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APPROXIMATING FIXED POINTS OF MAPPINGS SATISFYING CONDITION (CC) IN BANACH SPACES

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Abstract In this paper, we prove some weak and strong convergence results for mappings satisfying condition (CC) using Ishikawa iteration process in uniformly convex Banach spaces. The results obtained in the paper extend and improve many well-known results in the iterations theory.

1 Introduction

A mapping F on a subset K of a Banach space is called contraction if there is a real number $0 \leq a < 0$ such that

$$||Fu - Fv|| \leq a||u - v|| \text{ for all } u, v \in K. \tag{1.1}$$

If (1.1) is valid at $a = 1$ then F is called nonexpansive. A point $\omega \in K$ is called a fixed point of F is $\omega = F\omega$. The mapping F is called quasi-nonexpansive if for each $u \in K$,

$$||Fu - F\omega|| \leq ||u - \omega|| \text{ for all } \omega \in f(F).$$

In 1965, Kirk [12], Browder [8] and Gohde [9] independently proved that every nonexpansive mapping F has a fixed point provided that E is uniformly convex and S is closed convex and bounded.

In 2010, Takahashi [21] introduced the notion of hybrid mappings. A mapping F on a nonempty subset K of a Hlilbert space is called hybrid iff

$$3||Fu - Fv||^2 \leq ||u - v||^2 + ||Fu - v||^2 + ||u - Fv||^2 \text{ for all } u, v \in K.$$

In 2014, Suzuki [20], introduced the notion of Chatterjea mappings. A mapping F on a nonempty subset K of a Banach space is said to be a Chatterjea mapping with η (where $\eta : [0, \infty] \to [0, \infty)$ is strictly increasing and continuous and $\eta(0) = 0$) iff

$$2\eta(||Fu - Fv||) \leq \eta(||Fu - v||) + \eta(||u - Fv||) \text{ for all } u, v \in K.$$

In 2015, Kikkawa and Suzuki [11] introduced a new condition on mappings called condition (CC). They showed that condition (CC) is weaker than those of Chatterjea and hybrid mapping. A mapping F on a nonempty subset K of a Banach space is said to satisfy the (CC) condition if there is a continuous strictly-increasing function $\eta : [0, \infty) \to [0, \infty)$ with $\eta(0) = 0$ and $r, s \in [0, 1)$ such that $r + 2s = 1$ and

$$\eta(||Fu - Fv||) \leq r\eta(||u - v||) + s\eta(||Fu - v||) + s\eta(||u - Fv||) \text{ for all } u, v \in K.$$

They proved some characterizations and related fixed point results for these operators in Banach spaces. Here, we provide some convergence theorems (weak and strong) for these mapping under the two-step Ishikawa [10] iteration process, which is more general thana the Picard [16] and Mann [13] iterative processes.

The iterative scheme due to the Picard [16] is a simplest scheme, which is given as:

$$\begin{cases} u_1 \in K, \\ u_{n+1} = Fu_n, n \geq 1, \end{cases} \tag{1.2}$$

In the setting of nonexpansive operators, the Picard iterates generally not converge. Therefore, Mann [13] suggest a new scheme by the following formula:

$$\begin{cases} u_1 \in K, \\ u_{n+1} = \alpha_n Fu_n + (1-\alpha_n)u_n, n \geq 1, \end{cases} \tag{1.3}$$

where $\alpha_n \in [0,1]$. We may observe that the Picard iteration process (1.2) is a special case of Mann by putting $\alpha_n = 0$. In the setting of pseudo-contractive, the Mann iterative scheme not always converge. Thus in the year 1974, Ishikawa [10] constructed a new scheme by the following formula:

$$\begin{cases} u_1 \in K, \\ v_n = \beta_n Fu_n + (1-\beta_n)u_n, \\ u_{n+1} = \alpha_n Fv_n + (1-\alpha_n)Fu_n, n \geq 1, \end{cases} \tag{1.4}$$

where $\alpha_n, \beta_n \in [0,1]$. We may observe that the Mann iteration process (1.3) is a special case of Ishikawa by putting $\beta_n = 0$ and also Picard iteration process by putting $\alpha_n = \beta_n = 0$.

For more details and literature of these processes, one can cite the work [1, 2, 3, 4, 5, 6, 7, 15, 22, 23, 24, 25, 26, 27] and others.

In 1991, Rhoads [17] has noted some important facts about the iterative schemes as follows.

"The complete facts about the rate of convergence of the one iteration scheme to be different than the other scheme is not known because there is nothing available to cause the analysis of the one approximation scheme corresponding to the other scheme."

He also noted that the Mann iteration scheme is affective than Ishikawa iteration scheme in the setting of decreasing functions. While in the setting of increasing functions, the Ishikawa iteration scheme is effective than the Mann iteration scheme, and moreover, the Mann iteration scheme appears to be independent of the starting value (see also [18]).

2 Preliminaries

A complete normed space E is referred as a uniformly convex Banach space in the case when for $\varepsilon \in [0,1)$, some real constant $\delta > 0$ exists with the following facts

$$\frac{||u+v||}{2} \leq (1-\delta),$$

for each $u, v \in E$ with $||u|| \leq 1$, $||v|| \leq 1$ and $||u-v|| \geq \varepsilon$.

A complete normed space E is refered as a Banach space with the Opial condition [14] in the case when for a given weakly convergent $\{u_n\} \subseteq E$ having the weak limit $s \in E$, one obtain the following

$$\liminf_{n\to\infty} ||u_n - s|| < \liminf_{n\to\infty} ||u_n - l||.$$

for any $l \in E - \{s\}$.

The following facts are the about the condition (CC), which can be found in [11].

Proposition 2.1. Let K be a nonempty subset of a Banach space E and $F : K \to K$.

(i) If F enjoys condition (CC) with a nonempty fixed point set, then F is essentially quasi-nonexpansive on K.

(ii) If F enjoys the condition (CC) and $f(F)$ be its fixed point set. Then the fixed point set is closed in K and convex in the case when E is strictly convex and K is convex.

(iii) If E has Opial property, F satisfies the condition (CC), $\{u_n\}$ weakly convergent to a point p and $\lim_{n\to\infty} ||Fu_n - u_n|| = 0$, then $p \in f(F)$.

Xu [28] observed the following facts.

Lemma 2.2. *If a Banach space E is given. Then E is essentially uniformly convex iff there is real constant $c > 0$, one can find an increasing and continuous selfmap $\psi : [0, \infty) \to [0, \infty)$ in such a way that $\psi(0) = 0$ and the following hold*

$$||\alpha u + (1-\alpha)v||^2 \leq \alpha||u||^2 + (1-\alpha)||v||^2 - \alpha(1-\alpha)\psi\big(||u-v||\big),$$

for every two points $u, v \in E$ with $||u|| \leq c$, $||v|| \leq c$ and $\alpha \in [0,1]$.

3 Convergence theorems in uniformly convex Banach spaces

Lemma 3.1. *Let K be a nonempty closed convex subset of a Banach space. Assume that $F : K \to K$ satisfies the condition (CC) with $f(F) \neq \emptyset$. Let $\{u_n\}$ be the sequence of Ishikawa iterates defined by (1.4). Then $\lim_{n\to\infty} ||u_n - \omega||$ exists for each $\omega \in f(F)$.*

Proof. By choosing an element $\omega \in f(F)$ and then applying Proposition 2.1(i), the following are obtained

$$\begin{aligned} ||v_n - \omega|| &= ||\beta_n F u_n + (1-\beta_n)u_n - \omega|| \\ &\leq \beta_n||Fu_n - \omega|| + (1-\beta_n)||u_n - \omega|| \\ &\leq \beta_n||u_n - \omega|| + (1-\beta_n)||u_n - \omega|| \\ &\leq ||u_n - \omega||, \end{aligned}$$

which implies that

$$\begin{aligned} ||u_{n+1} - \omega|| &= ||\alpha_n F v_n + (1-\alpha_n)u_n - \omega|| \\ &\leq \alpha_n||Fv_n - \omega|| + (1-\alpha_n)||u_n - \omega|| \\ &\leq \alpha_n||v_n - \omega|| + (1-\alpha_n)||u_n - \omega|| \\ &\leq ||u_n - \omega||. \end{aligned}$$

Thus $\{||u_n - \omega||\}$ is essentially non-increasing and also bounded. We thus conclude that $\lim_{n\to\infty} ||u_n - \omega||$ exists whenever one choose any $\omega \in f(F)$. □

Theorem 3.2. *Let K be a nonempty subset of a Banach space having Opial property. Assume that $F : K \to K$ satisfies the condition (CC) with $f(F) \neq \emptyset$. Let $\{u_n\}$ be the sequence of Ishikawa iteration defined by (1.4) with $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then $\{u_n\}$ converges weakly to a fixed point of F.*

Proof. If we choose $\omega \in f(F)$ and applying Lemma 2.2, one can choose a strictly increasing continuous selfmap $\psi : [0,\infty) \to [0,\infty)$ in such a way that $\psi(0) = 0$ and so

$$\begin{aligned} ||v_n - \omega||^2 &= ||\beta_n F u_n + (1-\beta_n)u_n - \omega||^2 \\ &\leq \beta_n||Fu_n - \omega||^2 + (1-\beta_n)||u_n - \omega||^2 - \beta_n(1-\beta_n)\psi(||Fu_n - u_n||) \\ &\leq \beta_n||u_n - \omega||^2 + (1-\beta_n)||u_n - \omega||^2 - \beta_n(1-\beta_n)\psi(||Fu_n - u_n||) \\ &\leq ||u_n - \omega||^2 - \beta_n(1-\beta_n)\psi(|Fu_n - u_n||). \end{aligned}$$

Thus

$$\begin{aligned} ||u_{n+1} - q||^2 &= ||\alpha_n F v_n + (1-\alpha_n)Fu_n - \omega||^2 \\ &\leq \alpha_n||Fv_n - \omega||^2 + (1-\alpha_n)||u_n - \omega||^2 - \alpha_n(1-\alpha_n)\psi(||Fv_n - u_n||) \\ &\leq \alpha_n||v_n - \omega||^2 + (1-\alpha_n)||u_n - \omega||^2 - \alpha_n(1-\alpha_n)\psi(||Fv_n - u_n||) \\ &\leq \alpha_n||v_n - \omega||^2 + (1-\alpha_n)||u_n - \omega||^2 \\ &\leq \alpha_n||v_n - \omega||^2 + (1-\alpha_n)||u_n - \omega||^2 - \alpha_n\beta_n(1-\beta_n)\psi(||Fu_n - u_n||). \end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} a^2(1-b)\psi(||Fu_n - u_n|| \leq \sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)\psi(||Fu_n - u_n|| < \infty. \qquad (3.1)$$

Thus $\lim_{n\to\infty}\psi(||Fu_n - u_n||) = 0$. Since ψ is strictly increasing and continuous, we have

$$\lim_{n\to\infty}||Fu_n - u_n|| = 0 \tag{3.2}$$

We are going to provide the weak convergence of $\{u_n\}$ in the set $f(F)$. To achieve this aim, we may to prove that $\{u_n\}$ essentially endowed with a unique weak subsequential limit in the associated set $f(F)$. We now take two weakly convergent subsequences $\{u_{n_j}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ having weak limits p and q, respectively. Using (3.2), one has $\lim_{j\to\infty}||u_{n_j} - Fu_{n_j}|| = 0$. According to the Proposition 2.1(iii), one obtain $p \in f(F)$. Using the same techniques, it can be seen that $q \in f(F)$. The next aim is to establish that $p = q$. We assume that they are not same, then according to the Lemma 3.1 and also using the Opial condition of F, one have the following

$$\begin{aligned}\lim_{n\to\infty}||u_n - p|| = \lim_{j\to\infty}||u_{n_j} - p|| < \lim_{j\to\infty}||u_{n_j} - q|| &= \lim_{n\to\infty}||u_n - q|| \\ = \lim_{k\to\infty}||u_{n_k} - q|| < \lim_{k\to\infty}||u_{n_k} - p|| &= \lim_{n\to\infty}||u_n - p||.\end{aligned}$$

Hence we have reached to a contradiction. And so we have proved the result. □

Theorem 3.3. *Let K be a nonempty closed convex subset of a uniformly convex Banach space. Assume that $F : K \to K$ satisfies the condition (CC). If $f(F) \neq \emptyset$ and $\liminf_{n\to\infty} dist(u_n, f(F)) = 0$, then $\{u_n\}$ generated by (1.4) with $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ converges strongly to a fixed point of F.*

Proof. By Lemma 3.1, $\lim_{n\to\infty}||u_n - \omega||$ exists, for each $\omega \in f(F))$. So, $\lim_{n\to\infty} dist(u_n, f(F))$ exists, thus

$$\lim_{n\to\infty} dist(u_n, f(F)) = 0.$$

We must have two subsequences $\{u_{n_j}\}$ in $\{u_n\}$ and $\{s_j\}$ in the set $f(F)$ in the way that $||u_{n_j} - s_j|| \leq \frac{1}{2^j}$ for every choice of $j \geq 1$.But we have noted in the Lemma 3.1 that $\{u_n\}$ is nonicreasing. Consequently

$$||u_{n_{j+1}} - s_j|| \leq ||u_{n_j} - s_j|| \leq \frac{1}{2^j}.$$

Therefore,

$$\begin{aligned}||s_{j+1} - s_j|| &\leq ||s_{j+1} - u_{n_{j+1}}|| + ||u_{n_{j+1}} - s_j|| \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &\leq \frac{1}{2^{j-1}} \to 0, \text{ as } j \to \infty.\end{aligned}$$

We have noted from the above that, the sequence $\{s_j\}$ is in-fact a Cauchy in the set $f(F)$. According to the

Hence, we conclude that $\{s_j\}$ is a Cauchy sequence in $f(F)$ and hence strongly convergent to some ω. According to the Proposition 2.1(ii), the set $f(F)$ is closed in K, one must have $\omega \in f(F)$. If we apply Lemma 3.1, one must have $\lim_{n\to\infty}||u_n - \omega||$ exists. Thus we have noted that ω is a strong limit for $\{u_n\}$. □

Finally, we establish a strong convergence result using condition (I). Notice that for a selmap F of a subset K in the setting of a Banach space enjoys the the condition (I) [19] in the case, when one has a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ in the way that $\varphi(0) = 0$ and $\varphi(z) > 0$ for any $z > 0$ and the following facts are valid

$$||u - Fu|| \geq \varphi(dist(u, f(F))) \text{ for each } u \in K.$$

Theorem 3.4. *Let K be a nonempty closed convex subset of uniformly convex Banach space. Assume that $F : K \to K$ satisfies the condition with $f(F) \neq \emptyset$. Let $\{u_n\}$ be the sequence of Ishikawa iterates defined by (1.4) with $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. If F enjoys the condition (I) then $\{u_n\}$ converges strongly to a fixed point of F.*

Proof. From (3.2), it follows that

$$\liminf_{n\to\infty} ||Fu_n - u_n|| = 0. \tag{3.3}$$

The selfmap F enjoys condition (I), one has

$$||u_n - Fu_n|| \geq \varphi(\text{dist}(u_n, f(F))).$$

From (3.3), we conclude that

$$\liminf_{n\to\infty} \varphi(\text{dist}(u_n, f(F))) = 0.$$

But the map $\varphi : [0, \infty) \to [0, \infty)$ is nondecreasing and enjoy $\varphi(0) = 0$ and $\varphi(z) > 0$ for all $z \in (0, \infty)$, thus

$$\liminf_{n\to\infty} \text{dist}(u_n, f(F)) = 0.$$

Thus all the requirements of Theorem 3.3 are satisfied and so $\{u_n\}$ converges strongly to an element of $f(F)$. □

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