p}}, & 1<p<\infty\end{cases}\quad ;\ s=(s_k)\in l_p(\mathbb{BC}).$$

Proof. The proof is clear from Theorem 3.11. □

4 Concluding Remarks

In this paper, we have studied bicomplex sequence spaces with Euclidean norm in the set of bicomplex numbers. For the future, we will construct bicomplex sequence spaces with hyperbolic valued moduli of bicomplex numbers and we will investigate $\alpha-,\beta-$ and $\gamma-$ duals and multiplier spaces of them.

5 Acknowledgement

The authors are thankful to the worthy referee for his useful suggestions for the improvement of this paper.

References

[1] D. Alpay, M. E. Luna-Elizarrarás, M. Shapiro and D. C. Struppa, *Basics of functional analysis with bicomplex scalars, and bicomplex Schur analysis*, Springer Science & Business Media (2014).

[2] J. Banas and M. Mursaleen, *Sequence spaces and measure of noncompactness with applications to differential and integral equations*, New Delhi : Springer (2014).

[3] F. Basar, *Summability theory and its applications*, Bentham Science Publishers, e - books, Monographs, İstanbul (2012).

[4] R. E. Castillo and H. Rafeiro, *An introductory course in Lebesgue spaces*, Switzerland : Springer (2016).

[5] C. Duyar and O. Oğur, On a new space $m^2(M,A,\phi,p)$ of double sequences, *Journal of Function Spaces* **2013**, Article ID 509613, 8 pages (2013).

[6] R. Goyal, Bicomplex polygamma function, *Tokyo Journal of Mathematics* **30(2)**, 523–530 (2007).

[7] S. P. Goyal and R. Goyal, Bicomplex Hurwitz Zeta function, *South East Asian Journal of Mathematics and Mathematical Sciences* **4(3)**, 59–66 (2006).

[8] S. P. Goyal, T. Mathur and R. Goyal, Bicomplex gamma and beta function, *Journal of Raj. Academy Physical Sciences* **5(1)**, 131–142 (2006).

[9] R. Kumar and H. Saini, Topological bicomplex modules, *Advances in Applied Clifford Algebras* **26(4)**, 1249–1270 (2016).

[10] M. Nigam, *A study of certain sequence spaces of bicomplex numbers*, Phil dissertation, Dr. B. R. Ambedkar University, Agra. (2008).

[11] G. B. Price, *An introduction to multicomplex spaces and functions*, M. Dekker (1991).

[12] C. Segre, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, *Mathematische Annalen* **40(3)**, 413–467 (1892).

[13] R. K. Srivastava, Certain topological aspects of bicomplex space, *Bull. Pure & Appl. Math* **2(2)**, 222–234 (2008).

[14] R. K. Srivastava and N. K. Srivastava, On a class of entire bicomplex sequences, *South East Asian J. Math. & Math. Sc* **5(3)**, 47–68 (2007).

[15] M. A. Wagh, On certain spaces of bicomplex sequences, *Inter. J. Phy. Chem. and Math. fund* **7(1)**, 1–6 (2014).

[16] J. Yeh, *Real Analysis: Theory of measure and integration second edition*, World Scientific Publishing Company (2006).

Author information

Nilay Sager and Birsen Sağır, Ondokuz Mayis University Faculty of Art and Sciences Deparment of Mathematics Samsun, Turkey.
E-mail: `nilay.sager@omu.edu.tr`

Received: October 27, 2018.
Accepted: February 4, 2019.