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ON THE PAIR OF SPACELIKE BERTRAND-B CURVES WITH TIMELIKE PRINCIPAL NORMAL IN $\mathbb{R}_1^3$

F. Yerlikaya, S. Karaahmetoğlu and İ.Aydemir

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Abstract In this work, we introduce a pair of spacelike Bertrand-B curves with timelike principal normal and we divide the work into two subsections with respect to the given tables for the same (or different) types of vectors ζ_1 and ζ_2^*. Firstly, we obtain theorems concerning the pair of spacelike Bertrand-B curves. By using these theorems, we give propositions that express the condition for a pair of spacelike Bertrand-B curves to be planar, spatial and isometric. Finally, we conclude that Bertrand-B curves do not provide the Mannheim's theorem.

1 Introduction

In classical differential geometry, the theory of curves examines the geometric properties of curves by means of algebraic and calculus methods. In the theory of curves, the second derivative of a curve may be zero, that is, the curvature may vanish at some points along the curve. In such a situation, one may need for non-continuously differentiable curves to an alternative frame. The first author to construct the alternative frame called the Bishop frame in the three dimensional Euclidean space was Richard Lawrence Bishop. [3]. In applied sciences, this frame is used in engineering and biology, in particular in the study of DNA.

A new version of Bishop frame was first introduced and studied in three-dimensional Euclidean space by Yilmaz and Turgut in [7]. With this new version of Bishop frame, one means that the tangent vector ζ_1 and principal normal vector ζ_2 are considered as parallel transport plane while the binormal vector B remains fixed. Özyilmaz investigated some characterizations of curves with respect to this new frame in Euclidean space and gave some results [2]. Also, this frame has been investigated in semi- Euclidean spaces such as Minkowski space which is analogous to Euclidean space. Ünlütürk and Yılmaz obtained a new version of Bishop frame for spacelike curves in [4]. There is also a lot of surveys containing the studie of curves according to the new type of Bishop frame (see [4]-[8]). Lately, Yılmaz and et al obtained a new version of Bishop frame for timelike curves in Minkowski 3-space [9]. On the other hand, a pair of Bertrand-B curves was first introduced by Yerlikaya et al in three-dimensional Euclidean space[5] and was obtained some characterizations.

In this work, we introduce spacelike Bertrand B-curves and a pair of spacelike Bertrand-B curves by using the new version of Bishop frame. Then, we examine their relationship with each other. Also, we give propositions that express the condition for a pair of spacelike Bertrand-B curves to be planar, spatial and isometric.

2 Preliminaries

In this study, the 3-dimensional Minkowski space $\mathbb{R}_1^3$ is the pair $\left(\mathbb{R}^3, \langle , \rangle\right)$, in other words $\mathbb{R}_1^3$ is a three dimensional real vector space equipped with a Lorentz metric (inner product),

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 \tag{2.1}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. A vector $x \neq 0$ in $\mathbb{R}_1^3$ is called spacelike, timelike or null (lightlike), if $\langle x, x\rangle > 0$, $\langle x, x\rangle < 0$ or $\langle x, x\rangle = 0$, respectively. Let $x = (x_1, x_2, x_3) \in \mathbb{R}_1^3$ be any vector of $\mathbb{R}_1^3$, then its norm is defined by the following equality

$$\|x\| = |\langle x, x\rangle|^{\frac{1}{2}} = \sqrt{\left|-x_1^2 + x_2^2 + x_3^2\right|}, \tag{2.2}$$

and for any vector $y \in \mathbb{R}^3_1$ the vector product of x and y (in that order) is also defined as

$$x \times y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{2.3}$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3_1$. Any curve $\alpha = \alpha(s)$ can locally be a spacelike, timelike or null (lightlike) if the causal character of its velocity vector $\alpha'(s)$ are respectively spacelike, timelike or null. Let $\alpha(s)$ be a spacelike curve with timelike principal normal vector field and $\{T, N, B\}$ be its Frenet frame, then the matrix representation of Frenet formulas is

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.4}$$

Yılmaz et al. introduced a new type of Bishop frame by using binormal vector of a spacelike curve with spacelike principal normal as the common vector field [7]. The matrix representation of Type-2 Bishop formulas is expressed as

$$\begin{bmatrix} \zeta_1^{'} \\ \zeta_2^{'} \\ B^{'} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \epsilon_1 \\ 0 & 0 & -\epsilon_2 \\ -\epsilon_1 & -\epsilon_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ B \end{bmatrix} \tag{2.5}$$

Remark 2.1. Throughout this work, we recover the following type-2 Bishop frame formulas to investigate the properties of spacelike curves with timelike principal normal vector field:

$$\begin{bmatrix} \zeta_1^{'} \\ \zeta_2^{'} \\ B^{'} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\epsilon_{\zeta_1}\epsilon_1 \\ 0 & 0 & \epsilon_2 \\ \epsilon_1 & \epsilon_{\zeta_1}\epsilon_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ B \end{bmatrix} \tag{2.6}$$

where $\epsilon_{\zeta_1} = \langle \zeta_1, \zeta_1 \rangle$. In addition, we also recover a relationship between Frenet and type-2 Bishop vectors wrt the type of vector ζ_1 as follows:

$$\begin{cases} T = f_1(\theta(s))\,\zeta_1 + \epsilon_{\zeta_1} f_2(\theta(s))\,\zeta_2 & \begin{matrix} f_1 = \sinh\theta(s) \\ f_2 = \cosh\theta(s) \end{matrix} \quad \text{,if } \zeta_1 \text{ vector is timelike} \\ \\ N = f_2(\theta(s))\,\zeta_1 + \epsilon_{\zeta_1} f_1(\theta(s))\,\zeta_2 & \begin{matrix} f_1 = \cosh\theta(s) \\ f_2 = \sinh\theta(s) \end{matrix} \quad \text{,if } \zeta_1 \text{ vector is spacelike} \end{cases} \tag{2.7}$$

Definition 2.2. [3]. Let us consider two non-null vectors $\mathbb{R}^3_1$.

(a) Let x and y be positive or negative timelike vectors, then there is a unique nonnegative number θ such that

$$\langle x, y \rangle = \|x\| \, \|y\| \cosh\theta.$$

(b) Let x and y be spacelike vectors that span a spacelike plane, then there is a unique number $0 \leq \theta \leq \pi$ such that

$$\langle x, y \rangle = \|x\| \, \|y\| \cos\theta.$$

(c) Let x and y be spacelike vectors that span a timelike plane, then there is a unique real number $\theta \geq 0$ such that

$$|\langle x, y \rangle| = \|x\| \, \|y\| \cosh\theta.$$

(d) Let x be a spacelike vector and y be a timelike vector, then there is a unique number $\theta \geq 0$ such that

$$|\langle x, y \rangle| = \|x\| \, \|y\| \sinh\theta.$$

3 Results

Definition 3.1. Let α and β be spacelike curves with a timelike principal normal vector field in $\mathbb{R}_1^3$ and let $\{\zeta_1, \zeta_2, B\}$ and $\{\zeta_1^*, \zeta_2^*, B^*\}$ be type-2 Bishop frames along the curves α and β at the points $\alpha(s)$ and $\beta(s^*)$, respectively. If the vector fields B and B^* are linearly dependant for every $s, s^* \in I$, then the curve α called a spacelike Bertrand-B curve with a timelike principal normal vector field and the curve β called a spacelike Bertrand-B mate curve with a timelike principal normal vector field. Besides, (α, β) called a pair of Bertrand-B curve.

According to the type of curves α and β, one note that the following tables should be considered for such curves and due to this tables, note that there are four cases where we can evaluate for the pair of Bertrand-B curves (α, β) in the subsequent section. Also, since the curves α and β are the pair of Bertrand-B curves, we have the figure 1.

Table 1: The types of vectors for spacelike curve α with timelike principal normal vector field in $\mathbb{R}_1^3$

Frenet Frame T, N, B			Type2-Bishop Frame ζ_1,ζ_2,B
Spacelike	Timelike	Spacelike	Timelike Spacelike Spacelike
Spacelike	Timelike	Spacelike	Spacelike Timelike Spacelike

Table 2: The types of vectors for spacelike curve β with timelike principal normal vector field in $\mathbb{R}_1^3$

Frenet Frame T^*, N^*, B^*			Type2-Bishop Frame ζ_1^*,ζ_2^*,B^*
Spacelike	Timelike	Spacelike	Timelike Spacelike Spacelike
Spacelike	Timelike	Spacelike	Spacelike Timelike Spacelike

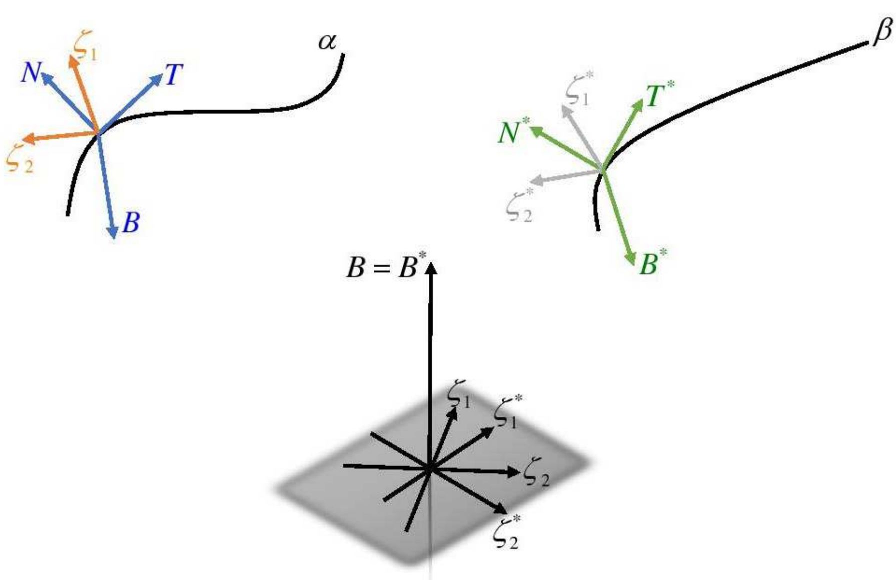

Fig. 1. The pair of Bertrand-B curves (α, β)

3.1 The pair of Bertrand-B curves with the Same type of vectors ζ_1 and ζ_1^*

Theorem 3.2. *The distance between corresponding points of the pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}_1^3$ is constant.*

Proof. Suppose that (α, β) is a pair of Bertrand-B curve, then we can write

$$\beta(s^*) = \alpha(s) + \lambda(s)B(s). \tag{3.1}$$

By taking derivative with respect to s and using Frenet formulas, we have

$$T^*(s^*)\ \tfrac{ds^*}{ds} = T(s) + \lambda'(s)B(s) + \lambda(s)\tau(s)N(s).$$

From 2.7 and necessary arrangemet, we get

$$f_1\left(\theta^*(s^*)\right)\zeta_1^* + f_2\left(\theta^*(s^*)\right)\zeta_2^* = \Big[\left\{f_1\left(\theta(s)\right) + \lambda(s)\epsilon_1(s)\right\}\zeta_1 + \epsilon_{\zeta_1}\left\{f_2\left(\theta(s)\right) + \lambda(s)\epsilon_2(s)\right\}\zeta_2 + \lambda'(s)B(s)\Big]. \tag{3.2}$$

Since B and B^* are linearly dependant, $\langle \zeta_1^*, B\rangle = 0$, $\langle \zeta_2^*, B\rangle = 0$. Hence

$$\lambda'(s) = 0,$$

and this proves the claim. □

Theorem 3.3. *Let (α, β) be a pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}_1^3$. Then the following properties hold:*

(a) The angle between ζ_1 and ζ_1^* vectors at corresponding points is constant.
(b) The angle between ζ_2 and ζ_2^* vectors at corresponding points is constant.

Proof. (a) Consider that $\langle \zeta_1, \zeta_1^*\rangle = \|\zeta_1\|\,\|\zeta_1^*\|\cosh\mu(s)$. By taking the derivative of both side with respect to s, from a straightforward computation, we have

$$\epsilon_{\zeta_1}\left\{\epsilon_1(s)\left\langle B, \zeta_1^*\right\rangle + \tfrac{ds^*}{ds}\epsilon_1^*(s^*)\left\langle B^*, \zeta_1\right\rangle\right\} = -\sinh\mu(s)\tfrac{d\mu}{ds}.$$

Since (α, β) is a pair of Bertrand-B curve, we get

$$\tfrac{d\mu}{ds} = 0.$$

That concludes the proof. The proof of (b) is made in a similar way. □

Theorem 3.4. *Let (α, β) be a pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}_1^3$. Then, there exists a relationship between the type-2 Bishop curvatures of the curves α and β such that the following equation holds:*

$$\tanh\left(\operatorname{arctan} h\frac{\epsilon_2^*}{\epsilon_1^*} + \epsilon_{\zeta_1^*}\mu\right) = \begin{cases} f(s) & ,\textit{if } \zeta_1 \textit{ and } \zeta_1^* \textit{ vectors are timelike} \\ \frac{1}{f(s)} & ,\textit{if } \zeta_1 \textit{ and } \zeta_1^* \textit{ vectors are spacelike} \end{cases}$$

where $f(s) = \frac{\epsilon_2 + \lambda\epsilon_1\sqrt{\left|\epsilon_2^2 - \epsilon_1^2\right|}}{\epsilon_1 + \lambda\epsilon_2\sqrt{\left|\epsilon_2^2 - \epsilon_1^2\right|}}$.

Proof. According to the previous theorem, since the distance function is constantit can easily see that the equation 3.2 is reduced to

$$f_1\left(\theta^*(s^*)\right)\zeta_1^* + f_2\left(\theta^*(s^*)\right)\zeta_2^* = \Big[\left\{f_1\left(\theta(s)\right) + \lambda(s)\epsilon_1(s)\right\}\zeta_1 + \epsilon_{\zeta_1}\left\{f_2\left(\theta(s)\right) + \lambda(s)\epsilon_2(s)\right\}\zeta_2\Big].$$

Since B and B^* are linearly dependant, $\zeta_1^*, \zeta_2^* \in span\{\zeta_1, \zeta_2\}$, namely ζ_1^* and ζ_2^* vectors are in the plane in which ζ_1 and ζ_2 vectors are spanned (see figure 1). Hence, above equation gives us way to the following equations, separately

$$\begin{aligned} \tfrac{ds^*}{ds}\left[f_1\left(\theta^*(s^*)\right)\cosh\mu + \epsilon_{\zeta_1} f_2\left(\theta^*(s^*)\right)\sinh\mu\right] &= f_1\left(\theta(s)\right) + \lambda\epsilon_1(s), \\ \tfrac{ds^*}{ds}\left[f_1\left(\theta^*(s^*)\right)\sinh\mu + \epsilon_{\zeta_1} f_2\left(\theta^*(s^*)\right)\cosh\mu\right] &= \epsilon_{\zeta_1}\left(f_2\left(\theta(s)\right) + \lambda\epsilon_2(s)\right). \end{aligned} \tag{3.3}$$

Then, the desired result is obtained by long separate computations considering f_1 and f_2 functions. □

Theorem 3.5. *Let (α, β) be a pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}_1^3$. Then the distance function of these curves at corresponding points is*

$$\lambda(s) = \frac{(\epsilon_1^*\epsilon_2 + \varepsilon_2^*\epsilon_1)\left\{\sqrt{\left|\epsilon_2^2 - \epsilon_1^2\right|} - \sqrt{\left|{\epsilon_2^*}^2 - {\epsilon_1^*}^2\right|}\right\}}{\left(\epsilon_1^*\epsilon_1 + \varepsilon_2^*\epsilon_2\right)\sqrt{\left|\epsilon_2^2 - \epsilon_1^2\right|}\sqrt{\left|{\epsilon_2^*}^2 - {\epsilon_1^*}^2\right|}}$$

where ϵ_1, ϵ_2, ϵ_1^* and ϵ_1^* denote the curvatures of α and β, respectively.

Proof. From the equation 2.7, we can write (when the vector ζ_1 is considered as timelike and spacelike) such that

$$\sinh\left(\theta^*(s^*) + \epsilon_{\zeta_1^*}\mu\right) = \tfrac{ds}{ds^*}\left(\sinh(\theta(s) + \lambda\epsilon_i(s)\right) \tag{3.4}$$

$$\cosh\left(\theta^*(s^*) + \epsilon_{\zeta_1^*}\mu\right) = \tfrac{ds}{ds^*}\left(\cosh(\theta(s) + \lambda\epsilon_j(s)\right) \tag{3.5}$$

where $\epsilon_i(s) = \begin{cases} i = 1 & \\ j = 2 & \text{, for timelike} \\ & \\ i = 2 & \text{, for spacelike.} \\ j = 1 & \end{cases}$

Furthermore, we can also write the equation 3.1 as

$$\alpha(s) = \alpha^*(s^*) - \lambda B^*(s^*).$$

By applying the similar process in the proof of the theorem 3.2, we have the following equations:

$$\sinh\left(\theta(s) - \epsilon_{\zeta_1^*}\mu\right) = \tfrac{ds^*}{ds}\left(\sinh^*(\theta(s^*) + \lambda\epsilon_i^*(s^*)\right) \tag{3.6}$$

$$\cosh\left(\theta(s) - \epsilon_{\zeta_1^*}\mu\right) = \tfrac{ds^*}{ds}\left(\cosh^*(\theta(s^*) + \lambda\epsilon_j^*(s^*)\right). \tag{3.7}$$

Multipying the equations 3.4,3.6 and 3.5,3.7 and in the sequel, summing the consequences, we get the desired distance function. □

We now can give two remarkable corollarys of the theorem 3.5 and can state a proposition based on these corollarys.

Corollary 3.6. *If $\epsilon_1 > \epsilon_2$, then the following relation*

$$\lambda^2\tau^2{\tau^*}^2 + \epsilon_{\zeta_1}{\tau^*}^2 + \epsilon_{\zeta_1}\tau^2 = 0$$

is satisfied, where τ and τ^* the torsion of the curves α and β, respectively.

Corollary 3.7. *If $\epsilon_2 > \epsilon_1$, then the following relation*

$$\lambda^2\tau^2{\tau^*}^2 - \epsilon_{\zeta_1}{\tau^*}^2 - \epsilon_{\zeta_1}\tau^2 = 0$$

is satisfied, where τ and τ^* the torsion of the curves α and β, respectively.

Proposition 3.8. *Let (α, β) be a pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}_1^3$. Then we have the following:*

(a) Let be $\epsilon_1 > \epsilon_2$ for the curvatures of the curve α.
(i) When $\langle\zeta_1, \zeta_1\rangle = -1$, the curve α is planar if and only if the curve β is planar.
(ii) When $\langle\zeta_1, \zeta_1\rangle = 1$, both the curve α and the curve β are planar.

(b) Let be $\epsilon_2 > \epsilon_1$ for the curvatures of the curve α.
(i) When $\langle\zeta_1, \zeta_1\rangle = -1$, both the curve α and the curve β are planar.
(ii) When $\langle\zeta_1, \zeta_1\rangle = 1$, the curve α is planar if and only if the curve β is planar.

Theorem 3.9. *Let (α, β) be a pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}^3_1$. Then there exists a constant μ such that the following relations hold:*

(a) $\epsilon_1(s) = \frac{ds^*}{ds}\left\{\cosh \mu\epsilon_1^*(s^*) + \epsilon_{\zeta_1^*} \sinh \mu\epsilon_2^*(s^*)\right\}.$

(b) $\epsilon_2(s) = -\frac{ds^*}{ds}\left\{\epsilon_{\zeta_2} \sinh \mu\epsilon_1^*(s^*) + \epsilon_{\zeta_2}\epsilon_{\zeta_1^*} \cosh \mu\epsilon_2^*(s^*)\right\}.$

(c) $\epsilon_1^*(s^*) = \frac{ds}{ds^*}\left\{\cosh \mu\epsilon_1(s) + \epsilon_{\zeta_1} \sinh \mu\epsilon_2(s)\right\}.$

(d) $\epsilon_2^*(s^*) = -\frac{ds}{ds^*}\left\{\epsilon_{\zeta_1} \sinh \mu\epsilon_1(s) + \cosh \mu\epsilon_2(s)\right\}.$

Proof. (b) Considering the fact that B and B^* vectors are linearly dependant, we can write $\langle \zeta_1, B^* \rangle = 0$. By taking the derivative with respect to s, we get

$$\left\langle \frac{d\zeta_1}{ds}, B^* \right\rangle + \left\langle \zeta_1, \frac{dB^*}{ds^*}\frac{ds^*}{ds} \right\rangle = 0.$$

Then, using the type-2 Bishop formulas and transition matrix one see that the proof is finished. Since other cases are similar in terms of the process, no need to prove. □

Corollary 3.10. *For two given Bertrand-B mates α and β, a relationship between their arc-parameters is given by*

$$\left|\epsilon_1^{*2} - \epsilon_2^{*2}\right| \left(\frac{ds^*}{ds}\right)^2 = \left|\epsilon_1^2 - \epsilon_2^2\right|.$$

Theorem 3.11. *Let (α, β) be a pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}^3_1$. Then, for the curvature centers M and M^* at the corresponding points of the curves α and β, respectively, the following ratio*

$$\frac{\|\alpha^*(s^*)M\|}{\|\alpha(s)M\|} : \frac{\|\alpha^*(s^*)M^*\|}{\|\alpha(s)M^*\|}$$

is not constant.

Corollary 3.12. *Mannheim's theorem is invalid for spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}^3_1$.*

3.2 The pair of Bertrand-B curves with the different type of vectors ζ_1 and ζ_1^*

In this subsection, we examine the properties of a pair of the Bertrand-B curves for different type ζ_1 and ζ_1^* vectors. Unlike other subsection, we here state the theorems without proof.

Theorem 3.13. *Let (α, β) be a pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}^3_1$. Then, we have the following:*

(a) the distance function $\lambda(s)$ is constant.
(b) the angle between ζ_1 and ζ_1^* vectors at corresponding points is constant.
(c) the angle between ζ_2 and ζ_2^* vectors at corresponding points is constant.
(d) Mannheim's theorem is invalid.

Theorem 3.14. *Let (α, β) be a pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}^3_1$. Then, there exists a relationship between the type-2 Bishop curvatures of the curves α and β such that the following equation holds:*

$$\coth\left(\arctan h\frac{\epsilon_2^*}{\epsilon_1^*} + \epsilon_{\zeta_1^*}\mu\right) = \begin{cases} f(s) & \text{, if } \zeta_2 \text{ and } \zeta_1^* \text{ vectors are timelike} \\ \frac{1}{f(s)} & \text{, if } \zeta_2 \text{ and } \zeta_1^* \text{ vectors are spacelike} \end{cases}$$

where $f(s) = -\frac{\epsilon_2 + \lambda\epsilon_1\sqrt{\left|\epsilon_2^2 - \epsilon_1^2\right|}}{\epsilon_1 + \lambda\epsilon_2\sqrt{\left|\epsilon_2^2 - \epsilon_1^2\right|}}$.

Theorem 3.15. *Let (α, β) be a pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}^3_1$. Then the distance function of these curves at corresponding points is*

$$\lambda(s) = \frac{(\epsilon_2^*\epsilon_2 - \varepsilon_1^*\epsilon_1)\left\{\sqrt{\left|\epsilon_2^{*2} - \epsilon_1^{*2}\right|} - \sqrt{\left|\epsilon_2^2 - \epsilon_1^2\right|}\right\}}{(\epsilon_2^*\epsilon_1 - \varepsilon_1^*\epsilon_2)\sqrt{\left|\epsilon_2^2 - \epsilon_1^2\right|}\sqrt{\left|\epsilon_2^{*2} - \epsilon_1^{*2}\right|}}$$

where ϵ_1, ϵ_2, ϵ_1^* and ϵ_1^* denote the curvatures of α and β, respectively.

Corollary 3.16. *If $\epsilon_1 > \epsilon_2$, then the following relation*

$$\epsilon_{\zeta_1}\lambda^2\tau^2{\tau^*}^2 + {\tau^*}^2 + \epsilon_{\zeta_1}\tau^2 = 0$$

is satisfied, where τ and τ^* the torsion of the curves α and β, respectively.

Corollary 3.17. *If $\epsilon_2 > \epsilon_1$, then the following relation*

$$\epsilon_{\zeta_1}\lambda^2\tau^2{\tau^*}^2 - {\tau^*}^2 - \epsilon_{\zeta_1}\tau^2 = 0$$

is satisfied, where τ and τ^* the torsion of the curves α and β, respectively.

Proposition 3.18. *Let (α, β) be a pair of spacelike Bertrand-B curves with timelike principal normal vector field in $\mathbb{R}^3_1$. Then, we have the following:*

(a) Let ζ_1 be a timelike vector.
(i) The curve α is planar if and only if the curve β is planar.
(ii) if $|\tau| = |\tau^*|$, then β is isometric to α.

(b) Let ζ_1 be a spacelike vector.
(i) When $\epsilon_1 > \epsilon_2$, both the curve α and the curve β are planar.
(ii) When $\epsilon_2 > \epsilon_1$, the curve α is planar if and only if the curve β is planar.

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Author information

F. Yerlikaya, S. Karaahmetoğlu and İ.Aydemir, Department of Mathematics, Ondokuz Mayıs University, 55139, Samsun, Turkey., T. R..

E-mail: `firat.yerlikaya@omu.edu.tr`, `satrask@omu.edu.tr` and `iaydemir@omu.edu.tr`

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