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SUBCLASSES OF JANOWSKI FUNCTIONS ASSOCIATED WITH (j,k)-SYMMETRICAL FUNCTIONS

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Abstract The objective of the present paper is to study results that are defined using notions of Janowski classes and (j,k)-symmetrical functions. We derive integral representations theorem, Polya-Schoenberg theorem and some interesting properties are also pointed out.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$.

For f and g be analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, if there exists an analytic function w in $\mathcal{U}$ such that $|w(z)| < 1$ with $w(0) = 0$, and $f(z) = g(w(z))$, and we denote this by $f(z) \prec g(z)$. If g is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. The convolution or Hadamard product of two analytic functions $f, g \in \mathcal{A}$ where f is defined by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, is

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

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Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive real parts.

Definition 1.1. [2] Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\operatorname{Re} p(z) > 0$, $z \in \mathcal{U}$.

Any function p in $\mathcal{P}$ has the representation $p(z) = \dfrac{1 + w(z)}{1 - w(z)}$, where $w \in \Omega$ and

$$\Omega = \{w \in \mathcal{A} : w(0) = 0, |w(z)| < 1\}. \tag{1.2}$$

The class of functions with positive real part $\mathcal{P}$ plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses like class of starlike $\mathcal{S}^*$, class of convex functions $\mathcal{C}$, class of starlike functions with respect to symmetric points have been defined by using the concept of class of functions with positive real part.

Like in [1], let $\mathcal{P}[A,B]$, with $-1 \leq B < A \leq 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z) = \dfrac{1+Aw(z)}{1+Bw(z)}$, $z \in \mathcal{U}$, where $w \in \Omega$. Remark that $p \in \mathcal{P}[A,B]$ if and only if $p(z) \prec \dfrac{1+Az}{1+Bz}$.
In [9] the class $\mathcal{P}[A,B,\alpha]$ of generalized Janowski functions was introduced. For arbitrary numbers A, B, α, with $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, a function p analytic in $\mathcal{U}$ with $p(0) = 1$ is in the class $\mathcal{P}[A,B,\alpha]$ if and only if $p(z) \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz}$.
In our work we will define the class $\mathcal{P}[A,B,\alpha,\beta]$ of generalized Janowski functions was introduced. For arbitrary numbers A, B, α, β with $-1 \leq B < A \leq 1$, $0 \leq \alpha, \beta < 1$, and $\alpha + \beta < 1$ a function p analytic in $\mathcal{U}$ with $p(0) = 1$ is in the class $\mathcal{P}[A,B,\alpha,\beta]$ if and only if

$$p(z) \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+[(1-\beta)B+\beta A]z} \Leftrightarrow p(z) = \frac{1+[(1-\alpha)A+\alpha B]w(z)}{1+[(1-\beta)B+\beta A]w(z)}, \quad w \in \Omega.$$

A function f is belongs to the class $\mathcal{S}^*[A,B,\alpha,\beta]$ if $\frac{zf'(z)}{f(z)} \in \mathcal{P}[A,B,\alpha,\beta]$.
In order to define a new class of generalized Janowski symmetrical functions defined in the open unit disk $\mathcal{U}$, we first recall the notion of k-fold symmetric functions defined in k-fold symmetric domain, where k is any positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{D}$ if for every z in $\mathcal{D}$ we have

$$f\left(e^{\frac{2\pi i}{k}}z\right) = e^{\frac{2\pi i}{k}}f(z), \ z \in \mathcal{D}.$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$, and for $k = 2$ we get class of odd univalent functions. In 1995, Liczberski and Polubinski [14] constructed the theory of (j,k)-symmetrical functions for $j = 0, 1, 2, \ldots, k-1)$ and $(k = 2, 3, \ldots$. If $\mathcal{D}$ is k-fold symmetric domain and j any integer, then a function $f : \mathcal{D} \to \mathbb{C}$ is called (j,k)-symmetrical if for each $z \in \mathcal{D}$, $f(\varepsilon z) = \varepsilon^j f(z)$. We note that the (j,k)-symmetrical functions is a generalization of the notions of even, odd, and k-symmetrical functions

The theory of (j,k)-symmetrical functions has many interesting applications; for instance, in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings, see [14].

Denote the family of all (j,k)-symmetrical functions by $\mathcal{S}^{(j,k)}$. We observe that , $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are the classes of even, odd and k-symmetric functions respectively. We have the following decomposition theorem:
Theorem 1 [14, Page 16] For every mapping $f : \mathcal{U} \mapsto \mathbb{C}$, and a k-fold symmetric set $\mathcal{U}$, there exists exactly one sequence of (j,k)-symmetrical functions $f_{j,k}$ such that

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z),$$

where

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f\left(\varepsilon^v z\right), \ z \in \mathcal{U}. \tag{1.3}$$

Al Sarari and Latha [3] introduced and studied the classes $\mathcal{S}^{(j,k)}(A,B)$ and $\mathcal{K}^{(j,k)}(A,B)$ which are starlike and convex with respect to (j,k)-symmetric points. For more details about the classes with (j,k)-symmetrical functions see [7, 8, 10].

Definition 1.2. A function $f \in \mathcal{A}$ is said to belongs to the class $\mathcal{S}^{(j,k)}[A,B,\alpha,\beta]$, with $-1 \leq B < A \leq 1$, $0 \leq \alpha, \beta < 1$, and $\alpha + \beta < 1$ if

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+[(1-\beta)B+\beta A]z},$$

where $f_{j,k}$ are defined by (1.3).

Remark 1.3. Using the definition of the subordination we can easily obtain that the equivalent condition for a function f belonging to the class $\mathcal{S}^{(j,k)}[A,B,\alpha,\beta]$, with $-1 \leq B < A \leq 1, 0 \leq \alpha,\beta < 1$ and $\alpha+\beta < 1$ is

$$\left|\frac{zf'(z)}{f_{j,k}(z)} - 1\right| < \left|[(1-\alpha)A + \alpha B] - [(1-\beta)B + \beta A]\frac{zf'(z)}{f_{j,k}(z)}\right|, \; z \in \mathcal{U}.$$

We note that special values of j, k, A, B, α and β yield the following classes: For $\alpha = \beta = 0$ we get the class introduced and studied by Alsarari and Latha [3]
For $j = 1$ and $\alpha = \beta = 0$ the class studied by Ohsang K and Yaungjae [5].
For $j = k = A = -B$ and $\beta = 0$ the class introduced by Polatoglu, Bolcal, Sen and Yavuz, [9], For $j = A = -B = 1, \alpha = 0$ and$\beta = 0$ the class is studied by Sakaguchi [12], etc. The second and third authors studied some classes with (j,k)-symmetrical functions [3, 7, 8, 10].
We need the following lemmas to prove our main results:

Lemma 1.4. *[11] Let ϕ be convex and g starlike. Then for F analytic in $\mathcal{U}$ with $F(0) = 1$,*

$$\frac{\phi * Fg}{\phi * g}(\mathcal{U}) \subset \overline{CO}(F(\mathcal{U}))$$

where $\overline{CO}(F(\mathcal{U}))$ denotes the closed convex hull of $F(\mathcal{U})$.

Lemma 1.5. *[6] If $f \in \mathcal{S}^*(A,B)$, then*

$$F(z) = \frac{m+1}{z^m}\int_0^z t^{m-1} f(t)dt \in \mathcal{S}^*(A,B)$$

where $\mathcal{S}^*(A,B)$ is defined by Janowski [1].

Lemma 1.6. *[9] Any function $f \in \mathcal{S}^*(A,B,\alpha)$ can be written in the form*

$$f(z) = \begin{cases} z\left(1 + Bw(z)\right)^{\frac{(1-\alpha)(A-B)}{B}}, & \textit{if} \quad B \neq 0, \\ z \,\exp\left[(1-\alpha)Aw(z)\right], & \textit{if} \quad B = 0, \end{cases}$$

where $w \in \Omega$.

Lemma 1.7. *[12]. Let N be regular and D starlike in $\mathcal{U}$ and $N(0) = D(0) = 0$. Then for $-1 \leq B < A \leq 1$,*

$$\frac{N'(z)}{D'(z)} \prec \frac{1+Az}{1+Bz}$$

implies that

$$\frac{N(z)}{D(z)} \prec \frac{1+Az}{1+Bz}.$$

2 Main results

Lemma 2.1. *Any function $f \in \mathcal{S}^*[A,B,\alpha,\beta]$ can be written in the form*

$$f(z) = \begin{cases} z\left(1 + [(1-\beta)B + \beta A]w(z)\right)^{\frac{(1-\alpha-\beta)(A-B)}{[(1-\beta)B+\beta A]}}, & \textit{if} \;\; B \neq 0 \neq \beta, \\ z\left(1 + Bw(z)\right)^{\frac{(1-\alpha)(A-B)}{B}}, & \textit{if} \;\; B \neq 0 = \beta, \\ z\left(1 + \beta Aw(z)\right)^{\frac{(1-\alpha-\beta)}{\beta}}, & \textit{if} \;\; B = 0 \neq \beta, \\ z \,\exp\left[(1-\alpha)Aw(z)\right], & \textit{if} \;\; B = 0 = \beta, \end{cases} \tag{2.1}$$

where $w \in \Omega$.

Proof. Suppose that $f \in \mathcal{S}^*[A,B,\alpha,\beta]$ and it is easy to note that

$$-1 \leq B \leq (1-\beta)B + \beta A < (1-\alpha)A + \alpha B \leq A \leq 1,$$

for $-1 \leq B < A \leq 1$, $0 \leq \alpha, \beta < 1$, and $\alpha + \beta < 1$.
Now but $(1-\beta)B + \beta A = C$ and apply the Lemma 1.6 we get (2.1). □

Theorem 2.2. *If $f \in \mathcal{S}^{(j,k)}[A,B,\alpha,\beta]$, then*

$$f_{j,k}(z) = \begin{cases} z\left(1 + [(1-\beta)B + \beta A]w(z)\right)^{\frac{(1-\alpha-\beta)(A-B)}{[(1-\beta)B+\beta A]}}, & \text{if } B \neq 0 \neq \beta, \\ z\left(1 + Bw(z)\right)^{\frac{(1-\alpha)(A-B)}{B}}, & \text{if } B \neq 0 = \beta, \\ z\left(1 + \beta A w(z)\right)^{\frac{(1-\alpha-\beta)}{\beta}}, & \text{if } B = 0 \neq \beta, \\ z\, \exp\left[(1-\alpha)Aw(z)\right], & \text{if } B = 0 = \beta, \end{cases} \tag{2.2}$$

where $w \in \Omega$, and $f_{j,k}$ are defined by (1.3).

Proof. Let $f \in \mathcal{S}^{(j,k)}[A,B,\alpha,\beta]$, by Definition 1.2 we have

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + [(1-\beta)B + \beta A]z}. \tag{2.3}$$

Substituting z by $\varepsilon^\nu z$ in (2.7), it follows

$$\frac{\varepsilon^\nu z f'(\varepsilon^v z)}{f_k(\varepsilon^\nu z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]\varepsilon^\nu z}{1 + [(1-\beta)B + \beta A]\varepsilon^\nu z} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + [(1-\beta)B + \beta A]z},$$

hence

$$\frac{\varepsilon^{\nu-\nu j} z f'(\varepsilon^\nu z)}{f'_{j,k}(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + [(1-\beta)B + \beta A]z}, \tag{2.4}$$

Letting $\nu = 0, 1, 2, \ldots, k-1$ in (2.4) and using the fact that $\mathcal{P}[A,B,\alpha,\beta]$ is a convex set, we deduce that

$$\frac{z\frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{\nu-\nu j} f'(\varepsilon^\nu z)}{f_{j,k}(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + [(1-\beta)B + \beta A]z},$$

or equivalently

$$\frac{zf'_{j,k}(z)}{f_{j,k}(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + [(1-\beta)B + \beta A]z}, \tag{2.5}$$

that is $f_{j,k} \in \mathcal{S}^*[A,B,\alpha,\beta]$, and by Lemma 2.1 we obtain our result. □

Theorem 2.3. *If $f \in \mathcal{S}^{(j,k)}[A,B,\alpha,\beta]$, then*

$$f(z) = \begin{cases} \int_0^z [(1 + [(1-\beta)B + \beta A]w(r))^{\frac{(1-\alpha-\beta)(A-B)}{[(1-\beta)B+\beta A]}} . \frac{1+[(1-\alpha)A+\alpha B]\widetilde{w}(r)}{1+[(1-\beta)B+\beta A]\widetilde{w}(r)} dr], & \text{if } B \neq 0 \neq \beta, \\ \int_0^z [(1 + Bw(r))^{\frac{(1-\alpha)(A-B)}{B}} . \frac{1+[(1-\alpha)A+\alpha B]\widetilde{w}(r)}{1+B\widetilde{w}(r)} dr], & \text{if } B \neq 0 = \beta, \\ \int_0^z [(1 + \beta A w(r))^{\frac{(1-\alpha-\beta)}{\beta}} . \frac{1+(1-\alpha)A\widetilde{w}(r)}{1+\beta A\widetilde{w}(r)} dr], & \text{if } B = 0 \neq \beta, \\ \int_0^z [\exp\left((1-\alpha)Aw(r)\right) .(1 + (1-\alpha)A\widetilde{w}(r))dr], & \text{if } B = 0 = \beta, \end{cases} \tag{2.6}$$

where $w, \widetilde{w} \in \Omega$.

Proof. Suppose that $f \in \mathcal{S}^{(j,k)}[A,B,\alpha,\beta]$, then

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]\widetilde{w}(z)}{1 + [(1-\beta)B + \beta A]\widetilde{w}(z)}, \quad \widetilde{w} \in \Omega. \tag{2.7}$$

By Theorem 2.2, we get

$$f'(z) = \begin{cases} (1 + [(1-\beta)B + \beta A]w(z))^{\frac{(1-\alpha-\beta)(A-B)}{[(1-\beta)B+\beta A]}} . \frac{1+[(1-\alpha)A+\alpha B]\widetilde{w}(z)}{1+[(1-\beta)B+\beta A]\widetilde{w}(z)}, & \text{if } B \neq 0 \neq \beta, \\ (1 + Bw(z))^{\frac{(1-\alpha)(A-B)}{B}} . \frac{1+[(1-\alpha)A+\alpha B]\widetilde{w}(z)}{1+B\widetilde{w}(z)}, & \text{if } B \neq 0 = \beta, \\ (1 + \beta A w(z))^{\frac{(1-\alpha-\beta)}{\beta}} . \frac{1+(1-\alpha)A\widetilde{w}(z)}{1+\beta A\widetilde{w}(z)}, & \text{if } B = 0 \neq \beta, \\ \exp\left[(1-\alpha)Aw(z)\right] .1 + (1-\alpha)A\widetilde{w}(z), & \text{if } B = 0 = \beta, \end{cases} \tag{2.8}$$

Integrating the above equation we complete the proof. □

Theorem 2.4. *Let $f \in \mathcal{S}^{(j,k)}[A, B, \alpha, \beta]$ and let ϕ be convex. Then*

$$(f * \phi) \in \mathcal{S}^{(j,k)}[A, B, \alpha, \beta].$$

Proof. To prove that $(f * \phi) \in \mathcal{S}^{(j,k)}[A, B, \alpha, \beta]$ it is sufficient to show that

$$\frac{z(f * \phi)'(z)}{(f * \phi)_{j,k}(z)} \subset \overline{CO}(F(\mathcal{U})),$$

where $F(z) = \frac{zf'(z)}{f_{j,k}(z)}$. Now

$$\frac{z(f * \phi)'(z)}{(f * \phi)_{j,k}(z)} = \frac{zf'(z) * \phi(z)}{f_{j,k}(z) * \phi(z)}$$

$$= \frac{\phi(z) * \frac{zf'(z)}{f_{j,k}(z)} . f_{j,k}(z)}{\phi(z) * f_{j,k}(z)},$$

by using Lemma 1.4 with $f_{j,k}(z) \in \mathcal{S}[A, B, \alpha, \beta], F \in \mathcal{P}[A, B, \alpha, \beta]$, that complete the proof.

Example. Let $f \in \mathcal{S}^{(j,k)}[A, B, \alpha, \beta]$, then so does

$$F(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} f(t) dt \tag{2.9}$$

for $m = 1, 2, 3, ...$
Proof. By using the equation (2.9), we have

$$F_{j,k}(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} f_{j,k}(t) dt,$$

and

$$\frac{zF'(z)}{F(z)} = -m + \frac{z^m f(z)}{\int_0^z t^{m-1} f(t)\, dt}.$$

Hence

$$\frac{zF'(z)}{F_{j,k}(z)} = \left(-m + \frac{z^m f(z)}{\int_0^z t^{m-1} f(t)\, dt}\right) \frac{F(z)}{F_{j,k}(z)} \tag{2.10}$$

$$= \frac{z^m f(z) - m \int_0^z t^{m-1} f(t)\, dt}{\int_0^z t^{m-1} f_{j,k}(t)\, dt} \tag{2.11}$$

$$:= \frac{N(z)}{D(z)}.$$

By (2.5) then $f_{j,k}(z) \in \mathcal{S}^*[A, B, \alpha, \beta]$, and by Lemma 1.5 we note that $F_{j,k}(z) \in \mathcal{S}^*[A, B, \alpha, \beta]$. Differentiating (2.11), we have

$$\frac{N'(z)}{D'(z)} = \frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + [(1-\beta)B + \beta A]z},$$

Lemma 1.7 gives $F(z) \in \mathcal{S}^{(j,k)}[A, B, \alpha, \beta]$.

Corollary 2.5. *Let $f \in \mathcal{S}^{(j,k)}[A, B, \alpha, \beta]$. Then*

$$F_i(z) \in \mathcal{S}^{(j,k)}[A, B, \alpha, \beta], (i = 1, 2, 3, 4),$$

where

$$F_1(z) = \int_0^z \frac{f(t)}{t} dt, \qquad F_2(z) = \int_0^z \frac{f(t) - f(xt)}{t - xt} dt, |x| \leq 1, x \neq 1,$$

$$F_3(z) = \frac{2}{z} \int_0^z f(t) dt, \qquad F_4(z) = \frac{m+1}{m} \int_0^z t^{m-1} f(t) dt, \Re m > 0.$$

Proof. Since

$$F_1(z)=\phi_1(z)*f(z),\qquad \phi_1(z)=\sum_0^\infty \frac{1}{n}z^n=\log(1-z)^{-1},$$

$$F_2(z)=\phi_2(z)*f(z),\qquad \phi_2(z)=\sum_0^\infty \frac{1-x^n}{n(1-x)}z^n=\frac{1}{1-x}\log(\frac{1-xz}{1-z}),|x|\le 1,x\neq 1,$$

$$F_3(z)=\phi_3(z)*f(z),\qquad \phi_3(z)=\sum_0^\infty \frac{2}{n+1}z^n=\frac{-2[z+\log(1-z)]}{z},$$

$$F_4(z)=\phi_4(z)*f(z),\qquad \phi_4(z)=\sum_0^\infty \frac{1+m}{n+m}z^n,\Re m>0.$$

We note that $\phi_i, i=1,2,3,4$ are convex. Now using Theorem 2.4.

Theorem 2.6. *Let $f\in\mathcal{S}^{(j,k)}[A,B,\alpha,\beta]$ and $f_\lambda(z):=(1-\lambda)z+\lambda f(z)$, with $0<\lambda<1$. Then,*

$$f_\lambda\in\mathcal{S}^{(j,k)}[A,0,\alpha,0].$$

Proof. Since $f\in\mathcal{S}^{(j,k)}[A,B,\alpha,\beta]$, then

$$\frac{zf'(z)}{f_{j,k}(z)}\prec\frac{1+[(1-\alpha)A+\alpha B]z}{1+[(1-\beta)B+\beta A]z},\quad\text{with}\quad f_{j,k,\lambda}(z)=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{-\nu j}f_\lambda(\varepsilon^\nu z).$$

Thus,

$$f_{j,k,\lambda}(z)=(1-\lambda)z+\lambda f_{j,k}(z),\quad zf'_\lambda(z)=(1-\lambda)z+\lambda zf'(z),$$

hence

$$\frac{zf'_\lambda(z)}{f_{j,k,\lambda}(z)}=\frac{(1-\lambda)\dfrac{z}{f_{j,k}(z)}+\lambda\dfrac{zf'(z)}{f_{j,k}(z)}}{(1-\lambda)\dfrac{z}{f_{j,k}(z)}+\lambda}. \tag{2.12}$$

Since $B=\beta=0$ it is sufficient to show that

$$\left|\frac{(1-\lambda)\dfrac{z}{f_{j,k}(z)}+\lambda\dfrac{zf'(z)}{f_{j,k}(z)}}{(1-\lambda)\dfrac{z}{f_{j,k}(z)}+\lambda}-1\right|<(1-\alpha)A,\ z\in\mathcal{U}. \tag{2.13}$$

From $f\in\mathcal{S}^{(j,k)}[A,B,\alpha,\beta]$ we have

$$\frac{zf'(z)}{f_{j,k}(z)}\prec 1+(1-\alpha)Az$$

which implies

$$\left|\frac{zf'(z)}{f_{j,k}(z)}-1\right|<(1-\alpha)A,\ z\in\mathcal{U},$$

and according to (2.2)

$$\frac{f_{j,k}(z)}{z}\prec\exp[(1-\alpha)Az],$$

hence, there exists a Schwarz function $w\in\Omega$ such that

$$\frac{f_{j,k}(z)}{z}=\exp[(1-\alpha)Aw(z)],\ z\in\mathcal{U}.$$

Thus,

$$\left| \frac{(1-\lambda)\dfrac{z}{f_{j,k}(z)} + \lambda \dfrac{zf'(z)}{f_{j,k}(z)}}{(1-\lambda)\dfrac{z}{f_{j,k}(z)} + \lambda} - 1 \right| = \lambda \left| \frac{\dfrac{zf'(z)}{f_{j,k}(z)} - 1}{(1-\lambda)\dfrac{z}{f_{j,k}(z)} + \lambda} \right| <$$

$$\frac{(1-\alpha)A\lambda}{|(1-\lambda)\exp[-(1-\alpha)Aw(z)] + \lambda|}, \ z \in \mathcal{U},$$

and using the fact that $|w(z)| < 1$ for all $z \in \mathcal{U}$, we may easily prove that

$$|(1-\lambda)\exp[-(1-\alpha)Aw(z)] + \lambda| > \lambda, \ z \in \mathcal{U}.$$

From the above two inequalities it follows

$$\left| \frac{(1-\lambda)\dfrac{z}{f_{j,k}(z)} + \lambda \dfrac{zf'(z)}{f_{j,k}(z)}}{(1-\lambda)\dfrac{z}{f_{j,k}(z)} + \lambda} - 1 \right| < (1-\alpha)A, \ z \in \mathcal{U},$$

and consequently, from (2.12) we obtain

$$\frac{zf'_\lambda(z)}{f_{j,k,\lambda}(z)} \prec 1 + (1-\alpha)Az,$$

that is $f_\lambda \in \mathcal{S}^{(j,k)}[A, 0, \alpha, 0]$. □

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