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Dynamics of modified fractional Illicit Drug Consumption Model

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Abstract We suggest a fractional differential equation describing the intake of illegal drugs in a population made up of non-users and drug consumers. Drug users are categorized into experimental users, recreational users, and addicts. This is an effort by analogy with the traditional multi-species logistic Holling type-III predator-prey models to suggest a model that considers non-users as prey, experimental and recreational users as predators as well as preys and addicts as predators. The proposed model is analyzed in terms of boundedness, existence, and uniqueness of solutions. We derive sufficient conditions for existence and stability of points of equilibrium in terms of some threshold parameters. The dynamics of these threshold parameters and the sensitivity indices of parameter values are investigated by relating them to basic reproduction numbers. The theoretical results are then validated by numerical simulations.

1 Introduction

A 'drug' is a chemical substance when consumed, it modifies the function of the human body and alters the mood. Drugs are of two kinds: legal (Eg: caffeine, tobacco, and alcohol) and illegal (Eg: marijuana, heroin, cannabis, cocaine, etc). As the people around the world are filled with temptation, consumption of illicit drugs has become as a matter of concern throughout the world. Abuse of drugs and their addiction lead to a serious damage in the individual's physical and mental health. Over the years, so many cases of HIV and deaths have been reported due to the same. They also lead to serious disorders in the nervous system and may lead the person to be a victim of crime and violence. There can be no question that over the last 30 years, substance use has risen dramatically, but it is not clear what has caused the rise or what preventive interventions might have been effective in the past or will be successful in the future. Due to its adverse effect, it has become necessary to study this area scientifically. To contribute to the social cause of drug use prevention, scientists from various fields are continuously putting their efforts in terms of modeling drug consumption behavior. Many mathematicians have modeled this phenomenon as mathematical models of illicit drug consumption and compared the drug consumption data of various countries to validate their models. Based on the calculations made about these drugs, some researchers have questioned the quality and reliability of the results [1, 2, 3]. In this context, since the beginning of the nineties, we get to see the evidence of proposing a good number of time dependent continuous models to explain the dynamics of illegal drug uses [4, 5, 6, 7, 8, 9, 10, 11, 12]. They prominently considered the second-order nonlinear equations involving two variables for describing long term dynamics of drug market dealers and addicts. Gragnani et al. [9] extended the model to a third-order model and the additional ordinary differential equation represents the constraints imposed by the authorities. In 2013, Dauhoo et al. [13] have formulated a model depicting drug consumption, which describes the dynamics of non-users(N), experimental users(E), recreational users(R) and addicts(A), respectively, and termed it as NERA model. Then, by the analogy with the classical prey-predator model, a modified illicit drug consumption model was presented by Ginoux et al. [14] in view of a third-order differential equation. Grass et al. [15] have studied the optimal control theory and descriptive model of drug epidemics. Baveja et al. [16] have discussed some policies relating to drug markets and their local enforcement by implementing crackdowns in some cities. Adam et al. [17] have considered the person-to-person unpredictable contacts of illicit drugs and analyzed random factors of the continuous spectrum of the drugs.

Exploration of fractional calculus has become one of the most common trends among applied mathematics researchers in recent years. It is gaining attention as a result of its more realistic interpretation of physical phenomena. Many researchers have recently pointed out and demonstrated that integer-order differential operators are not always suitable methods for modeling complex and nonlinear events. Classical derivatives fail to confine important physical properties such as anomalous diffusion, non-Markovian processes, random walk, long-range, and most importantly heterogeneous behaviors. Due to these drawbacks, mathematicians and physicists are in the continuous effort to develop sophisticated and scientific mathematical operators to effectively reproduce and capture the above-mentioned natural processes. In this strive of new contribution, the definition of local differential operators, along with the power-law setting and non-local differential operators such as Caputo, Riemann-Liouville, Gr$\ddot{u}$nwald Letnikov, Jumarie, Atangana-Baleanu, Riesz, Riesz-Caputo, etc. were proposed. Elaborate theories of the FDEs can be found in [18, 19, 20, 21, 22, 23, 24]. Conformal fractional derivatives, introduced by Khalil et al. [25] is applied by many researchers to give the differential equations different physical interpretations [26, 27, 28]. Incorporation of the fractional derivative is significant in the case of evolution equations because the definition holds integration and hence the function holds the information about the memory. Many phenomena of mathematical biology and their interdisciplinary fields [29, 30, 31, 32, 33, 34] have been studied in a better way by using these fractional derivatives. Since the outcome of highly non-linear fractional derivatives are very complicated, solving these systems analytically is very difficult. Thus, many researchers have contributed towards introducing new numerical techniques that can accommodate fractional differential equations [35, 36, 37, 38, 39, 40].

In our present work, we have analyzed a modified NERA model incorporating Caputo fractional derivative and Holling type-III functional response. We have established the theoretical aspects related to the solutions of the proposed model taking into our consideration the uniqueness, existence, and boundedness of our proposed model. We have derived the basic reproduction(BR) number related to some threshold parameters and we have also established a relationship of BR number and sensitivity indices, which is a novel idea of investigating the strength of the parameters in a narcotic mathematical model. A numerical validation of the analytical results in this work is performed by taking suitable values of the parameters. Using the Colorado and Washington states' marijuana drug consumption data from Hanley [41], we have analyzed the impact of fractional derivative and Holling Type-III functional response on the obtained results. Inclusion of fractional derivative in drug consumption model where the influence of one category of the population on the other category is defined by Holling type-III functional is original in the literature.

2 Some Essential Theorems

Since integer order initial conditions are physically more authentic, and Caputo fractional derivative(CFD) supports them, in this work fractional derivative is considered in Caputo sense. Here, we have mentioned few theorems that we are going to apply for establishing the uniqueness, existence, and boundedness of solutions. We will denote the CFD by the capital letter with the upper-left index ${}^{C}D$.

Definition 2.1. (Caputo Fractional Derivative) [18] Suppose $g(t)$ is m times continuously differentiable function and $g^{(m)}(t)$ is integrable in $[t_0, T]$. Then, Caputo fractional derivative of order α for a function $g(t)$ is defined as

$$ {}^{C}_{t_0}D^{\alpha}_{t} g(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{g^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, $$

Here, $t > a$ and m is a positive integer with $m-1 < \alpha < m$. $\Gamma(\cdot)$ refers to Gamma function.

Lemma 2.2. *[42] Consider the FDE*

$$ {}^{C}_{t_0}D^{\alpha}_{t} g(t) = y(t,x), t > t_0, \tag{2.1} $$

with the starting values $g(t_0)$, $0 < \alpha \leq 1$ *and* $y : [t_0, \infty] \times \Omega \to \mathbb{R}^n, \Omega \in \mathbb{R}^n$. *There exists a unique solution of Eq. (2.1) on* $[t_0, \infty) \times \Omega$ *provided that* $y(t,x)$ *satisfies the locally Lipchitz condition with respect to* x.

Table 1. Biological meanings of the Symbols

λ_1	Rate at which $E(t)$ influences $N(t)$
α_1	Rate at which $R(t)$ influences $N(t)$
α_2	Rate at which $R(t)$ influences $E(t)$
γ_1	Rate at which $A(t)$ influences $N(t)$
γ_2	Rate at which $A(t)$ influences $E(t)$
γ_3	Rate at which $A(t)$ influences $R(t)$
β_1	Growth rate of the non-user.
β_2	Natural mortality rate of experimental users in absence of all others.
β_3	Natural mortality rate recreational users in absence of all others.
β_4	Natural mortality rate addicts in absence of all others.

Lemma 2.3. *[43] If $g(t)$ is a continuous function on $[t_0, +\infty)$ satisfying*

$$ {}_{t_0}^{C}D_t^{\alpha} g(t) \leq -\lambda g(t) + \mu, g(t_0) = g_{t_0}, $$

where $0 < \alpha \leq 1,\ (\lambda, \mu) \in \mathbb{R}^2$ and $\lambda \neq 0$ and $t_0 \geq 0$ is the initial time, then

$$ g(t) \leq (g(t_0) - \frac{\mu}{\lambda})E_\alpha[-\lambda(t - t_0)^\alpha] + \frac{\mu}{\lambda}. $$

3 Model Formulation

Motivated by the work proposed by Dauhoo et al. [13] and Ginox et al.[14], where the authors have adapted the structure from the classical predator-prey model representing non-users of drugs as preys and users of drugs as predators, in this paper, we have proposed a fractional illicit drug consumption model by incorporating Holling type-III functional response. We denote non-users, experimental users, recreational users, and addicts in a proportion by $E(t)$, $R(t)$, and $A(t)$ such that $N(t) + E(t) + R(t) + A(t) = 1$. However, experimental users and recreational users play the role of both predators and prey. The proposed fractional mathematical model is as follows:

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha} N &= \beta_1 N(1 - N) - \lambda_1 \frac{EN}{1 + N^2} - \alpha_1 \frac{RN}{1 + N^2} - \gamma_1 \frac{AN}{1 + N^2}, \\ {}_{t_0}^{C}D_t^{\alpha} E &= \lambda_1 \frac{EN}{1 + N^2} - \alpha_2 \frac{RE}{1 + E^2} - \gamma_2 \frac{AE}{1 + E^2} - \beta_2 E, \\ {}_{t_0}^{C}D_t^{\alpha} R &= \alpha_1 \frac{RN}{1 + N^2} + \alpha_2 \frac{RE}{1 + E^2} - \gamma_3 \frac{AR}{1 + R^2} - \beta_3 R, \\ {}_{t_0}^{C}D_t^{\alpha} A &= \gamma_1 \frac{AN}{1 + N^2} + \gamma_2 \frac{AE}{1 + E^2} + \gamma_3 \frac{AR}{1 + R^2} - \beta_4 A \end{aligned} \tag{3.1} $$

with initial positive conditions $N(t_0), E(t_0), R(t_0), A(t_0)$ where t_0 is the initial time. ${}_{t_0}^{C}D_t^{\alpha}$ represents Caputo fractional derivative operator. All the parameters $\lambda_1, \alpha_1, \alpha_2, \gamma_1, \gamma_2, \gamma_3,$ $\beta_1, \beta_2, \beta_3, \beta_4$ are considered to be positive. In Table 1, parameters' biological meanings are presented.

4 Uniqueness and Existence

The uniqueness and existence of the solutions of the fractional system 3.1 are established in this section.

Theorem 4.1. *In the region $\Omega \times [t_0, T]$, where $\Omega = \{(N, E, R, A) \in \mathbb{R}^4 : max\{|N|, |E|, |R|, |A|\} \leq 1\}$ and $T < +\infty$, the solution of the system 3.1 exists and is unique.*

Proof. Let us consider $X = (N, E, R, A)$, $\bar{X} = (\bar{N}, \bar{E}, \bar{R}, \bar{A})$, and a mapping

$$ \Pi(X) \quad = \quad (\Pi_1(X), \Pi_2(X), \Pi_3(X), \Pi_4(X)), $$

where

$$\begin{aligned}\Pi_1(X) &= \beta_1 N(1-N) - \lambda_1 \frac{EN}{1+N^2} - \alpha_1 \frac{RN}{1+N^2} - \gamma_1 \frac{AN}{1+N^2}\\ \Pi_2(X) &= \lambda_1 \frac{EN}{1+N^2} - \alpha_2 \frac{RE}{1+E^2} - \gamma_2 \frac{AE}{1+E^2} - \beta_2 E\\ \Pi_3(X) &= \alpha_1 \frac{RN}{1+N^2} + \alpha_2 \frac{RE}{1+E^2} - \gamma_3 \frac{AR}{1+R^2} - \beta_3 R\\ \Pi_4(X) &= \gamma_1 \frac{AN}{1+N^2} + \gamma_2 \frac{AE}{1+E^2} + \gamma_3 \frac{AR}{1+R^2} - \beta_4 A\end{aligned}$$

We shall show that there exists some Υ such that

$$||\Pi(X) - \Pi(\bar{X})|| \leq \Upsilon ||X - \bar{X}||$$

Consider,

$$\begin{aligned}& ||\Pi(X) - \Pi(\bar{X})||\\ = & |\Pi_1(X) - \Pi_1(\bar{X})| + |\Pi_2(X) - \Pi_2(\bar{X})| + |\Pi_3(X) - \Pi_3(\bar{X})| + |\Pi_4(X) - \Pi_4(\bar{X})|\\ = & |\beta_1 N(1-N) - \lambda_1 \frac{EN}{1+N^2} - \alpha_1 \frac{RN}{1+N^2} - \gamma_1 \frac{AN}{1+N^2} - \beta_1 \bar{N}(1-\bar{N}) + \lambda_1 \frac{\bar{E}\bar{N}}{1+\bar{N}^2}\\ & +\alpha_1 \frac{\bar{R}\bar{N}}{1+\bar{N}^2} + \gamma_1 \frac{\bar{A}\bar{N}}{1+\bar{N}^2}| + |\lambda_1 \frac{EN}{1+N^2} - \alpha_2 \frac{RE}{1+E^2} - \gamma_2 \frac{AE}{1+E^2} - \beta_2 E\\ & -\lambda_1 \frac{\bar{E}\bar{N}}{1+\bar{N}^2} + \alpha_2 \frac{\bar{R}\bar{E}}{1+\bar{E}^2} + \gamma_2 \frac{\bar{A}\bar{E}}{1+\bar{E}^2} + \beta_2 \bar{E}| + |\alpha_1 \frac{RN}{1+N^2} + \alpha_2 \frac{RE}{1+E^2} - \gamma_3 \frac{AR}{1+R^2}\\ & -\beta_3 R - \alpha_1 \frac{\bar{R}\bar{N}}{1+\bar{N}^2} - \alpha_2 \frac{\bar{R}\bar{E}}{1+\bar{E}^2} + \gamma_3 \frac{\bar{A}\bar{R}}{1+\bar{R}^2} + \beta_3 \bar{R}| + |\gamma_1 \frac{AN}{1+N^2} + \gamma_2 \frac{AE}{1+E^2}\\ & +\gamma_3 \frac{AR}{1+R^2} - \beta_4 A - \gamma_1 \frac{\bar{A}\bar{N}}{1+\bar{N}^2} - \gamma_2 \frac{\bar{A}\bar{E}}{1+\bar{E}^2} - \gamma_3 \frac{\bar{A}\bar{R}}{1+\bar{R}^2} + \beta_4 \bar{A}|\end{aligned}$$

$$\begin{aligned}\leq & \beta_1 |(N-\bar{N})(1-(N+\bar{N}))| + \lambda_1 |(EN - \bar{E}\bar{N}) + N\bar{N}(E\bar{N} - \bar{E}N)| + \alpha_1 |(RN - \bar{R}\bar{N})\\ & +N\bar{N}(R\bar{N} - \bar{R}N)| + \gamma_1 |(AN - \bar{A}\bar{N}) + N\bar{N}(A\bar{N} - \bar{A}N)| + \lambda_1 |(EN - \bar{E}\bar{N})\\ & +N\bar{N}(E\bar{N} - \bar{E}N)| + \alpha_2 |(RE - \bar{R}\bar{E}) + E\bar{E}(R\bar{E} - \bar{R}E)| + \gamma_2 |(AE - \bar{A}\bar{E})\\ & +E\bar{E}(A\bar{E} - \bar{A}E)| + \beta_2 |E - \bar{E}| + \alpha_1 |(RN - \bar{R}\bar{N}) + N\bar{N}(R\bar{N} - \bar{R}N)|\\ & +\alpha_2 |(RE - \bar{R}\bar{E}) + E\bar{E}(R\bar{E} - \bar{R}E)| + \gamma_3 |(AR - \bar{A}\bar{R}) + R\bar{R}(A\bar{R} - \bar{A}R)|\\ & +\beta_3 |R - \bar{R}| + \gamma_1 |(AN - \bar{A}\bar{N}) + N\bar{N}(A\bar{N} - \bar{A}N)| + \gamma_2 |(AE - \bar{A}\bar{E})\\ & +E\bar{E}(A\bar{E} - \bar{A}E)| + \gamma_3 |(AR - \bar{A}\bar{R}) + R\bar{R}(A\bar{R} - \bar{A}R)| + \beta_4 |A - \bar{A}|\end{aligned}$$

$$\begin{aligned}\leq & 3\beta_1 |N - \bar{N}| + 2\lambda_1 (|N - \bar{N}| + |E - \bar{E}|) + 2\alpha_1 (|N - \bar{N}| + |R - \bar{R}|)\\ & +2\gamma_1 (|N - \bar{N}| + |A - \bar{A}|) + 2\lambda_1 (|N - \bar{N}| + |E - \bar{E}|) + 2\alpha_2 (|R - \bar{R}| + |E - \bar{E}|)\\ & +2\gamma_2 (|A - \bar{A}| + |E - \bar{E}|) + \beta_2 |E - \bar{E}| + 2\alpha_1 (|R - \bar{R}| + |N - \bar{N}|) + 2\alpha_2 (|R - \bar{R}) + |E - \bar{E}|)\\ & +2\gamma_3 (|A - \bar{A}| + |R - \bar{R}|) + \beta_3 |R - \bar{R}| + 2\gamma_1 (|A - \bar{A}| + |N - \bar{N}|)\\ & +2\gamma_2 (|A - \bar{A}| + |E - \bar{E}|) + 2\gamma_3 (|A - \bar{A}| + |R - \bar{R}|) + \beta_4 |A - \bar{A}|\end{aligned}$$

This implies,

$$\begin{aligned}||\Pi(X) - \Pi(\bar{X})|| &\leq (3\beta_1 + 4\lambda_1 + 4\alpha_1 + 4\gamma_1)|N - \bar{N}| + (4\lambda_1 + 4\lambda_2 + 4\gamma_2 + \beta_2)|E - \bar{E}|\\ & +(4\alpha_1 + 4\alpha_2 + 4\gamma_3 + \beta_3)|R - \bar{R}| + (4\gamma_1 + 4\gamma_2 + 4\gamma_3 + \beta_4)|A - \bar{A}|\\ &\leq \Upsilon_1 |N - \bar{N}| + \Upsilon_2 |E - \bar{E}| + \Upsilon_3 |R - \bar{R}| + \Upsilon_4 |A - \bar{A}|\end{aligned}$$

Where,

$$\begin{aligned}\Upsilon_1 &= 3\beta_1 + 4\lambda_1 + 4\alpha_1 + 4\gamma_1,\\ \Upsilon_2 &= 4\lambda_1 + 4\lambda_2 + 4\gamma_2 + \beta_2,\\ \Upsilon_3 &= 4\alpha_1 + 4\alpha_2 + 4\gamma_3 + \beta_3,\\ \Upsilon_4 &= 4\gamma_1 + 4\gamma_2 + 4\gamma_3 + \beta_4\end{aligned}$$

Let, $\Upsilon = max\{\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4\}$
$\therefore ||\Pi(X) - \Pi(\bar{X})|| \leq \Upsilon||X - \bar{X}||$
Hence, by lemma 2.2, the solution exists and is unique. □

5 Boundedness

In this section we show that the solutions of the system 3.1 are bounded.

Theorem 5.1. *The solutions of the system 3.1 are uniformly bounded.*

Proof. Let us define a function, $F(t) = N(t) + E(t) + R(t) + A(t)$
by using the lemma 2.3, we get,

$$\begin{aligned}
& {}_{t_0}^{C}D_t^{\alpha} F(t) + \beta_2 F(t)\\
=\ & {}_{t_0}^{C}D_t^{\alpha}(N(t) + E(t) + R(t) + A(t)) + \beta_2(N(t) + E(t) + R(t) + A(t))\\
=\ & \beta_1 N(1-N) - \lambda_1\frac{EN}{1+N^2} - \alpha_1\frac{RN}{1+N^2} - \gamma_1\frac{AN}{1+N^2} + \lambda_1\frac{EN}{1+N^2} - \alpha_2\frac{RE}{1+E^2} - \gamma_2\frac{AE}{1+E^2}\\
& -\beta_2 E + \alpha_1\frac{RN}{1+N^2} + \alpha_2\frac{RE}{1+E^2} - \gamma_3\frac{AR}{1+R^2} - \beta_3 R + \gamma_1\frac{AN}{1+N^2} + \gamma_2\frac{AE}{1+E^2} + \gamma_3\frac{AR}{1+R^2}\\
& -\beta_4 A + \beta_2(N + E + R + A)\\
=\ & \beta_1 N(1-N) - \beta_2 E - \beta_3 R - \beta_4 A + \beta_2(N + E + R + A)\\
=\ & \beta_1 N(1-N) - \beta_3 R - \beta_4 A + \beta_2(N + R + A)\\
\leq\ & \beta_1 N(1-N) + \beta_2(N + R + A)\\
=\ & -\beta_1(N - \frac{1}{2})^2 + \frac{1}{4}\beta_1 + 3\beta_2\\
\leq\ & \frac{1}{4}\beta_1 + 3\beta_2
\end{aligned}$$

The solution exists and is unique in
$\Omega = \{(N, E, R, A) : \quad max\{|N|, |E|, |R|, |A|\} \leq 1\}$
The above inequality yields,
${}_{t_0}^{C}D_t^{\alpha} F(t) + \beta_2 F(t) \leq \frac{1}{4}\beta_1 + 3\beta_2$
By the lemma 2.3, we get
${}_{t_0}^{c}D_t^{\alpha} F(t) \leq \big(F(t_0) - \frac{1}{\beta_2}\big(\frac{1}{4}\beta_1 + 3\beta_2\big)\big)E_\alpha[-\beta_2(t-t_0)^\alpha] + \frac{1}{\beta_2}\big(\frac{1}{4}\beta_1 + 3\beta_2\big) \to \frac{1}{4}\beta_1 + 3\beta_2, t \to \infty$
Therefore, all solutions of the system 3.1 that were initiated in Ω remained bounded in
$\Theta = \{(N, E, R, A) \in \Omega_+ | F(t) \leq \frac{1}{4}\beta_1 + 3\beta_2 + \epsilon, \quad \epsilon > 0\}$ □

6 Existence of points of equilibrium and their local stability

The system 3.1 has a set of interesting points of equilibrium. Their stability criteria are investigated in this section.

(i) Axial equilibrium point is $\xi = (1, 0, 0, 0)$, and it always exists.

Theorem 6.1. *Let* $\Re_0 = \frac{\alpha_1}{2\beta_3}, \Re_1 = \frac{\lambda_1}{2\beta_2}, \Re_2 = \frac{\gamma_1}{2\beta_4}$.
Then, the axial equilibrium point $\xi = (1, 0, 0, 0)$ *is stable if* $\Re_0 < 1, \Re_1 < 1, \Re_2 < 1$.

Proof. Eigen values at the axial equilibrium point ξ are $\Lambda_{00} = -2\beta_1$, $\Lambda_{01} = \alpha_1 - 2\beta_3$, $\Lambda_{02} = -2\beta_2 + \lambda_1$, $\Lambda_{03} = -2\beta_4 + \gamma_1$. ξ will be stable if $|arg(\Lambda_{0i})| > \frac{\alpha\pi}{2}$, $i = 0,1,2,3$ i.e., we must have eigenvalues as negative. Λ_{00} is always negative. $\Lambda_{01}, \Lambda_{02}, \Lambda_{03}$ will be negative if $\Re_0 < 1, \Re_1 < 1, \Re_2 < 1$ respectively.

□

(ii) Experimental and Recreational free equilibrium point is

$$\bar{\xi} = (\bar{N}, 0, 0, \bar{A}) = \left(\frac{\gamma_1 + \sqrt{-4\beta_4^2 + \gamma_1^2}}{2\beta_4}, 0, 0, \frac{2\beta_1\beta_4 + \beta_1\gamma_1 - \frac{\beta_1\gamma_1^2}{\beta_4} + \beta_1\sqrt{-4\beta_4^2 + \gamma_1^2} - \frac{\beta_1\gamma_1\sqrt{-4\beta_4^2+\gamma_1^2}}{\beta_4}}{2\beta_4^2}\right).$$

$\bar{\xi}$ exists if $\gamma_1 > 2\beta_4$.

Theorem 6.2. *Let* $\bar{\Re}_0 = \frac{\lambda_1\beta_4}{\gamma_1(\beta_2 + \bar{A}\gamma_2)}$ *and* $\bar{\Re}_1 = \frac{\beta_4\alpha_1}{\gamma_1(\beta_3 + \bar{A}\gamma_3)}$. *The experimental and recreational free equilibrium point* $\bar{\xi} = (\bar{N}, 0, 0, \bar{A})$ *is stable if* $\bar{\Re}_0 < 1$, $\bar{\Re}_1 < 1$, $\bar{A}\gamma_1 \geq \frac{1}{(1-\bar{N})^2}\beta_4$.

Proof. Eigen values at $\bar{\xi}$ are

$\Lambda_{10} = \frac{1}{2}(-P - \sqrt{(P^2 - 4Q)})$, $\quad \Lambda_{11} = \frac{1}{2}(-P + \sqrt{(P^2 - 4Q)})$

$\Lambda_{12} = \frac{\bar{N}\gamma_1}{\beta_4}\left(\frac{\beta_4\lambda_1}{\gamma_1} - (\beta_2 + \bar{A}\gamma_2)\right)$, $\quad \Lambda_{13} = \frac{\bar{N}\gamma_1}{\beta_4}\left(\frac{\beta_4\alpha_1}{\gamma_1} - (\beta_3 + \bar{A}\gamma_3)\right)$

where,

$P = -\frac{1}{(1-\bar{N})^2}\beta_4 + \bar{A}\gamma_1$

$Q = \bar{A}(1 - \bar{N}^2)\beta_1\gamma_1$

We have, $\frac{\bar{N}}{1+\bar{N}^2} = \frac{\beta_4}{\gamma_1}$. Λ_{12} and Λ_{13} will be negative if $\bar{\Re}_0 < 1$ and $\bar{\Re}_1 < 1$ respectively. Clearly, Q is positive. Therefore $\sqrt{(P^2 - 4Q)}$ will be either complex number or a real number less than P. Hence the eigenvalues Λ_{10} and Λ_{11} will be negative or complex conjugate with negative real part if $P \geq 0$.

Hence, the sufficient conditions for $\bar{\xi}$ to be stable are, $\bar{\Re}_0 < 1$, $\bar{\Re}_1 < 1$ and $\bar{A}\gamma_1 \geq \frac{1}{(1-\bar{N})^2}\beta_4$. □

(iii) Recreational and Addiction free equilibrium point is

$$\tilde{\xi} = (\tilde{N}, \tilde{E}, 0, 0) = \left(\frac{\lambda_1 - \sqrt{-4\beta_2^2 + \lambda_1^2}}{2\beta_2}, \frac{2\beta_1\beta_2 + \beta_1\lambda_1 - \frac{\beta_1\lambda_1^2}{\beta_2} + \beta_1\sqrt{-4\beta_2^2 + \lambda_1^2} - \frac{\beta_1\lambda_1\sqrt{-4\beta_2^2+\lambda_1^2}}{\beta_2}}{2\beta_2^2}, 0, 0\right).$$

$\tilde{\xi}$ exists if $\lambda_1 > 2\beta_2$.

Theorem 6.3. *Let* $\tilde{\Re}_0 = \frac{\tilde{E}\alpha_2}{1+\tilde{E}^2}$ *and* $\tilde{\Re}_1 = \frac{\tilde{E}\gamma_2}{1+\tilde{E}^2}$. *The recreational and addiction free equilibrium point* $\tilde{\xi} = (\tilde{N}, \tilde{E}, 0, 0)$ *is stable if* $\tilde{\Re}_0 < \beta_3 - \frac{\alpha_1\beta_2}{\lambda_1}$, $\tilde{\Re}_1 < \beta_4 - \frac{\beta_2\gamma_1}{\lambda_1}$ *and* $\tilde{E}\lambda_1 \geq \frac{1}{(1-\tilde{N})^2}\beta_1$.

Proof. Eigen values at $\tilde{\xi}$ are

$\Lambda_{20} = \frac{\tilde{N}\lambda_1}{\beta_2(1+\tilde{E}^2)}\left(\left(\frac{\beta_2\alpha_1}{\lambda_1} - \beta_3\right) + \frac{\tilde{E}\alpha_2}{1+\tilde{E}^2}\right)$,

$\Lambda_{21} = \frac{1}{2}(-X - \sqrt{(X^2 - 4Y)})$, $\quad \Lambda_{22} = \frac{1}{2}(-X + \sqrt{(X^2 - 4Y)})$.

$\Lambda_{23} = \frac{\tilde{N}\lambda_1}{(1+\tilde{E}^2)\beta_2}\left(\left(\frac{\beta_2\gamma_1}{\lambda_1} - \beta_4\right) + \frac{\tilde{E}\gamma_2}{1+\tilde{E}^2}\right)$

where,

$X = -\frac{1}{(1-\tilde{N})^2}\beta_1 + \tilde{E}\lambda_1$

$Y = \tilde{E}\lambda_1((1 - \tilde{N}^2 + \tilde{E}^2(3 + \tilde{N}^2))\beta_2 + 2\tilde{E}^2\tilde{N}\lambda_1)$

We have, $\frac{\tilde{N}}{1+\tilde{N}^2} = \frac{\beta_2}{\lambda_1}$. Λ_{20} and Λ_{23} will be negative if $\tilde{\Re}_0 < \beta_3 - \frac{\alpha_1\beta_2}{\lambda_1}$ and $\tilde{\Re}_1 < \beta_4 - \frac{\beta_2\gamma_1}{\lambda_1}$ respectively. Clearly, Y is positive. Therefore, $\sqrt{(X^2 - 4Y)}$ is either a complex number or a real number less than X. So, the eigenvalues Λ_{21} and Λ_{22} will be negative if $X \geq 0$.

Hence, the sufficient conditions for $\tilde{\xi}$ to be stable are, $\tilde{\Re}_0 < \beta_3 - \frac{\alpha_1\beta_2}{\lambda_1}$, $\tilde{\Re}_1 < \beta_4 - \frac{\beta_2\gamma_1}{\lambda_1}$ and $\tilde{E}\lambda_1 \geq \frac{1}{(1-\tilde{N})^2}\beta_1$. □

7 Dynamics of the system based on the threshold parameters

Threshold parameters defined in section 6 have been validated in this section.

(i) $\Re_0 = \frac{\alpha_1}{2\beta_3}, \Re_1 = \frac{\lambda_1}{2\beta_2}$ and $\Re_2 = \frac{\gamma_1}{2\beta_4}$ determine the local stability of the axial equilibrium ξ. $\frac{1}{2\beta_3}$ is the average life span of a recreational user, and α_1 is the rate at which non-users are being influenced by each recreational user. Hence, the product of $\frac{1}{2\beta_3}$ and α_1 gives the mean number of population going to the recreational user category which can be referred as BR number of the recreational user category in ξ.
$\frac{1}{2\beta_2}$ is the average life span of an experimental user category, and λ_1 is the rate at which non-users are being influenced by each experimental user. Hence, the product of $\frac{1}{2\beta_2}$ and λ_1 gives the mean number of population going to the experimental user category which can be referred as the BR number of the experimental user category in ξ.
$\frac{1}{2\beta_4}$ is the average life span of an addict and γ_1 is the rate at which non-users are being influenced by each addict. Hence, the product of $\frac{1}{2\beta_4}$ and γ_1 gives the mean number of population going to the addicts category which can be referred as the BR number of the addicts category in ξ. If $\Re_0 < 1, \Re_1 < 1$ and $\Re_2 < 1$, the recreational users, experimental users, and addict population go to extinction.

(ii) $\bar{\Re}_0 = \frac{\lambda_1\beta_4}{\gamma_1(\beta_2+A\gamma_2)}$ and $\bar{\Re}_1 = \frac{\beta_4\alpha_1}{\gamma_1(\beta_3+A\gamma_3)}$ control the stability of experimental and recreational free equilibrium point $\bar{\xi}$. We have, $\frac{\bar{N}}{1+\bar{N}^2} = \frac{\beta_4}{\gamma_1}$. $\frac{1}{\beta_2+\bar{A}\gamma_2}$ is the total population number retaining in the experimental user category, and $\frac{\lambda_1\bar{N}}{1+\bar{N}^2}$ is the total number of non-users being influenced by each experimental user. Hence, the product of $\frac{1}{\beta_2+\bar{A}\gamma_2}$ and $\frac{\lambda_1\bar{N}}{1+\bar{N}^2}$ gives the mean number of population going to the experimental user category which can be referred as the BR number of the experimental user category in $\bar{\xi}$.

On the other hand, $\frac{1}{\beta_3+\bar{A}\gamma_3}$ is the total number of population retaining in the recreational user category and $\frac{\alpha_1\bar{N}}{1+\bar{N}^2}$ is the total number of non-users being influenced by each recreational user. Hence, the product of $\frac{1}{\beta_3+\bar{A}\gamma_3}$ and $\frac{\alpha_1\bar{N}}{1+\bar{N}^2}$ gives the mean number of population going to the recreational user category which can be referred as the BR number of the recreational user category in $\bar{\xi}$. If $\bar{\Re}_0 < 1$ and $\bar{\Re}_1 < 1$, the experimental and recreational users category go to extinction.

(iii) $\tilde{\Re}_0 < \beta_3 - \frac{\alpha_1\beta_2}{\lambda_1}$ and $\tilde{\Re}_1 < \beta_4 - \frac{\beta_2\gamma_1}{\lambda_1}$ determine the local stability of the recreational and addiction free equilibrium point $\tilde{\xi}$ where, $\tilde{\Re}_0 = \frac{\tilde{E}\alpha_2}{1+\tilde{E}^2}$ and $\tilde{\Re}_1 = \frac{\tilde{E}\gamma_2}{1+\tilde{E}^2}$. We have $\frac{1}{2} > \frac{\beta_2}{\lambda_1}$. $\tilde{\Re}_0 = \frac{\tilde{E}\alpha_2}{1+\tilde{E}^2}$ is the total number of experimental users being influenced by each recreational user, which can be referred as the BR number of the recreational user category at $\tilde{\xi}$. β_3 is the natural death rate of recreational users. $\frac{\alpha_1\beta_2}{\lambda_1}$ indicates the rate of influence of recreational users on the population who quit recreational category. Hence, $\beta_3 - \frac{\alpha_1\beta_2}{\lambda_1}$ is a quantity less than the death rate of recreational users.
On the other hand, $\tilde{\Re}_1 = \frac{\tilde{E}\gamma_2}{1+\tilde{E}^2}$ is the total number of experimental user population being influenced by each addict, which can be referred as the BR number of the addict category in $\tilde{\xi}$. β_4 is the natural death rate of addicts. $\frac{\beta_2\gamma_1}{\lambda_1}$ indicates the rate of influence of addicts on the population who quit recreational category. Hence, $\beta_4 - \frac{\beta_2\gamma_1}{\lambda_1}$ is a quantity less than the death rate of addicts. If $\tilde{\Re}_0 < \beta_3 - \frac{\alpha_1\beta_2}{\lambda_1}$ and $\tilde{\Re}_1 < \beta_4 - \frac{\beta_2\gamma_1}{\lambda_1}$, the recreational and addict user category go to extinction.

8 Sensitivity indices of illicit drug usage

Sensitivity indices analysis is generally used to determine the accuracy and effectiveness of the BR number. In our drug model, the BR number describes the mean number of population staying in the particular drug user category. Analysis of the sensitivity indices for the parameters of the model 3.1 is evaluated by relating them to the BR number $\Re$. The sensitivity indices of the variable v are given as follows:

$$S_v^{\Re} = \frac{\partial \Re}{\partial_v} * \frac{v}{\Re}$$

The sign of sensitivity indices of each parameter indicates their contribution to the drug usage. Here, we are going to determine the reduction in drug usage by formulating the sensitivity indices for all the BR numbers $\Re_0, \Re_1, \Re_2, \bar{\Re}_0, \bar{\Re}_1, \tilde{\Re}_0, \tilde{\Re}_1$ of the model 3.1 with respect to the parameter values. By considering nine different parameters, we have derived the sensitivity indices for them which are given in the table 2. The sign of each parameter in the sensitivity indices table

Table 2. Sensitivity indices of parameter values

Parameter	Description	Sensitivity Index (+ve, -ve)
α_1	Rate at which $R(t)$ influences $N(t)$	-ve
λ_1	Rate at which $E(t)$ influences $N(t)$	-ve
γ_1	Rate at which $A(t)$ influences $N(t)$	-ve
β_3	Rate at which recreational users stop drug consumption	+ve
β_2	Rate at which experimental users stop drug consumption	+ve
β_4	Rate at which addicts stop drug consumption	+ve
γ_2	Rate at which $A(t)$ influences $E(t)$	-ve
γ_3	Rate of conversion of recreational users into addicts	-ve
α_2	Rate at which $R(t)$ influences $E(t)$	-ve

represents their contribution to illicit drug usage. As the value of $\alpha_1, \lambda_1, \gamma_1, \gamma_2, \gamma_3, \alpha_2$ decreases, the basic reproduction number decreases which makes the mean number of population staying in particular drug user category less. Also, the increase in the value of $\beta_2, \beta_3, \beta_4$ helps to reduce the drug usage. Hence, the usage of illicit drugs can be reduced.

9 Numerical Simulation

The obtained theoretical results for the fractional-order model 3.1 in Section 6 are numerically investigated in this section. The trapezoidal-based homotopy perturbation method [44] is applied to solve the system 3.1 with the help of mathematical software Mathematica. The role of various parameters involved in the system and fractional order α are discussed to establish the validity of the model 3.1 in the ecological environment.

9.1 Dynamics of the system 3.1 around various points of equilibrium.

(i) **Axial equilibrium**
For the parameter values $\lambda_1 = 0.2, \alpha_1 = 0.3, \alpha_2 = 0.5, \gamma_1 = 0.3, \gamma_2 = 0.9, \gamma_3 = 0.3, \beta_1 = 0.4, \beta_2 = 0.2, \beta_3 = 0.2, \beta_4 = 0.3$, we get the axial equilibrium point $\xi = (1, 0, 0, 0)$. For this parameter set, we get $\Re_0 = 0.75 < 1, \Re_1 = 0.5 < 1$ and $\Re_2 = 0.5 < 1$, which satisfies the conditions of the theorem 6.1. The eigenvalues for Jacobian matrix at ξ are $(-0.8, -0.3, -0.2, -0.)$ and hence the equilibrium point ξ is stable. Figure 1 indicates the stability profile of the axial equilibrium point. We can see that, for all the values of α, the equilibrium point is stable and converges to ξ.

(ii) **Experimental and Recreational free equilibrium**
For the parameter values $\lambda_1 = 0.2, \alpha_1 = 0.6, \alpha_2 = 0.5, \gamma_1 = 0.6, \gamma_2 = 0.9, \gamma_3 = 0.3, \beta_1 = 0.4, \beta_2 = 0.2, \beta_3 = 0.2, \beta_4 = 0.1$, we get the experimental and recreational free equilibrium point $\bar{\xi} = (0.171573, 0, 0, 0.568542)$. For this parameter set, we get $\bar{\Re}_0 = 0.04683 < 1$ and

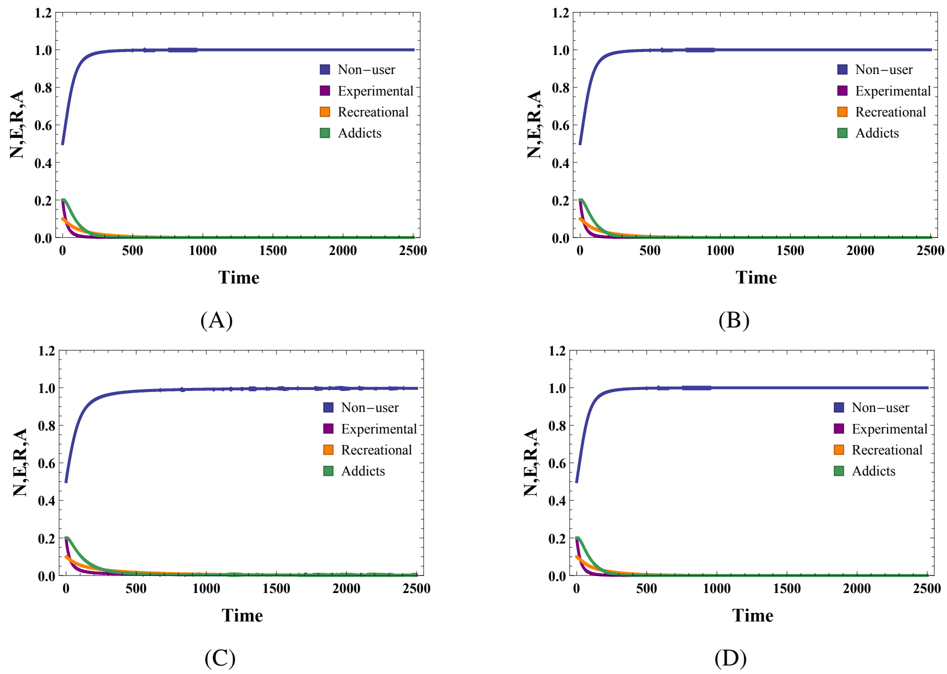

Figure 1. Stability profile of the axial equilibrium point $\xi = (1,0,0,0)$. (A) $\alpha = 1$, (B) $\alpha = 0.95$, (C) $\alpha = 0.9$, (D) $\alpha = 0.8$

$\bar{\Re}_1 = 0.2698 < 1$, which satisfies the conditions of the theorem 6.2. The eigenvalues for Jacobian matrix at $\bar{\xi}$ are $(-0.698324, -0.278527, -0.0255701 + i0.180151, -0.0255701 - i0.180151)$ and hence the equilibrium point $\bar{\xi}$ is stable. Figure 2 gives a clear observation of experimental and recreational free equilibrium point. It can be observed that as the value of fractional derivative is diverging from 1, oscillatory behavior of the solution gradually reduced to smooth stability. For $\alpha = 1$, the non-user and addict population are converging to the desired values that were obtained theoretically. As the value of α decreases, the convergence rate is even faster, and the stability is achieved quickly.

(iii) **Recreational and Addiction free equilibrium**
For the parameter values $\lambda_1 = 0.8, \alpha_1 = 0.3, \alpha_2 = 0.5, \gamma_1 = 0.2, \gamma_2 = 0.9, \gamma_3 = 0.3, \beta_1 = 0.4, \beta_2 = 0.2, \beta_3 = 0.3, \beta_4 = 0.4$, we get the recreational and addiction free equilibrium point $\tilde{\xi} = (0.267949, 0.392305, 0, 0)$. For this parameter set, we get $\tilde{\Re}_0 = 0.16999 < 0.225 = \beta_3 - \frac{\alpha_1\beta_2}{\lambda_1}$ and $\tilde{\Re}_1 = 0.3059 < 0.35 = \beta_4 - \frac{\beta_2\gamma_1}{\lambda_1}$, which satisfies the conditions of the theorem 6.3. The eigenvalues for Jacobian matrix at $\tilde{\xi}$ are $(-0.0364139 + i0.256716, -0.0364139 - i0.256716, -0.0680331, -0.0544384)$ and hence the equilibrium point $\tilde{\xi}$ is stable. In Figure 3, we can see that for $\alpha = 1$, the non-user and experimental user population are converging to the stable point of equilibrium with oscillation. As the value of α decreases, the convergence rate becomes faster and smooth.

10 Applications of fractional NERA model

We have used the marijuana drug consumption data in the states of Washington and Colorado from Hanley [41], and we have performed the numerical experiments. While collecting the data in the Washington state, four key indicators of marijuana use are taken into consideration. They are current marijuana use, lifetime marijuana use, marijuana abuse, and age of initiation. In the fractional NERA model 3.1, the first three categories are related to the recreational, experimental, and addict categories. The aim of the numerical experiments here is to analyze the behavioral change of the solution after implementing the data collected from Hanley [41] in the model 3.1.

(i) **Fractional NERA model for Colorado**
The parameter values $\lambda_1 = 0.44, \alpha_1 = 0.043, \alpha_2 = 0.193, \gamma_1 = 0.103, \gamma_2 = 0.031, \gamma_3 =$

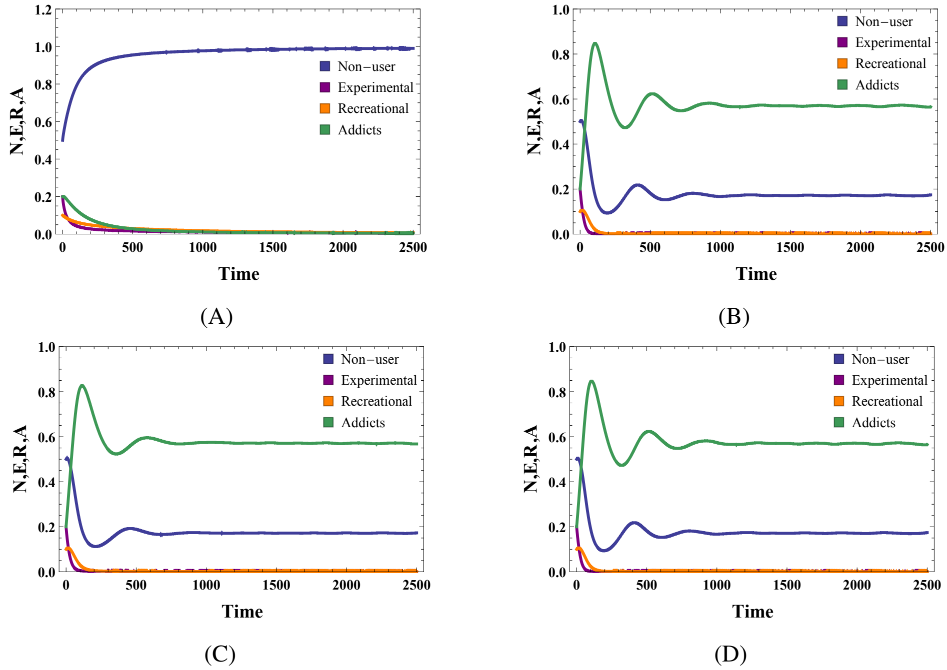

Figure 2. Stability profile of the experimental and recreational free equilibrium point $\bar{\xi} = (0.171573, 0, 0, 0.568542)$. (A) $\alpha = 1$, (B) $\alpha = 0.95$, (C) $\alpha = 0.9$, (D) $\alpha = 0.8$

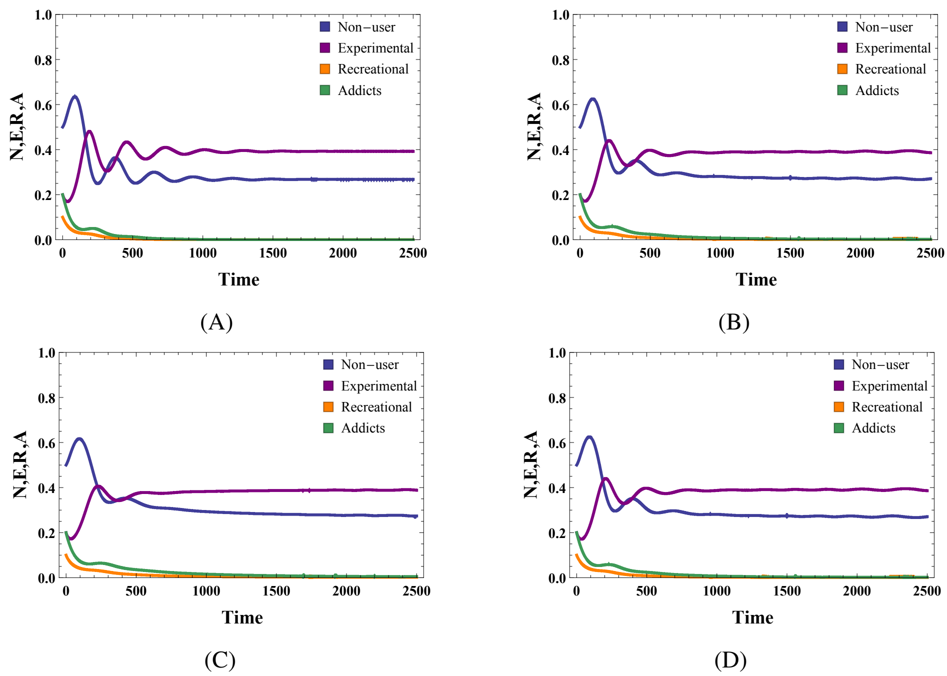

Figure 3. Stability profile of the recreational and addiction free equilibrium point $\tilde{\xi} = (0.267949, 0.392305, 0, 0)$. (A) $\alpha = 1$, (B) $\alpha = 0.95$, (C) $\alpha = 0.9$, (D) $\alpha = 0.8$

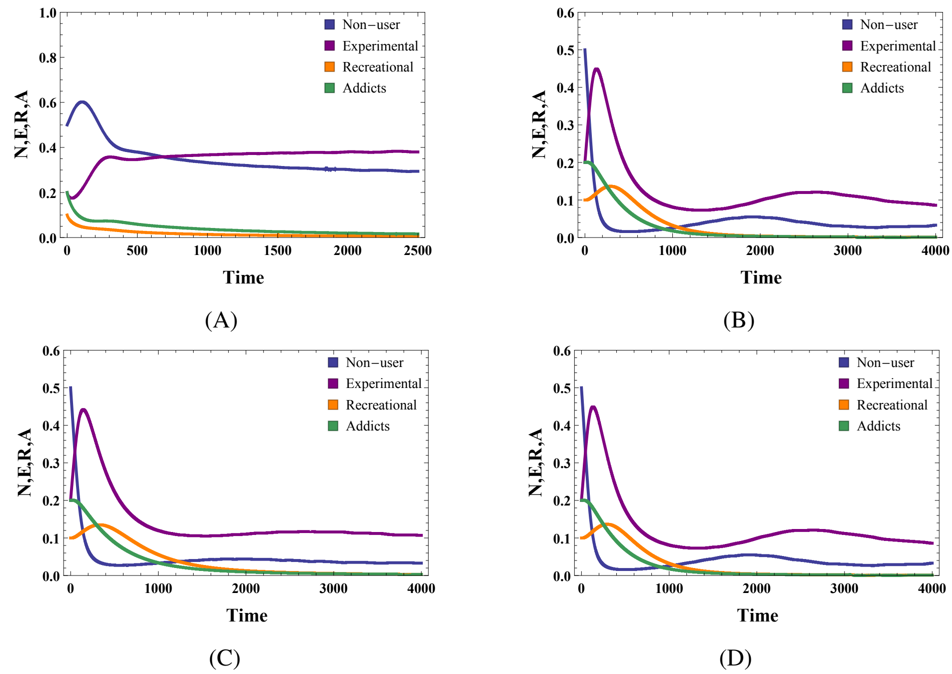

Figure 4. Stability profile of the Colorado model. (A) $\alpha = 1$, (B) $\alpha = 0.95$, (C) $\alpha = 0.9$, (D) $\alpha = 0.8$

0.029, $\beta_1 = 0.042, \beta_2 = 0.016, \beta_3 = 0.052, \beta_4 = 0.047$ correspond to the Colorado model [41]. For the above parameter values, we have three equilibrium points (0.03641, 0.0921, 0, 0), (0.64779, 0, 0, 0.20388), and (1, 0, 0, 0) and the eigenvalues for Jacobian matrix at these points are (-0.0404234, -0.0328104,-0.000711+0.0254i, -0.000711-0.0254i), (0.178456, -0.0382913, -0.00923+0.0141i, -0.00923-0.0141i), and (0.204, -0.042, -0.0305, 0.0045) respectively. Hence, the equilibrium point (0.03641, 0.0921, 0, 0) is the stable point of equilibrium. In Figure 4, it is observed that the fractional NERA model 3.1 with integer-order derivative (Figure 4(A)) for Colorado data does not give the scope to study the coexistence pattern for all of the four population. As the value of α reduces, we can observe the coexistence behavior (Figure 4(D)). Also, it can be noticed that the influences of R on N, A on E and A on R are very less. Therefore, experimental users play a dominating role here.

(ii) **Fractional NERA model for Washington**
For the parameter values $\lambda_1 = 0.38, \alpha_1 = 0.112, \alpha_2 = 0.142, \gamma_1 = 0.099, \gamma_2 = 0.032, \gamma_3 = 0.034, \beta_1 = 0.015, \beta_2 = 0.03, \beta_3 = 0.066, \beta_4 = 0.039$ correspond to the Washington model. For the above parameter values, we have three equilibrium points (0.07944, 0.03656,0,0), (0.4875,0,0,0.09609) and (1,0,0,0) and the eigenvalues for Jacobian matrix for these equilibrium points are (-0.0519723, -0.0300156, -0.0005 + 0.0202i, -0.0005 -0.0202i), (0.116622, -0.025146, -0.00218+ 0.01341i, -0.00218-0.01341i) and (0.16, -0.015, 0.0105, -0.01) respectively. Hence, the equilibrium point (0.07944, 0.03656,0,0) is stable. For Washington data, we have observed that for $\alpha = 1$ initially drug users have huge influence on non-users. But in long run non-users only dominate the population. However, as α started decreasing, experimental users dominance becomes visible. Again, the influence of fractional derivative facilitates the study of the coexistence pattern in the proposed model 3.1(Figure 5).

11 Conclusion

By classifying the drug users in four categories viz. non-users (N), experimental users (E), recreational users (R), and addicts (A), we have presented a modified drug consumption model by introducing the influence of one category on other in terms of Holling Type-III functional

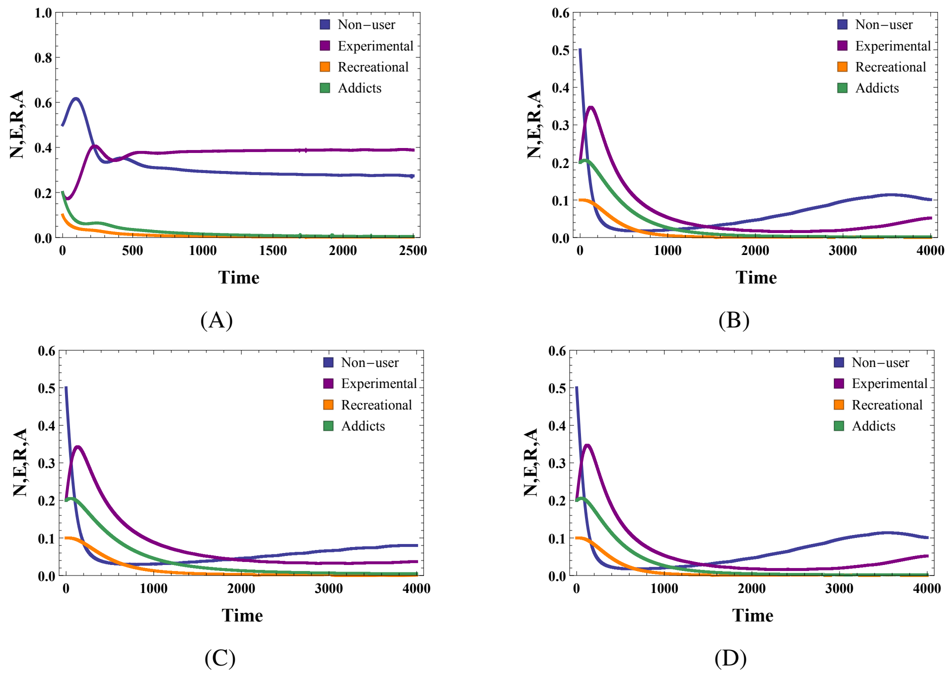

Figure 5. Stability profile of the Washington model. (A) $\alpha = 1$, (B) $\alpha = 0.95$, (C) $\alpha = 0.9$, (D) $\alpha = 0.8$

response and by incorporating Caputo fractional order derivative. We have analyzed the theoretical aspects such as uniqueness, existence, and boundedness of the solutions of the system 3.1. For various equilibrium points, conditions for local stability are set by defining certain threshold parameters, the dynamics of which in turn are analyzed concerning BR number. We have calculated the sensitivity indices and established a good connection among threshold parameters, BR number, and sensitivity indices. The performed numerical analysis for various set of values of parameters make it clear that fractional derivative has a great impact on modeling drug consumption behavior, and it can be a fantastic tool for a deeper understanding of illegal drug use.

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